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Fundamental properties of Tsallis relative entropy

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Fundamental properties for the Tsallis relative entropy in both classical and quantum systems are studied. As one of our main results, we give the parametric extension of the trace inequality between the quantum relative entropy and the minus of the trace of the relative operator entropy given by Hiai and Petz. The monotonicity of the quantum Tsallis relative entropy for the trace preserving completely positive linear map is also shown without the assumption that the density operators are invertible. The generalized Tsallis relative entropy is defined and its subadditivity is shown by its joint convexity. Moreover, the generalized Peierls–Bogoliubov inequality is also proven. © *2004 American Institute of Physics.*
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I. INTRODUCTION

In the field of the statistical physics, Tsallis entropy was defined in Ref. 28 by $S_q(X) = -\Sigma_x p(x)^q \ln_q p(x)$ with one parameter q as an extension of Shannon entropy, where q-logarithm is defined by $\ln_q(x) \equiv (x^{1-q}-1)/(1-q)$ for any non-negative real number q and x, and $p(x) \equiv p(X = x)$ is the probability distribution of the given random variable X. We easily find that the Tsallis entropy $S_q(X)$ converges to the Shannon entropy $-\Sigma_x p(x) \log p(x)$ as $q \to 1$, since q-logarithm uniformly converges to natural logarithm as $q \to 1$. Tsallis entropy plays an essential role in nonextensive statistics, which is often called Tsallis statistics, so that many important results have been published from the various points of view.[29] As a matter of course, the Tsallis entropy and its related topics are mainly studied in the field of statisitical physics. However the concept of entropy is important not only in thermodynamical physics and statistical physics but also in information theory and analytical mathematics such as operator theory and probability theory. Recently, information theory has been in progress as quantum information theory[19] with the help of the operator theory[5,12] and the quantum entropy theory.[20] To study a certain entropic quantity is important for the development of information theory and the mathematical interest itself. In particular, the relative entropy is fundamental in the sense that it produces the entropy and the mutual information as special cases. Therefore in the present paper, we study properties of the Tsallis relative entropy in both the classical and quantum systems.

In the rest of this section, we will review several fundamental properties of the Tsallis relative entropy, as giving short proofs for the convenience of the readers. See Refs. 7, 27, and 26, for the pioneering works of the Tsallis relative entropy and their applications in the classical system.

Definition 1.1: We suppose a_j and b_j are probability distributions satisfying $a_j \geqslant 0$, $b_j \geqslant 0$, and

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$\Sigma_{j=1}^{n}a_j=\Sigma_{j=1}^{n}b_j=1$. Then we define the Tsallis relative entropy between $A=\{a_j\}$ and $B=\{b_j\}$, for any $q \geqslant 0$ as

$$D_q(A|B) \equiv -\sum_{j=1}^{n} a_j \ln_q \frac{b_j}{a_j}, \tag{1}$$

where q-logarithm function is defined by $\ln_q(x) \equiv (x^{1-q}-1)/(1-q)$ for non-negative real number x and q, and we make a convention $0 \ln_q \infty \equiv 0$.

Note that $\lim_{q\to 1} D_q(A|B)=D_1(A|B) \equiv \Sigma_{j=1}a_j \log(a_j/b_j)$, which is known as relative entropy (which is often called Kullback–Leibler information, divergence or cross entropy). For the Tsallis relative entropy, the following proposition is known.

Proposition 1.2:

(1) (Non-negativity) $D_q(A|B) \geqslant 0$.

(2) (Symmetry) $D_q(a_{\pi(1)},\ldots,a_{\pi(n)}|b_{\pi(1)},\ldots,b_{\pi(n)})=D_q(a_1,\ldots,a_n|b_1,\ldots,b_n)$.

(3) (Possibility of extention) $D_q(a_1,\ldots,a_n,0|b_1,\ldots,b_n,0)=D_q(a_1,\ldots,a_n|b_1,\ldots,b_n)$.

(4) (Pseudoadditivity)

$$D_q(A^{(1)}\times A^{(2)}|B^{(1)}\times B^{(2)})=D_q(A^{(1)}|B^{(1)})+D_q(A^{(2)}|B^{(2)})+(q-1)D_q(A^{(1)}|B^{(1)})D_q(A^{(2)}|B^{(2)}),$$

where

$$A^{(1)}\times A^{(2)}=\{a_j^{(1)}a_j^{(2)}|a_j^{(1)}\in A^{(1)},a_j^{(2)}\in A^{(2)}\},$$

$$B^{(1)}\times B^{(2)}=\{b_j^{(1)}b_j^{(2)}|b_j^{(1)}\in B^{(1)},b_j^{(2)}\in B^{(2)}\}.$$

(5) (Joint convexity) For $0 \leqslant \lambda \leqslant 1$, any $q \geqslant 0$ and the probability distributions $A^{(i)}=\{a_j^{(i)}\}$, $B^{(i)}=\{b_j^{(i)}\}$ $(i=1,2)$, we have

$$D_q(\lambda A^{(1)}+(1-\lambda)A^{(2)}|\lambda B^{(1)}+(1-\lambda)B^{(2)}) \leqslant \lambda D_q(A^{(1)}|B^{(1)})+(1-\lambda)D_q(A^{(2)}|B^{(2)}).$$

(6) (Strong additivity)

$$D_q(a_1,\ldots,a_{i-1},a_{i_1},a_{i_2},a_{i+1},\ldots,a_n|b_1,\ldots,b_{i-1},b_{i_1},b_{i_2},b_{i+1},\ldots,b_n)$$
$$=D_q(a_1,\ldots,a_n|b_1,\ldots,b_n)+b_i^{1-q}a_i^qD_q\left(\frac{a_{i_1}}{a_i},\frac{a_{i_2}}{a_i}\middle|\frac{b_{i_1}}{b_i},\frac{b_{i_2}}{b_i}\right),$$

where $a_i=a_{i_1}+a_{i_2}$, $b_i=b_{i_1}+b_{i_2}$.

Proof: (1) follows from the convexity of the function $-\ln_q(x)$:

$$D_q(A|B) \equiv -\sum_{j=1}^{n} a_j \ln_q \frac{b_j}{a_j} \geqslant -\ln_q\left(\sum_{j=1}^{n} a_j \frac{b_j}{a_j}\right)=0.$$

(2) and (3) are trivial. (4) follows by the direct calculation. (5) follows from the generalized log-sum inequality:[7]

$$\sum_{i=1}^{n} \alpha_i \ln_q\left(\frac{\beta_i}{\alpha_i}\right) \leqslant \left(\sum_{i=1}^{n} \alpha_i\right)\ln_q\left(\frac{\sum_{i=1}^{n}\beta_i}{\sum_{i=1}^{n}\alpha_i}\right), \tag{2}$$

for non-negative numbers α_i,β_i $(i=1,2,\ldots,n)$ and any $q \geqslant 0$. We define the function L_q for $q \geqslant 0$ to prove (6) as

$$L_q(x,y) \equiv -x \ln_q \frac{y}{x}$$

and

$$a_{i_1} = a_i(1-s), \quad b_{i_1} = b_i(1-t),$$

$$a_{i_2} = a_i s, \quad b_{i_2} = b_i t.$$

Then we have

$$L_q(x_1x_2,y_1y_2) = x_2L_q(x_1,y_1) + x_1L_q(x_2,y_2) + (q-1)L_q(x_1,y_1)L_q(x_2,y_2),$$

which implies the claim with easy calculations. ∎

Remark 1.3: 1. (1) of Proposition 1.2 implies

$$S_q(A) \leqslant \ln_q n,$$

since we have

$$D_q(A|U) = -n^{q-1}(S_q(A) - \ln_q n),$$

for any $q \geqslant 0$ and two probability distributions $A=\{a_j\}$ and $U=\{u_j\}$, where $u_j=1/n$, $(^\forall j)$, where the Tsallis entropy is represented by

$$S_q(A) \equiv -\sum_{j=1}^{n} a_j^q \ln_q a_j.$$

2. (4) of Proposition 1.2 is reduced to the pseudoadditivity for the Tsallis entropy:

$$S_q(A^{(1)} \times A^{(2)}) = S_q(A^{(1)}) + S_q(A^{(2)}) + (1-q)S_q(A^{(1)})S_q(A^{(2)}). \tag{3}$$

3. (5) of Proposition 1.2 recover the concavity for the Tsallis entropy, by setting $B^{(1)} = \{1,0,\ldots,0\}$, $B^{(2)}=\{1,0,\ldots,0\}$.

4. (6) of Proposition 1.2 is reduced to the strong additivity for the Tsallis entropy:

$$S_q(a_1,\ldots,a_{i-1},a_{i_1},a_{i_2},a_{i+1},\ldots,a_n) = S_q(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n) + a_i^q S_q\left(\frac{a_{i1}}{a_i},\frac{a_{i2}}{a_i}\right).$$

We finally show the monotonicity for the Tsallis relative entropy. To this end, we introduce some notations. We consider the transition probability matrix W: $\mathcal{A} \to \mathcal{B}$, which can be identified to the matrix having the conditional probability W_{ji} as elements, where $\mathcal{A}$ and $\mathcal{B}$ are alphabet sets (finite sets) and $\sum_{j=1}^{m} W_{ji}=1$ for all $i=1,\ldots,n$. By $A=\{a_i^{(\mathrm{in})}\}$ and $B=\{b_i^{(\mathrm{in})}\}$, two distinct probability distributions in the input system $\mathcal{A}$ are denoted. Then the probability distributions in the output system $\mathcal{B}$ are represented by $WA=\{a_j^{(\mathrm{out})}\}$, $WB=\{b_j^{(\mathrm{out})}\}$, where $a_j^{(\mathrm{out})}=\sum_{i=1}^{n} a_i^{(\mathrm{in})}W_{ji}$, $b_j^{(\mathrm{out})} = \sum_{i=1}^{n} b_i^{(\mathrm{in})}W_{ji}$, in terms of $W=\{W_{ji}\}$ $(i=1,\ldots,n; j=1,\ldots,m)$. Then we have the following.

Proposition 1.4: In the above notation, for any $q \geqslant 0$, we have

$$D_q(WA|WB) \leqslant D_q(A|B).$$

Proof: Applying the generalized log-sum inequality Eq. (2), we have

$$D_q(WA|WB) = -\sum_{j=1}^{m} a_j^{(\mathrm{out})} \ln_q \frac{b_j^{(\mathrm{out})}}{a_j^{(\mathrm{out})}} = -\sum_{j=1}^{m}\sum_{i=1}^{n} a_i^{(\mathrm{in})} W_{ji} \ln_q \frac{\sum_{i=1}^{n} b_i^{(\mathrm{in})} W_{ji}}{\sum_{i=1}^{n} a_i^{(\mathrm{in})} W_{ji}} \leq$$

$$-\sum_{j=1}^{m}\sum_{i=1}^{n} a_i^{(\mathrm{in})} W_{ji} \ln_q \frac{b_i^{(\mathrm{in})} W_{ji}}{a_i^{(\mathrm{in})} W_{ji}} = -\sum_{i=1}^{n} a_i^{(\mathrm{in})} \ln_q \frac{b_i^{(\mathrm{in})}}{a_i^{(\mathrm{in})}} = D_q(A|B).$$

■

We note that the above proposition is a special case of the monotonicity of f divergence[9] for the convex function f. Closing the introduction, we should also note here that the Tsallis entropy can be derived by a simple transformation from Rényi entropy which was used before the Tsallis one in the mathematical literature. See Ref. 4 on the details of Rényi entropy, in particular see pp. 184–191 of Ref. 4 for the relation to the structural a-entropy[14] [or called the entropy of type β (Ref. 10)], which is one of the nonextensive entropies including the Tsallis entropy.

II. QUANTUM TSALLIS RELATIVE ENTROPY AND ITS PROPERTIES

In Refs. 1 and 2, the quantum Tsallis relative entropy was defined by

$$D_q(\rho|\sigma) \equiv \frac{1 - \mathrm{Tr}[\rho^q \sigma^{1-q}]}{1-q} \tag{4}$$

for two density operators ρ and σ and $0 \leq q < 1$, as one parameter extension of the definition of the quantum relative entropy by Umegaki[30]

$$U(\rho|\sigma) \equiv \mathrm{Tr}[\rho(\log \rho - \log \sigma)]. \tag{5}$$

See Chap. II written by Rajagopal in Ref. 29, for the quantum version of Tsallis entropies and their applications.

For the quantum Tsallis relative entropy $D_q(\rho|\sigma)$ and the quantum relative entropy $U(\rho|\sigma)$, the following relations are known.

Proposition 2.1 [Ruskai–Stillinger[24] (see also Ref. 21)]: For the strictly positive operators with a unit trace ρ and σ, we have

(1) $D_q(\rho|\sigma) \leq U(\rho|\sigma) \leq D_{2-q}(\rho|\sigma)$ for $0 \leq q < 1$.
(2) $D_{2-q}(\rho|\sigma) \leq U(\rho|\sigma) \leq D_q(\rho|\sigma)$ for $1 < q \leq 2$.

Note that both sides in both inequalities converge to $U(\rho|\sigma)$ as $q \to 1$. We must extend this definition of the Tsallis relative entropy Eq. (4) for $0 \leq q \leq 2$ and impose the invertibility on the density operators of $D_{2-q}(\rho|\sigma)$ for $0 \leq q < 1$ and of $D_q(\rho|\sigma)$ for $1 < q \leq 2$.

Proof: Since we have for any $x > 0$ and $t > 0$,

$$\frac{1 - x^{-t}}{t} \leq \log x \leq \frac{x^t - 1}{t},$$

the following inequalities hold for any $a, b, t > 0$:

$$a\left(\frac{1 - a^{-t} b^t}{t}\right) \leq a \log \frac{a}{b} \leq a\left(\frac{a^t b^{-t} - 1}{t}\right). \tag{6}$$

Let $\rho = \Sigma_i \lambda_i P_i$ and $\sigma = \Sigma_j \mu_j Q_j$ be the spectral decompositions. Since $\Sigma_i P_i = \Sigma_j Q_j = I$, then we have

$$\mathrm{Tr}\left[\frac{\rho^{1+t}\sigma^{-t}-\rho}{t}-\rho(\log\rho-\log\sigma)\right]=\sum_{i,j}\mathrm{Tr}\left[P_i\left\{\frac{\rho^{1+t}\sigma^{-t}-\rho}{t}-\rho(\log\rho-\log\sigma)\right\}Q_j\right]$$
$$=\sum_{i,j}\mathrm{Tr}\left[P_i\left(\frac{1}{t}\lambda_i^{1+t}\mu_j^{-t}-\frac{1}{t}\lambda_i-\lambda_i\log\lambda_i+\lambda_i\log\mu_j\right)Q_j\right]$$
$$=\sum_{i,j}\left(\frac{1}{t}\lambda_i^{1+t}\mu_j^{-t}-\frac{1}{t}\lambda_i-\lambda_i\log\lambda_i+\lambda_i\log\mu_j\right)\mathrm{Tr}[P_iQ_j]\geqslant 0.$$

The last inequality in the above is due to the inequality of the right-hand side of Eq. (6). Thus we have

$$\mathrm{Tr}[\rho(\log\rho-\log\sigma)]\leqslant\frac{1}{t}\mathrm{Tr}[\rho^{1+t}\sigma^{-t}-\rho].$$

The left-hand side inequality is proven by a similar way. Thus setting $1-q=t(>0)$ in the above, we have (1) in Proposition 2.1. Also we have (2) in Proposition 2.1, by setting $q-1=t(>0)$. ■

We next consider another relation on the quantum Tsallis relative entropy. In Ref. 11, the relative operator entropy was defined by

$$S(\rho|\sigma)\equiv\rho^{1/2}\log(\rho^{-1/2}\sigma\rho^{-1/2})\rho^{1/2},$$

for two strictly positive operators ρ and σ. If ρ and σ are commutative, then we have $U(\rho|\sigma)=-\mathrm{Tr}[S(\rho|\sigma)]$. For this relative operator entropy and the quantum relative entropy $U(\rho|\sigma)$, Hiai and Petz proved the following relation:

$$U(\rho|\sigma)\leqslant-\mathrm{Tr}[S(\rho|\sigma)],\tag{7}$$

in Ref. 15 (see also Ref. 16).

In our previous papers,[32] we introduced the Tsallis relative operator entropy $T_q(\rho|\sigma)$ as a parametric extension of the relative operator entropy $S(\rho|\sigma)$ such as

$$T_q(\rho|\sigma)\equiv\frac{\rho^{1/2}(\rho^{-1/2}\sigma\rho^{-1/2})^{1-q}\rho^{1/2}-\rho}{1-q},$$

for $0\leqslant q<1$ and strictly positive operators ρ and σ, in the sense that

$$\lim_{q\to1}T_q(\rho|\sigma)=S(\rho|\sigma).\tag{8}$$

Actually we should note that there is a slight difference between the two parameters q in the present paper and λ in the previous paper,[32] which is an extension of Ref. 13. If ρ and σ are commutative, then we have $D_q(\rho|\sigma)=-\mathrm{Tr}[T_q(\rho|\sigma)]$. Also we now have that

$$\lim_{q\to1}D_q(\rho|\sigma)=U(\rho|\sigma).\tag{9}$$

These relations, Eq. (7), Eq. (8), and Eq. (9) naturally lead us to show the following theorem as a parametric extension of Eq. (7).

Theorem 2.2: For $0\leqslant q<1$ and any strictly positive operators with unit trace ρ and σ, we have

$$D_q(\rho|\sigma)\leqslant-\mathrm{Tr}[T_q(\rho|\sigma)].\tag{10}$$

Proof: We denote the α-power mean $\sharp_\alpha$ by $A\sharp_\alpha B\equiv A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$. From Theorem 3.4 of Ref. 16, we have

$$\mathrm{Tr}[e^A \sharp_\alpha e^B] \leqslant \mathrm{Tr}[e^{(1-\alpha)A+\alpha B}]$$

for any $\alpha \in [0,1]$. Setting $A=\log \rho$ and $B=\log \sigma$, we have

$$\mathrm{Tr}[\rho \sharp_\alpha \sigma] \leqslant \mathrm{Tr}[e^{\log \rho^{1-\alpha}+\log \sigma^\alpha}].$$

Since the Golden–Thompson inequality $\mathrm{Tr}[e^{A+B}] \leqslant \mathrm{Tr}[e^A e^B]$ holds for any Hermitian operators A and B, we have

$$\mathrm{Tr}[e^{\log \rho^{1-\alpha}+\log \sigma^\alpha}] \leqslant \mathrm{Tr}[e^{\log \rho^{1-\alpha}} e^{\log \sigma^\alpha}] = \mathrm{Tr}[\rho^{1-\alpha}\sigma^\alpha].$$

Therefore

$$\mathrm{Tr}[\rho^{1/2}(\rho^{-1/2}\sigma\rho^{-1/2})^\alpha \rho^{1/2}] \leqslant \mathrm{Tr}[\rho^{1-\alpha}\sigma^\alpha]$$

which implies the theorem by taking $\alpha = 1-q$. ∎

Corollary 2.3 (Hiai–Petz[15,16]): For any strictly positive operators with unit trace ρ and σ, we have

$$\mathrm{Tr}[\rho(\log \rho - \log \sigma)] \leqslant \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]. \tag{11}$$

Proof: It follows by taking the limit as $q \to 1$ in both sides of Eq. (10). ∎

Thus the result proved by Hiai and Petz in Refs. 15 and 16 is recovered as a special case of Theorem 2.2.

For the quantum Tsallis relative entropy $D_q(\rho|\sigma)$, (i) pseudoadditivity and (ii) non-negativity are shown in Ref. 1, moreover (iii) joint convexity and (iv) monotonicity for projective mesurements, are shown in Ref. 2 Here we show the unitary invariance of $D_q(\rho|\sigma)$ and the monotonicity of that for the trace-preserving completely positive linear map.

Proposition 2.4: For $0 \leqslant q < 1$ and any density operators ρ and σ, the quantum relative entropy $D_q(\rho|\sigma)$ has the following properties.

(1) (Non-negativity) $D_q(\rho|\sigma) \geqslant 0$.
(2) (Pseudoadditivity) $D_q(\rho_1 \otimes \rho_2 | \sigma_1 \otimes \sigma_2) = D_q(\rho_1|\sigma_1) + D_q(\rho_2|\sigma_2) + (q-1)D_q(\rho_1|\sigma_1)D_q(\rho_2|\sigma_2)$.
(3) (Joint convexity) $D_q(\Sigma_j \lambda_j \rho_j | \Sigma_j \lambda_j \sigma_j) \leqslant \Sigma_j \lambda_j D_q(\rho_j|\sigma_j)$.
(4) The quantum Tsallis relative entropy is invariant under the unitary transformation U:

$$D_q(U\rho U^* | U\sigma U^*) = D_q(\rho|\sigma).$$

Proof: Since it holds that $f(q,x,y) \equiv (x - x^q y^{1-q})/(1-q) - (x-y) \geqslant 0$ for $x \geqslant 0$, $y \geqslant 0$, and $0 \leqslant q < 1$, we have $D_q(\rho|\sigma) \geqslant \mathrm{Tr}[\rho - \sigma]$, which implies (1), since ρ and σ are density operators. (See Proposition 3.16 of Ref. 21 on the so-called Klein inequality.)

(2) follows by the direct calculation.

(3) follows from the Lieb's theorem that for any operator Z and and $0 \leqslant t \leqslant 1$, the functional $f(A,B) \equiv \mathrm{Tr}[Z^* A^t Z B^{1-t}]$ is joint concave with respect to two positive operators A and B.

(4) is obvious by the use of Stone–Weierstrass approximation theorem. (It also can be shown by the application of Theorem 2.5.) ∎

(1) of the above proposition follows from the generalized Peierls–Bogoliubov inequality which will be shown in the next section.

In Ref. 22, the monotonicity for more generalized relative entropy was shown under the assumption of the invertibility of the density operators. Here we show the monotonicity for the quantum Tsallis relative entropy in the case of $0 \leqslant q < 1$ without the assumption of the invertibility of the density operators.

Theorem 2.5: For any trace-preserving completely positive linear map Φ, any density operators ρ and σ and $0 \leqslant q < 1$, we have

$$D_q(\Phi(\rho)|\Phi(\sigma)) \leqslant D_q(\rho|\sigma).$$

Proof: We prove this theorem in a similar way as Ref. 18. To this end, we first prove the monotonicity of $D_q(\rho|\sigma)$ for the partial trace Tr_B in the composite sysytem AB. Let ρ^{AB} and σ^{AB} be density operators in the composite system AB. From Refs. 20 and 31, there exists unitary operators U_j and the probability p_j such that

$$\rho^A \otimes \frac{I}{n} = \sum_j p_j (I \otimes U_j)\rho^{AB}(I \otimes U_j)^*,$$

where n and I present the dimension and identity operator of the system B, $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$ and $\sigma^A = \mathrm{Tr}_B[\sigma^{AB}]$. By the help of the joint concavity and the unitary invariance of the Tsallis relative entropy, we thus have

$$D_q\left(\rho^A \otimes \frac{I}{n} \middle| \sigma^A \otimes \frac{I}{n}\right) \le \sum_j p_j D_q((I \otimes U_j)\rho^{AB}(I \otimes U_j)^* | (I \otimes U_j)\sigma^{AB}(I \otimes U_j)^*)$$
$$= \sum_j p_j D_q(\rho^{AB}|\sigma^{AB}) = D_q(\rho^{AB}|\sigma^{AB}).$$

Since

$$D_q\left(\rho^A \otimes \frac{I}{n} \middle| \sigma^A \otimes \frac{I}{n}\right) = D_q(\rho^A|\sigma^A),$$

we thus have

$$D_q(\mathrm{Tr}_B(\rho^{AB})|\mathrm{Tr}_B(\sigma^{AB})) \le D_q(\rho^{AB}|\sigma^{AB}). \tag{12}$$

It is known[25] (see also Refs. 8, 18, and 19) that every trace-preserving completely positive linear map Φ has the following representation with some unitary operator U^{AB} on the total system AB and the projection (pure state) P^B on the subsystem B,

$$\Phi(\rho^A) = \mathrm{Tr}_B\, U^{AB}(\rho^A \otimes P^B)U^{AB*}.$$

Therefore we have the following result, by the result of Eq. (12) and the unitary invariance of $D_q(\rho|\sigma)$ again,

$$D_q(\Phi(\rho^A)|\Phi(\sigma^A)) \le D_q(U^{AB}(\rho^A \otimes P^B)U^{AB*}|U^{AB}(\sigma^A \otimes P^B)U^{AB*}) = D_q(\rho^A \otimes P^B|\sigma^A \otimes P^B)$$

which implies our claim, since $D_q(\rho^A \otimes P^B|\sigma^A \otimes P^B) = D_q(\rho^A|\sigma^A)$. ■

Setting $\sigma = (1/n)I$ in Theorem 2.5, we have the following corollary.

Corollary 2.6: For any trace-preserving completely positive linear unital map Φ, any density operator ρ and $0 \le q < 1$, we have

$$H_q(\Phi(\rho)) \ge H_q(\rho),$$

where $H_q(X) = (\mathrm{Tr}[X^q] - 1)/(1-q)$ represents the Tsallis entropy for density operator X, which is often called the quantum Tsallis entropy.

We note that Theorem 2.5 for the fixed σ, namely the monotonicity of the quantum Tsallis relative entropy in the case of $\Phi(\sigma) = \sigma$, was proved in Ref. 3 to establish Clausius' inequality.

Remark 2.7: It is known[19] (see also Ref. 23) that there is an equivalent relation between the monotonicity for the quantum relative entropy and the strong subadditivity for the quantum entropy. However in our case, we have not yet found such a relation. Because the pseudoadditivity of the q-logarithm function,

$$\ln_q xy = \ln_q x + \ln_q y + (1-q)\ln_q x \ln_q y,$$

disturbs us to derive the beautiful relation such as

$$D_q(p(x,y)|p(x)p(y)) = S_q(p(x)) + S_q(p(y)) - S_q(p(x,y))$$

for the Tsallis relative entropy $D_q(p(x,y)\,|\,p(x)p(y))$, the Tsallis entropy $S_q(p(x))$, $S_q(p(y))$, and the Tsallis joint entropy $S_q(p(x,y))$, even if our stage is in the classical system.

III. GENERALIZED TSALLIS RELATIVE ENTROPY

For any two positive operators A, B and any real number $q \in [0,1)$, we can define the generalized Tsallis relative entropy.

Definition 3.1:

$$D_q(A\|B) \equiv \frac{\mathrm{Tr}[A] - \mathrm{Tr}[A^q B^{1-q}]}{1-q}.$$

To avoid the confusions of readers, we use the different symbol $D_q(\cdot\|\cdot)$ for the generalized Tsallis relative entropy.

Since Lieb's concavity theorem is available for any positive operators A and B, the generalized Tsallis relative entropy has a joint convexity,

$$D_q\left(\sum_j \lambda_j A_j \middle\| \sum_j \lambda_j B\right) \leqslant \sum_j \lambda_j D_q(A_j\|B_j), \tag{13}$$

for the positive number λ_j satisfying $\Sigma_j \lambda_j = 1$ and any positive operators A_j and B_j. Then we have the subadditivity of the generalized Tsallis relative entropy between A_1+A_2 and B_1+B_2.

Theorem 3.2: For any positive operators A_1, A_2, B_1, and B_2, and $0 \leqslant q < 1$, we have the subadditivity

$$D_q(A_1 + A_2\|B_1 + B_2) \leqslant D_q(A_1\|B_1) + D_q(A_2\|B_2). \tag{14}$$

Proof: First we note that we have the following relation for any numbers α and β, and two positive operators A and B,

$$D_q(\alpha A\|\beta B) = \alpha D_q(A\|B) - \alpha \ln_q \frac{\beta}{\alpha} \mathrm{Tr}[A^q B^{1-q}]. \tag{15}$$

Now from Eq. (13), we have

$$D_q(\lambda_1 X_1 + \lambda_2 X_2\|\lambda_1 Y_1 + \lambda_2 Y_2) \leqslant \lambda_1 D_q(X_1\|Y_1) + \lambda_2 D_q(X_2\|Y_2)$$

for any positive operators X_1, X_2, Y_1, and Y_2, and λ_1, λ_2 $(\lambda_1+\lambda_2=1)$. Setting $A_i=\lambda_i X_i$ and $B_i=\lambda_i Y_i$ for $i=1,2$ in the above inequality, we have

$$D_q(A_1 + A_2\|B_1 + B_2) \leqslant \lambda_1 D_q\left(\frac{A_1}{\lambda_1}\middle\|\frac{B_1}{\lambda_1}\right) + \lambda_2 D_q\left(\frac{A_2}{\lambda_2}\middle\|\frac{B_2}{\lambda_2}\right).$$

Thus we have our claim due to Eq. (15). ∎

As a famous inequality in statistical physics, the Peierls–Bogoliubov inequality[17,6] is known. Finally, we prove the generalized Peierls–Bogoliubov inequality for the generalized Tsallis relative entropy in the following.

Theorem 3.3: For any positive operators A and B, $0 \leqslant q < 1$,

$$D_q(A\|B) \geqslant \frac{\mathrm{Tr}[A] - (\mathrm{Tr}[A])^q (\mathrm{Tr}[B])^{1-q}}{1-q}.$$

Proof: In general, we have the following Hölder's inequality:

$$|\mathrm{Tr}[XY]| \leq \mathrm{Tr}[|X|^s]^{1/s}\,\mathrm{Tr}[|Y|^t]^{1/t}, \tag{16}$$

for any bounded linear operators X and Y satisfying $\mathrm{Tr}[|X|^s]<\infty$ and $\mathrm{Tr}[|Y|^t]<\infty$ and for any $1<s<\infty$ and $1<t<\infty$ satisfying $(1/s)+(1/t)=1$. By setting $X=A^q$, $Y=B^{1-q}$, and $s=1/q$, $t=1/(1-q)$ in Eq. (16), we have

$$\mathrm{Tr}[A^q B^{1-q}] \leq (\mathrm{Tr}[A])^q (\mathrm{Tr}[B])^{1-q},$$

which implies our claim. ■

Note that Theorem 3.3 can be considered a noncommutative version of Eq. (2). If A and B are density operators, then the non-negativity of the quantum Tsallis relative entropy follows from Theorem 3.3.

IV. CONCLUSION

As we have seen, the monotonicity of the quantum Tsallis relative entropy for the trace-preserving completely positive map was shown. Also the trace inequality between the Tsallis quantum relative entropy and the Tsallis relative operator entropy was shown. It is remarkable that our inequality recovers the famous inequality shown by Hiai–Petz as $q \to 1$.

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