A sufficient condition on concavity of the auxiliary function appearing in quantum reliability function

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Abstract A sufficient condition on concavity of the auxiliary function which appears in the random coding exponent as the lower bound of the quantum reliability function for general quantum states is noted. The validity of its sufficient condition is shown by some numerical computations.

Keywords : Quantum reliability function, random coding exponent and quantum information theory.

1 Introduction

In quantum information theory, it is important to study the properties of the auxiliary function $E_q(\pi, s)$, which will be defined in the below, appearing in the lower bound with respect to the random coding in the reliability function for general quantum states. In classical information theory [1], the random coding exponent $E_r^c(R)$, the lower bound of the reliability function, is defined by

$$E_r^c(R) = \max_{p,s} \left[E_c(p,s) - sR\right].$$

As for the classical auxiliary function $E_c(p,s)$, it is well-known the following properties [1].

(a) $E_c(p,0) = 0$.

(b) $\frac{\partial E_c(p,s)}{\partial s}|_{s=0} = I(X;Y)$, where $I(X;Y)$ presents the classical mutual information.

(c) $E_c(p,s) > 0$ $(0 < s \leq 1)$. $E_c(p,s) < 0$ $(-1 < s \leq 0)$.

(d) $\frac{\partial E_c(p,s)}{\partial s} > 0$, $(-1 < s \leq 1)$.

(e) $\frac{\partial^2 E_c(p,s)}{\partial s^2} \leq 0$, $(-1 < s \leq 1)$.

In figure 1, we suppose that p^* is *a priori* probability which attains the maximum of the classical mutual information. We then find that there exists a code satisfying $E_r^c(R) > 0$ by the above properties. Thus the upper bound [1] of the error probability P_e due to the random coding and the maximum likelihood decoding

$$P_e \leq \exp\left[-nE_r^c(R)\right], \quad (0 \leq s \leq 1)$$

goes to 0 as the code length $n \to \infty$.

In quantum case, the corresponding properties to (a),(b),(c) and (d) have been shown in [3, 5]. Also the concavity of the auxiliary function $E_q(\pi, s)$ is shown in the case when the signal states are pure [4], and when the expurgation method is adopted [5]. However, for general signal states, the concavity of the auxiliary function $E_q(\pi, s)$ which corresponds to (e) in the above has remained as an open question [3, 5].

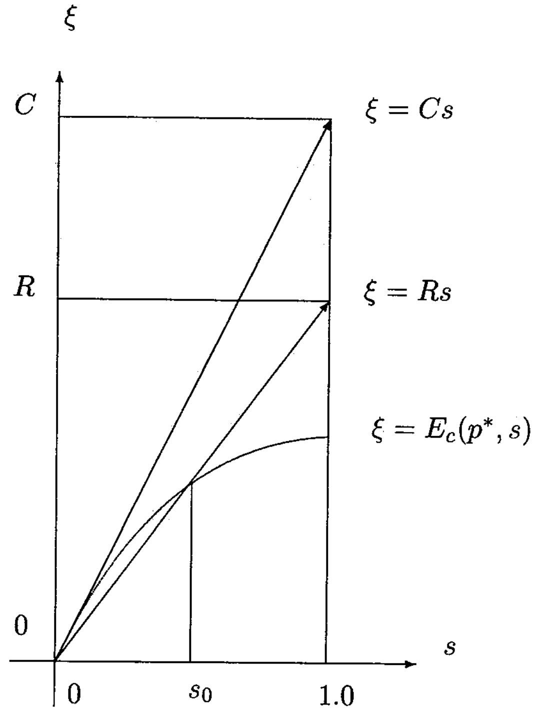

Figure 1: The sketch of the auxiliary function $\xi = E_c(p^*, s)$ in $0 \le s \le 1$.

2 Setting it up

The reliability function of classical-quantum channel is defined by

$$E(R) \equiv -\liminf_{n \to \infty} \frac{1}{n} \log P_e(2^{nR}, n), \quad 0 < R < C, \tag{1}$$

where C is a classical-quantum capacity, R is a transmission rate $R = \frac{\log_2 M}{n}$ (n and M represent the number of the code words and the messages, respectively), $P_e(M, n)$ can be taken any minimal error probabilities of $\min_{\mathcal{W},\mathcal{X}} \bar{P}(\mathcal{W}, \mathcal{X})$ or $\min_{\mathcal{W},\mathcal{X}} P_{\max}(\mathcal{W}, \mathcal{X})$. These error probabilities are defined by

$$\begin{aligned} \bar{P}(\mathcal{W}, \mathcal{X}) &= \frac{1}{M} \sum_{j=1}^{M} P_j(\mathcal{W}, \mathcal{X}), \\ P_{\max}(\mathcal{W}, \mathcal{X}) &= \max_{1 \le j \le M} P_j(\mathcal{W}, \mathcal{X}), \end{aligned}$$

where

$$P_j(\mathcal{W}, \mathcal{X}) = 1 - \mathrm{Tr} S_{w^j} X_j$$

is the usual error probability associated with the positive operator valued measurement $\mathcal{X} = \{X_j\}$ satisfying $\sum_{j=1}^{M} X_j \le I$. Here we note S_{w^j} represents the density operator corresponding to the code word w^j chosen from the code(blook) $\mathcal{W} = \left\{w^1, w^2, \cdots, w^M\right\}$. For details, see [2, 3, 5].

The lower bound for the quantum reliability function defined in Eq.(1), when we use random coding, is given by

$$E(R) \geq E_r^q(R) \equiv \max_{\pi} \sup_{0<s\leq 1} \left[E_q(\pi, s) - sR\right],$$

where $\pi = \{\pi_1, \pi_2, \cdots, \pi_a\}$ is *a priori* probability distribution satisfying $\sum_{i=1}^{a} \pi_i = 1$ and

$$\begin{aligned} E_q(\pi, s) &= -\log G(s), \\ G(s) &= \mathrm{Tr}\left[A(s)^{1+s}\right], \\ A(s) &= \sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}, \end{aligned}$$

where each S_i is density operator which corresponds to the output state of the classical-quantum channel $i \to S_i$ from the set of the input alphabet $A = \{1, 2, \cdots, a\}$ to the set of the output quantum states in the Hilbert space $\mathcal{H}$.

3 A sufficient condition on concavity of the auxiliary function $E_q(\pi, s)$

Proposition 3.1 For any real number s $(-1 < s \leq 1)$, density operators $S_i (i = 1, \cdots, a)$ and *a priori* probability $\pi = \{\pi_i\}_{i=1}^{a}$, if the operator inequality

$$\left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}} \left\{\sum_{j=1}^{a} \pi_j S_j^{\frac{1}{1+s}} \left(\log S_j^{\frac{1}{1+s}}\right)^2\right\} \left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}} \geq \left\{\sum_{i=1}^{a} \pi_i H\left(S_i^{\frac{1}{1+s}}\right)\right\}^2 \tag{2}$$

holds, then the auxiliary function

$$E_q(\pi, s) = -\log\left[\mathrm{Tr}\left\{\left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{1+s}\right\}\right]$$

is concave in s. Where $H(x) = -x \log x$ is the operator entropy.

(Proof) Since

$$\frac{\partial E_q(\pi, s)}{\partial s} = -G(s)^{-1} G'(s),$$

we have

$$\frac{\partial^2 E_q(\pi, s)}{\partial s^2} = G(s)^{-2}\left(G'(s)^2 - G(s) G''(s)\right).$$

By the use of the formula [5] for the operator valued function $A(s)$ w.r.t. the real number s,

$$\frac{d}{ds} \mathrm{Tr} f(s, A(s)) = \mathrm{Tr} f_s'(s, A(s)) + \mathrm{Tr} f_A'(s, A(s)) A'(s),$$

we have

$$\begin{aligned} G'(s) &= \mathrm{Tr}\left[A(s)^s \left(A(s) \log A(s) + (1+s) A'(s)\right)\right] \\ &= -\mathrm{Tr}\left[A(s)^s \Delta H(\pi, s)\right], \end{aligned} \tag{3}$$

where

$$\Delta H(\pi, s) = H(A(s)) - \sum_{i=1}^{a} \pi_i H(S_i^{\frac{1}{1+s}}).$$

By some simple calculations, we have

$$\begin{aligned} G''(s) &= \mathrm{Tr}\left[A(s)^{-1+s}\left\{A(s)^2(\log A(s))^2 + s(1+s)A'(s)^2\right\}\right] \\ &+ \mathrm{Tr}\left[A(s)^{-1+s}\left\{A(s)(2(1+(1+s)\log A(s))A'(s) + (1+s)A''(s))\right\}\right], \end{aligned} \tag{4}$$

where

$$A'(s) = -\frac{1}{(1+s)^2}\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}} \log S_i, \tag{5}$$

$$A''(s) = \frac{1}{(1+s)^4}\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}} \left(2(1+s)\log S_i + (\log S_i)^2\right). \tag{6}$$

Substituting (5) and (6) into (4), we have

$$\begin{aligned} G''(s) &= \mathrm{Tr}\left[A(s)^{-1+s}\left\{H(A(s))^2 + \frac{s}{1+s}\left(\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right)^2 - 2H(A(s))\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right.\right. \\ &\left.\left. + \frac{1}{1+s}\sum_{i=1}^{a}\pi_i S_i^{\frac{1}{1+s}}\sum_{j=1}^{a}\pi_j S_j^{\frac{1}{1+s}}(\log S_j^{\frac{1}{1+s}})^2\right\}\right] \\ &= \mathrm{Tr}\left[A(s)^{-1+s}\left\{H(A(s))^2 - 2H(A(s))\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}}) + \left(\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right)^2\right.\right. \\ &\left.\left. + \frac{1}{1+s}\sum_{i=1}^{a}\pi_i S_i^{\frac{1}{1+s}}\sum_{j=1}^{a}\pi_j S_j^{\frac{1}{1+s}}(\log S_j^{\frac{1}{1+s}})^2 - \frac{1}{1+s}\left(\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right)^2\right\}\right] \end{aligned} \tag{7}$$

By Cauchy-Schwarz inequality, we have

$$G'(s)^2 - G(s)\widetilde{G''}(s) \le 0, \tag{8}$$

where

$$\widetilde{G''}(s) = \mathrm{Tr}\left[A(s)^{-1+s}\Delta H(\pi, s)^2\right]. \tag{9}$$

Therefore if we have

$$G'(s)^2 - G(s)G''(s) \le G'(s)^2 - G(s)\widetilde{G''}(s) \tag{10}$$

that is,

$$\widetilde{G''}(s) \le G''(s), \tag{11}$$

then the theorem holds. From (7) and (9), (11) can be deformed,

$$\begin{aligned} 0 &\le \mathrm{Tr}\left[A(s)^{-1+s}\left\{-H(A(s))\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}}) + \sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})H(A(s))\right\}\right] \\ &+ \frac{1}{1+s}\mathrm{Tr}\left[A(s)^{-1+s}\left\{\sum_{i=1}^{a}\pi_i S_i^{\frac{1}{1+s}}\sum_{j=1}^{a}\pi_j S_j^{\frac{1}{1+s}}(\log S_j^{\frac{1}{1+s}})^2 - \left(\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right)^2\right\}\right]. \end{aligned} \tag{12}$$

Since $H(A(s))$ is commutative with $A(s)^{-1+s}$, the first term of (12) is equal to 0 so that (12) can be rewriten down

$$\frac{1}{1+s}\mathrm{Tr}\left[A(s)^{-1+s}\left\{\sum_{i=1}^{a}\pi_i S_i^{\frac{1}{1+s}}\sum_{j=1}^{a}\pi_j S_j^{\frac{1}{1+s}}(\log S_j^{\frac{1}{1+s}})^2-\left(\sum_{i=1}^{a}\pi_i H(S_i^{\frac{1}{1+s}})\right)^2\right\}\right]\geq 0 \quad (13)$$

which implies the proposition follows. ■

4 Numerical check

In this section, we check the validity of our sufficient condition by taking a simple example. We set $a=2$ and take two noncommutative density matrices such that

$$S_1=\begin{pmatrix}\frac{1}{5} & 0\\ 0 & \frac{4}{5}\end{pmatrix},\quad S_2=\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}.$$

Then the two eigenvalues of the following matrix

$$\left(\sum_{i=1}^{2}\pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}}\left\{\sum_{j=1}^{2}\pi_j S_j^{\frac{1}{1+s}}\left(\log S_j^{\frac{1}{1+s}}\right)^2\right\}\left(\sum_{i=1}^{2}\pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}}-\left\{\sum_{i=1}^{2}\pi_i H\left(S_i^{\frac{1}{1+s}}\right)\right\}^2. \quad (14)$$

are plotted by some numerical computations. From these figures, we may find the operator in (14) is positive. We are now trying to prove the inequality in (2) or (13). We hope to report on this proof in a forthcoming paper.

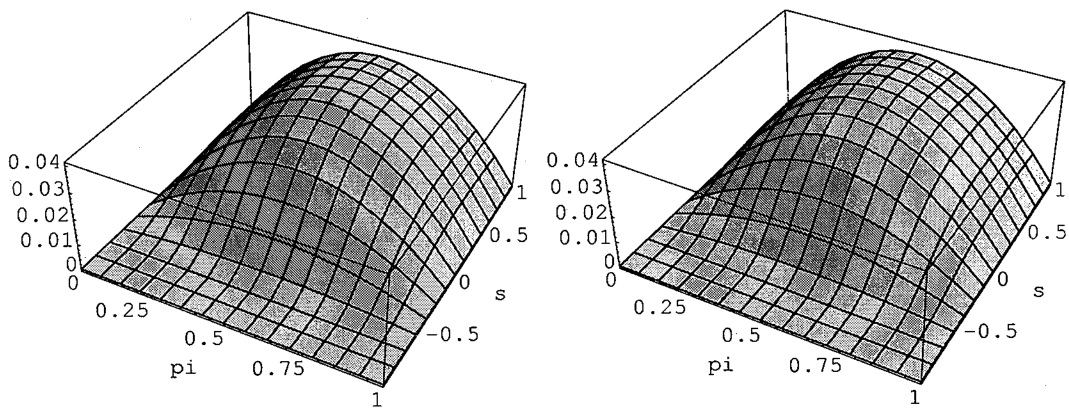

Figure 2: Eigenvalues are plotted as a function of the parameter s and *a priori* probability π.

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INFORMATION, Vol.6, No.1, January 2003

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