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ON LEFT HILBERT ALGEBRAS WITH RESPECT TO MINKOWSKY FORMS

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1. Introduction

In 1967, Professor Tomita wrote unpublished papers [4] and [5]. In those papers he introduced a notion, called a left Hilbert algebra, and proved the commutation theorem. That is, let $\mathfrak{A}$ be a left Hilbert algebra and $\mathscr{L}(\mathfrak{A})$ be the left von Neumann algebra associated with $\mathfrak{A}$, then the commutation theorem $J\mathscr{L}(\mathfrak{A})J=\mathscr{L}(\mathfrak{A})'$ holds.

In this paper we define a left Hilbert algebra with respect to a Minkowsky form and give a commutation theorem.

2. Adjoint Operations with respect to Minkowsky forms

Let $\mathscr{H}$ be a Hilbert space with an inner product (|). An hermitian sesquilinear form [,] is said to be a Minkowsky form if the following condition is satisfied:

$$\sup\{|[\xi, \eta]|;\ \|\eta\|\leqq 1\}=\|\xi\|.$$

For the Minkowsky form [,], there exists uniquely a unitary hermitian operator U such that $[\xi, \eta]=(U\xi|\eta)$ for all $\xi, \eta\in\mathscr{H}$.

DEFINITION 2.1. Let T be a densely defined linear (respectively, conjugate linear) operator, then T^{U} is defined as follows.

$$[T\xi, \eta]=[\xi, T^{\mathrm{U}}\eta] \quad ([T\xi, \eta]=[T^{\mathrm{U}}\eta, \xi]), \quad \xi\in\mathscr{D}(T), \quad \eta\in\mathscr{D}(T^{\mathrm{U}}).$$

It is evident that T^{U} is a closed operator and $T^{\mathrm{U}}=UT^{*}U$.

PROPOSITION 2.2. Let T be a densely defined operator. Then T is closable if and only if T^U is densely defined. If T is closable, T^{UU} is the closure $\overline{T}$ of T.

PROOF. If T is a densely defined closable operator, T^* is densely defined. Since $\mathscr{D}(T^U)=U^{-1}\mathscr{D}(T^*)=U\mathscr{D}(T^*)$, T^U is densely defined. Conversely if T^U is densely defined, so is T^* and hence T is closable. If T is closable, $T^{UU} = T^{**}=\overline{T}$.

DEFINITION 2.3. A linear (conjugate linear) operator T is said to be reflexive if $T\mathscr{D}(T)\subset\mathscr{D}(T)$ and $TT\xi=\xi$ for all $\xi\in\mathscr{D}(T)$.

PROPOSITION 2.4. If T is a reflexive closable operator, $\overline{T}$ is reflexive.

PROOF. For any $\xi\in\mathscr{D}(\overline{T})$, there exists a sequence $\{\xi_n\}$ in $\mathscr{D}(T)$ such that $\xi_n\to\xi$ and $T\xi_n\to\overline{T}\xi$. Since $\mathscr{D}(T)\ni T\xi_n\to\overline{T}\xi$ and $\xi_n=T(T\xi_n)\to\xi$, we have $\overline{T}\xi \in\mathscr{D}(\overline{T})$ and $\overline{T}(\overline{T}\xi)=\xi$. Therefore $\overline{T}$ is reflexive.

PROPOSITION 2.5. If T is a densely defined reflexive operator, T^* and T^U are reflexive.

PROOF. For any $\xi\in\mathscr{D}(T^U)$ and any $\eta\in\mathscr{D}(T)$, we have $[T\eta, T^U\xi]= [T(T\eta), \xi]=[\eta, \xi]$.

Thus $T^U\xi\in\mathscr{D}(T^U)$ and $T^U(T^U\xi)=\xi$ for any $\xi\in\mathscr{D}(T^U)$. Consequently T^U is reflexive.

3. U-homomorphisms of Left Hilbert Algebras with respect to Minkowsky Forms

Let $\mathfrak{A}$ be a $*$-algebra with an inner product (1) and a Minkowsky form [,] with respect to (1). Let $\mathscr{H}$ be the completion of $\mathfrak{A}$ with respect to (1) and U be the unitary hermitian operator associated with Minkowsky form [,]. From now on, we denote the involution by $\sharp$.

DEFINITION 3.1. Let $\mathfrak{A}$ be a $*$-algebra with an inner product (1) and a Minkowsky form [,]. $\mathfrak{A}$ is said to be a left Hilbert algebra with respect to the Minkowsky form if the following conditions are satisfied:

(1) $[\xi\eta, \zeta]=[\eta, \xi^\sharp\zeta]$ for any $\xi, \eta, \zeta\in\mathfrak{A}$;
(2) For any $\xi\in\mathfrak{A}$
a mapping: $\mathfrak{A}\ni\eta\to\xi\eta$ is continuous.

(3) $\mathfrak{A}^2$ is dense in $\mathfrak{A}$.
(4) The mapping: $\mathfrak{A} \in \xi \to \xi^\#$ is closable as conjugate linear operator on $\mathscr{H}$.

EXAMPLE. Let $\mathscr{H}$ be a Hilbert space, U be a unitary hermitian operator on $\mathscr{H}$ and M be a U-involutive algebra with a cyclic separating vector ξ_0, where the U-involution means $X^{\mathrm{U}} = UX^*U$. We set $\mathfrak{A} = M\xi_0$ and define the product and the involution by;

$$(X\xi_0)(Y\xi_0) = XY\xi_0 \qquad \text{for all} \quad X,\ Y \in M$$

$$(X\xi_0)^\# = X^{\mathrm{U}}\xi_0 \qquad \text{for all} \quad X \in M$$

Then $\mathfrak{A}$ is a left Hilbert algebra with respect to the Minkosky form associated with U.

PROOF. For any X, Y and $Z \in M$, we have

$$[(X\xi_0)(Y\xi_0),\ Z\xi_0] = [XY\xi_0,\ Z\xi_0] = [Y\xi_0,\ X^{\mathrm{U}}Z\xi_0] = [Y\xi_0,\ (X\xi_0)^\#(Z\xi_0)]\,.$$

Hence the equation (1) of definition 3.1. is satisfied. For each $X \in M$ and $Y \in M'$, we have

$$[(X\xi_0)^\#,\ Y\xi_0] = [X^{\mathrm{U}}\xi_0 Y\xi_0] = [\xi_0,\ XY\xi_0] = [\xi_0,\ YX\xi_0] = [Y^{\mathrm{U}}\xi_0,\ X\xi_0]\,.$$

Therefore the map: $X\xi_0 \to (X\xi_0)^\#$ is closable by proposition 2.2 and the density of $M'\xi_0$. Consequently we get that $\mathfrak{A}$ is a left Hilbert algebra with respect to the Minkowsky form. Q. E. D.

We denote the continuous extension of the map: $\mathfrak{A} \ni \eta \to \xi\eta$ by $\pi(\xi)$.

PROPOSITION 3.2. Let $\mathfrak{A}$ be a left Hilbert algebra with respect to a Minkowsky form. Then π is a non-degenerate U-involutive representation of $\mathfrak{A}$ on $\mathscr{H}$.

PROOF. For any ξ, η and $\zeta \in \mathfrak{A}$, we have

$$[\pi(\xi)\eta,\ \zeta] = [\xi\eta,\ \zeta] = [\eta,\ \xi^\#\zeta] = [\eta,\ \pi(\xi^\#)\zeta]$$

so that $\pi(\xi^\#) = \pi(\xi)^{\mathrm{U}}$. Q. E. D.

We denote the U-adjoint of the map: $\mathfrak{A} \in \xi \to \xi^\#$ by F and the closure of the map by S respectively.

DEFINITION 3.3. Let $\mathscr{D}^\#$ and $\mathscr{D}^b$ denote domains of S and F. We denote $S\xi$ by $\xi^{\#}$, $\xi\in\mathscr{D}^{\#}$, and $F\xi$ by ξ^{b}, $\xi\in\mathscr{D}^{b}$. Take and fix an η in $\mathscr{D}^{b}$. Define operators a and b by:

$$a\xi=\pi(\xi)\eta \quad \text{and} \quad b\xi=\pi(\xi)\eta^{b}, \qquad \xi\in\mathfrak{A}.$$

Then a and b are both densely defined operators.

PROPOSITION 3.4. a is closable, and $b\subset a^{\cup}$.

PROOF. We have, for each $\xi,\zeta\in\mathfrak{A}$,

$$\begin{aligned}[a\xi,\zeta]&=[\pi(\xi)\eta,\zeta]=[\eta,\pi(\xi^{\#})\zeta]=[\eta,\xi^{\#}\zeta]\\ &=[\zeta^{\#}\xi,\eta^{b}]=[\xi,\pi(\zeta)\eta^{b}]=[\xi,b\zeta]\,;\end{aligned}$$

so that $a^{\cup}\supset b$. Therefore it follows from proposition 2.2, that a is closable.

Let $\pi'(\eta)$ denote closure of a.

DEFINITION 3.5. If $\pi'(\eta)$, $\eta\in\mathscr{D}^{b}$ is bounded, then η is called π'-bounded. Let $\mathfrak{A}'$ denote the set of all π'-bounded elements. For each $\xi\in\mathscr{H}$ and $\eta\in\mathfrak{A}'$, define a product of ξ and η by: $\xi\eta=\pi'(\eta)\xi$.

PROPOSITION 3.6. If η belongs to $\mathscr{D}^{b}$ and x belongs to the strong closure of $\pi(\mathfrak{A})$, then $\pi'(\eta)$ commutes with x. In particular, we have $\pi'(\mathfrak{A}')\subset\pi(\mathfrak{A})'$.

PROOF. The Proposition is proved analogously with Lemma 3.1 in [3], For each $\xi_0\in\mathfrak{A}$ and $\eta\in\mathscr{D}^{b}$, we show $\pi(\xi_0)\pi'(\eta)\subset\pi'(\eta)\pi(\xi_0)$. For any ξ in $\mathfrak{A}$ and ζ in definition domain of $\pi'(\eta)^{\cup}$, we have

$$\begin{aligned}[\pi'(\eta)\xi,\pi(\xi_0)\zeta]&=[\pi(\xi_0^{\#}\xi)\eta,\zeta]=[\pi'(\eta)\xi_0^{\#}\xi,\zeta]\\ &=[\xi_0^{\#}\xi,\pi'(\eta)^{\cup}\zeta]=[\xi,\pi(\xi_0)\pi'(\eta)^{\cup}\zeta]\,.\end{aligned}$$

Hence we have

$$[\pi'(\eta)\xi,\pi(\xi_0)\zeta]=[\xi,\pi(\xi_0)\pi'(\eta)^{\cup}\zeta]$$

$$\text{for any} \quad \xi\in\mathscr{D}(\pi'(\eta)) \quad \text{and} \quad \zeta\in\mathscr{D}(\pi'(\eta)^{\cup})\,.$$

Thus we have $\pi(\xi_0)\pi'(\eta)^{\cup}\subset\pi'(\eta)^{\cup}\pi(\xi_0)$, and $\pi(\xi_0)\pi'(\eta)\subset\pi'(\eta)\pi(\xi_0)$.

In the next place, we show that for each η in $\mathscr{D}^{b}$ and x in the strong closure of $\pi(\mathfrak{A})$, $\pi'(\eta)$ commutes with x.

We can find a sequence $\{\xi_n\}$ in $\mathfrak{A}$ such that

$$\lim \pi(\xi_n)\zeta = x\zeta ;$$

$$\lim \pi(\xi_n)\pi'(\eta) = x\pi'(\eta)\zeta .$$

Then we get

$$\lim \pi'(\eta)\pi(\xi_n)\zeta = \lim \pi(\xi_n)\pi'(\eta)\zeta = x\pi'(\eta)\zeta .$$

From the closedness of $\pi'(\eta)$, $x\zeta$ belongs to $\mathscr{D}(\pi'(\eta))$ and $\pi'(\eta)x\zeta = x\pi'(\eta)\zeta$. Thus $\pi'(\eta)$ commutes with x. This completes the proof.

PROPOSITION 3.7. $\mathfrak{A}'$ is a $*$-algebra with a involution: $\eta \rightarrow \eta b$. Furthermore π' is a U-involutive anti-representation of $\mathfrak{A}'$ on $\mathscr{H}$.

PROOF. This is proved analogously with Lemma 3.2 in [3]. If η is π'-bounded, η^b belongs to $\mathscr{D}^b$ by Proposition 2.5. And it is trivial $\pi'(\eta^b) = \pi'(\eta)^U$. Hence we get η^b is π'-bounded. Take any two elements η_1 and η_2 in $\mathfrak{A}'$. We prove $\eta_1\eta_2$ belongs to $\mathfrak{A}'$ and

$$(\eta_1\eta_2)^b = \eta_2^b\eta_1^b$$

$$\pi'(\eta_1\eta_2) = \pi'(\eta_2)\pi'(\eta_1) .$$

For each $\xi \in \mathfrak{A}$, we have

$$\begin{aligned}[\xi, \eta_1\eta_2] &= [\xi, \pi'(\eta_2)\eta_1] = [\pi'(\eta_2)\xi, \eta_1] \\ &= [\pi'(\eta_2^b)\xi, \eta_1] = [\pi(\xi)\eta_2^b, \eta_1] \\ &= [\eta_2^b, \pi'(\eta_1)\xi^\#] = [\pi'(\eta_1^b)\eta_2^b, \xi^\#] \\ &= [\eta_2^b\eta_1^b, \xi^\#],\end{aligned}$$

so that $\eta_1\eta_2$ belongs to $\mathscr{D}^b$ and $(\eta_1\eta_2)^b = \eta_2^b\eta_1^b$. Moreover we have, for each $\xi \in \mathfrak{A}$,

$$\begin{aligned}\|\pi'(\eta_1\eta_2)\xi\| &= \|\pi(\xi)\eta_1\eta_2\| = \|\pi(\xi)\pi'(\eta_2)\eta_1\| \\ &= \|\pi'(\eta_2)\pi(\xi)\eta_1\| = \|\pi'(\eta_2)\pi'(\eta_1)\xi\| \\ &\leqq \|\pi'(\eta_2)\| \, \|\pi'(\eta_1)\| \, \|\xi\| ,\end{aligned}$$

so that $\eta_1\eta_2$ is π'-bounded. Thus $\eta_1\eta_2$ belongs to $\mathfrak{A}'$. It is trivial that $\pi'(\eta_1\eta_2) = \pi'(\eta_2)\pi'(\eta_1)$. This completes the proof.

If $\mathfrak{A}'$ is dense in the Hilbert space $\mathscr{D}^b$, we can define a closed operator $\pi(\xi)$, $\xi \in \mathscr{D}^\#$ as the closure of a operator:

$$\mathfrak{A}' \in \eta \longrightarrow \pi'(\eta)\xi.$$

DEFINITION 3.8. Let $\mathfrak{A}'$ be dense in $\mathscr{D}^b$. If $\pi(\xi)$, $\xi \in \mathscr{D}^\#$ is bounded, then ξ is called π-bounded. Let $\mathfrak{A}''$ denote the set of all π-bounded elements. For each $\xi \in \mathfrak{A}''$ and $\eta \in \mathscr{H}$, define a product of ξ and η by:

$$\xi\eta = \pi(\xi)\eta.$$

For $\mathfrak{A}''$ we obtain the same properties as $\mathfrak{A}'$.

PROPOTION 3.10. If $\mathfrak{A}'$ is dense in $\mathscr{D}^b$, then we get the following results:

(1) $\mathfrak{A}''$ is a left Hilbert algebra with respect to the Minkowsky form with the involution: $\mathfrak{A}'' \ni \xi \to \xi^\#$;

(2) $\mathfrak{A}$ is contained in $\mathfrak{A}''$ as a $*$-subalgebra;

(3) π is a U-involutive representation of $\mathfrak{A}''$ on $\mathscr{H}$;

(4) $\pi(\mathfrak{A}'') \subset \pi'(\mathfrak{A}')'$.

PROPOSITION 3.10. If $\mathfrak{A}'$ is dense in $\mathscr{D}^b$, then we get

$$\mathfrak{A}' = \mathfrak{A}''' = \mathfrak{A}^{(5)} = \cdots$$

$$\mathfrak{A}'' = \mathfrak{A}^{(4)} = \cdots .$$

PROOF. It is trivial that $\mathfrak{A}'''$ is contained in $\mathfrak{A}'$. Take an η in $\mathfrak{A}'$. From the π'-boundedness of η, there exists $\gamma > 0$ such that $\|\pi(\xi)\eta\| \leqq \gamma\|\xi\|$, $\xi \in \mathfrak{A}$. For any ξ in $\mathfrak{A}''$, we can choose a sequence $\{\xi_n\}$ in $\mathfrak{A}$ with

$$\lim \xi_n = \xi,$$

then we have

$$\begin{aligned} \|\pi'(\eta)\xi_n - \pi'(\eta)\xi_m\| &= \|\pi'(\eta)(\xi_n - \xi_m)\| \\ &= \|\pi(\xi_n - \xi_m)\eta\| \\ &\leqq \gamma\|\xi_n - \xi_m\|, \end{aligned}$$

so that $\{\pi'(\eta)\xi_n\}$ is a convergence sequence. From the closedness of $\pi'(\eta)$, ξ belongs to the domain of $\pi'(\eta)$ and we have

$$\pi'(\eta)\xi = \lim \pi'(\eta)\xi_n.$$

Hence

$$\|\pi'(\eta)\xi\| = \lim \|\pi'(\eta)\xi_n\| \leqq \gamma \lim \|\xi_n\| = \gamma\|\xi\| .$$

Therefore η belongs to $\mathfrak{A}'''$. This completes the proof.

4. A Commutation Theorem

PROPOSITION 4.1. Let $\mathfrak{A}$ be a left Hilbert algebra with respect to a Minkowsky form. If $\mathfrak{A}$ contains a unit element l, then $\pi(\mathfrak{A})'=\pi'(\mathfrak{A}')=\pi'(\mathfrak{A}')''$.

PROOF. From Proposition 3.6, we obtain $\pi'(\mathfrak{A}')$ is contained in $\pi(\mathfrak{A})'$. Take an element x in $\pi(\mathfrak{A})'$. We have, for each ξ in $\mathfrak{A}$,

$$\begin{aligned}[\xi^{\#}, xl]&=[\pi(\xi^{\#})l, xl]=[x^{\cup}\pi(\xi^{\#})l, l]\\ &=[\pi(\xi^{\#})x^{\cup}l, l]=[x^{\cup}l, \pi(\xi)l]\\ &=[x^{\cup}l, \xi],\end{aligned}$$

so that xl belongs to $\mathscr{D}^b$ and $(xl)^b=x^{\cup}l$. Furthermore we have, for each ξ in $\mathfrak{A}$,

$$\pi'(xl)\xi=\pi(\xi)xl=x\pi(\xi)l=x\xi,$$

so that xl belongs to $\mathfrak{A}'$ and $\pi'(xl)=x$. Hence x belongs to $\pi'(\mathfrak{A}')$. This completes the proof.

PROPOSITION 4.2. Let $\mathfrak{A}$ be a left Hilbert algebra with respect to a Minkowsky form which satisfies the following conditions:

(1) $\mathfrak{A}$ contains a unit element l;
(2) For each η in $\mathfrak{A}$,
 the map: $\mathfrak{A}\ni\xi\rightarrow\xi\eta$ is continuous;
(3) The involution: $\mathfrak{A}\ni\xi\rightarrow\xi^{\#}$ is continuous.
 Then we have $\pi(\mathfrak{A})'=\pi'(\mathfrak{A})''=\pi'(\mathfrak{A}')$.

PROOF. It follows from the condition (2) that $\mathfrak{A}'$ contains $\mathfrak{A}$. Since $\mathfrak{A}'$ is dence in the Hilbert space $\mathscr{D}^b$, we can define $\mathfrak{A}''$. From Proposition 4.1, we have

$$\pi(\mathfrak{A})''=\pi'(\mathfrak{A}')'=\pi(\mathfrak{A}'')=\pi(\mathfrak{A}'')''.$$

Now take x in $\pi'(\mathfrak{A})'$. Then we have, for each η in $\mathfrak{A}$,

$$\pi(xl)\eta=\pi'(\eta)xl=x\pi'(\eta)l=x\eta,$$

in the proof of Proposition 3.10, we find that xl belongs to $\mathfrak{A}''$ and $x=\pi(xl)$.

Hence x belongs to $\pi(\mathfrak{A}'')$.

On the other hand, $\pi(\mathfrak{A}'')$ is contained in $\pi'(\mathfrak{A})'$. Therefore we obtain

$$\pi'(\mathfrak{A})''=(\pi'(\mathfrak{A})')'=\pi(\mathfrak{A}'')'=\pi(\mathfrak{A})'''=\pi(\mathfrak{A})'.$$

This completes the proof.

We denote $\pi(\mathfrak{A})''$ by $\mathscr{L}(\mathfrak{A})$.

THEOREM 4.3. *Let $\mathfrak{A}$ be as in Proposition 4.2. Then there exists a reflexive bounded operator S on $\mathscr{H}$ such that $S\mathscr{L}(\mathfrak{A})S=\mathscr{L}(\mathfrak{A})'$.*

PROOF. Let S be the continuous extension of the map:

$$\mathfrak{A}\ni\xi\longrightarrow\xi^{\#}.$$

From Proposition 2.4., S is reflexive i.e.,

$$S^2=1.$$

Take a ξ in $\mathfrak{A}$. We have, for each η in $\mathfrak{A}$,

$$\begin{aligned}S\pi(\xi)S\eta&=S\pi(\xi)\eta^{\#}=S(\xi\eta^{\#})\\&=\eta\xi^{\#}=\pi'(S\xi)\eta,\end{aligned}$$

so that

$$S\pi(\xi)S=\pi'(S\xi).$$

Hence we get $S\pi(\mathfrak{A})S=\pi'(\mathfrak{A})$. Therefore we have, from Proposition 4.2.,

$$\begin{aligned}S\pi(\mathfrak{A})''S&=(S\pi(\mathfrak{A})S)''=\pi'(\mathfrak{A})''\\&=\pi(\mathfrak{A})',\end{aligned}$$

so that

$$S\mathscr{L}(\mathfrak{A})S=\mathscr{L}(\mathfrak{A})'.$$

COROLLARY. Let $\mathfrak{A}$ be as in Theorem 4.3. If $\mathfrak{A}$ satisfies the following condition:

$$[\xi^{\#},\eta^{\#}]=[\eta,\xi],\qquad \xi,\eta\in\mathfrak{A},$$

then $\mathscr{L}(\mathfrak{A})$ is anti-$*$-isomorphic to $\mathscr{L}(\mathfrak{A})'$.

PROOF. We easily obtain

$$[S\xi, S\eta]=[\eta, \xi], \qquad \xi, \eta \in \mathscr{H}.$$

That is, S is a U-unitary operator. Hence the map: $\mathscr{L}(\mathfrak{A}) \ni x \to SxS$ is an anti-$*$-isomorphism.

References

[1] J. DIXMIER, *Les algèbre d'opérateurs dans l'espace hilbertien,* Gauthier-villars, Paris, 2 é edition 1969.
[2] M. G. KREIN, *Introduction to the geometry of indefinite J-spaces and to the theory of operators in those spaces,* Amer. Math. Soc. Transl. **93** (1970), 103–176.
[3] M. TAKESAKI, *Tomita's Theory of Modular Hilbert Algebras and its Applications,* Lecture Notes, Springer **128** (1970).
[4] M. TOMITA, *Quasi-standard von Neumann algebras,* mimeographed note, (1967).
[5] M. TOMITA, *Standard forms of von Neumann algebras.* The Vth Functional Analysis Symposium of the Math. Soc. of Japan, Sendai. (1967).