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TOPOLOGIES ON UNBOUNDED OPERATOR ALGEBRAS

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Abstract

In this paper we study several topologies on unbounded operator algebras. The first purpose is to discuss the relations between the topologies on general unbounded operator algebras. Secondly we study the topologies on special unbounded operator algebras in details. Finally we study the relations between locally convex $*$-algebras and unbounded operator algebras.

§ 1. Introduction

In recent years several authors have investigated unbounded operator algebras in various situations [2, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18]. In particular, it may be important to study the topologies of them. Lassner has introduced in [13] a *uniform topology* t_u and a *quasi-uniform topology* t_{qu} on an unbounded operator algebra. Arnal and Jurzak [2, 12] have defined the topologies called *ρ-topology* t_ρ and *λ-topology* t_λ, which are in general different from the uniform and quasi-uniform topologies, however each topology mentioned above equals the operator-norm topology in case of the algebra of bounded operators. They have given necessary and sufficient conditions under which the ρ-topology is equal to the λ-topology.

In this paper we shall define the other topologies called *weak*, *quasi-weak*, *σ-weak*, *quasi-σ-weak*, *strong* and *σ-strong* (which are denoted by t_w, t_{qw}, $t_{\sigma w}$, $t_{q\sigma w}$, t_s and $t_{\sigma s}$, *respectively*). We shall obtain in Section 2 the following relations between their topologies:

$$\begin{array}{cccccccccccc} t_{qu} & \geqq & t_u & \geqq & t_w & \leqq & t_{qw} & \leqq & t_s & \leqq & t_{qu} \\ \wedge\!\| & & \wedge\!\| & & \wedge\!\| & & \wedge\!\| & & \wedge\!\| & & \wedge\!\| \\ t_\lambda & \geqq & t_\rho & \geqq & t_{\sigma w} & \leqq & t_{q\sigma w} & \leqq & t_{\sigma s} & \leqq & t_\lambda \end{array}$$

where the symbols $\tau_1 \geqq \tau_2$, $\tau_2 \leqq \tau_1$ and $\begin{array}{c}\tau_2\\ \wedge\!\| \\ \tau_1\end{array}$ mean "τ_1 is finer than τ_2".

It is well known that on infinite-dimensional algebras of bounded operators the following connections hold:

$$\left.\begin{array}{ccc} t_w & \lneqq & t_s \\ \wedge\!\| & & \wedge\!\| \\ t_{\sigma w} & \lneqq & t_{\sigma s} \end{array}\right\} \lneqq t_u = t_{qu} = t_\rho = t_\lambda = \textit{operator-norm topology}.$$

In general, various cases may be happened for unbounded operator algebras. We shall give in Section 3 four particular unbounded operator algebras and discuss their topologies in details.

Finally we shall discuss the relations between locally convex $*$-algebras and unbounded operator algebras. An unbounded operator algebra $\mathscr{A}$ equipped with the topology t_w (*resp.* $t_{\sigma w}$, t_u, t_ρ) is a locally convex $*$-algebra. We shall show in Section 4 that the locally convex $*$-algebra $(\mathscr{A}; t_w)$ (*resp.* $(\mathscr{A}; t_{\sigma w})$, $(\mathscr{A}; t_u)$, $(\mathscr{A}; t_\rho)$) is a GB^*-algebra defined by Allan [1] and Dixon [4] if and only if the unbounded operator algebra $\mathscr{A}$ is an $EC^\sharp$-algebra defined in [7].

§2. Topologies on $\sharp$-algebras

In this section we shall introduce various topologies on an unbounded operator algebra called a $\sharp$-algebra and study the relations between their topologies.

We begin with the definitions and notations of $\sharp$-algebras. Let $\mathfrak{D}$ be a pre-Hilbert space with inner product $(\ |\ )$ and $\mathfrak{h}$ the completion of $\mathfrak{D}$. By $\mathscr{L}(\mathfrak{D})$ we denote the set of all linear operators on $\mathfrak{D}$ and by $\mathscr{L}^\sharp(\mathfrak{D})$ we denote the set of all linear operators $A \in \mathscr{L}(\mathfrak{D})$ for which there exists an element $A^\sharp \in \mathscr{L}(\mathfrak{D})$ such that $(A\xi \mid \eta) = (\xi \mid A^\sharp \eta)$ for every $\xi, \eta \in \mathfrak{D}$. Each element A of $\mathscr{L}^\sharp(\mathfrak{D})$ is a closable operator in $\mathfrak{h}$ and $A^\sharp$ equals the restriction of the hermitian adjoint A^* of A onto $\mathfrak{D}$. Equipped with the involution $A \to A^\sharp$, $\mathscr{L}^\sharp(\mathfrak{D})$ is a $*$-algebra with the identity operator I. A $*$-subalgebra $\mathscr{A}$ of $\mathscr{L}^\sharp(\mathfrak{D})$ is called a *$\sharp$-algebra* on $\mathfrak{D}$. The $\sharp$-algebra $\mathscr{L}^\sharp(\mathfrak{D})$ is *maximal* among $\sharp$-algebras on $\mathfrak{D}$, which is called the *maximal $\sharp$-algebra* on $\mathfrak{D}$. Let $\mathscr{B}(\mathfrak{h})$ be the set of all bounded linear operators on the Hilbert space $\mathfrak{h}$. Let $\mathscr{A}$ be a $\sharp$-algebra on $\mathfrak{D}$. We set

$$\mathscr{A}_b = \{A \in \mathscr{A};\ \bar{A} \in \mathscr{B}(\mathfrak{h})\},$$

where $\bar{A}$ is the closure of a closable operator A. Then, $\mathscr{A}_b$ is a *#-subalgebra* of $\mathscr{A}$, which is called the bounded part of $\mathscr{A}$.

Throughout this section let $\mathscr{A}$ be a #-algebra on a pre-Hilbert space $\mathfrak{D}$ and $\mathfrak{h}$ the completion of $\mathfrak{D}$.

[I] Weak, quasi-weak and strong topologies

For each $\xi \in \mathfrak{D}$ and $x \in \mathfrak{h}$ we set

$$P_{\xi,x}(A) = |(A\xi \mid x)|,$$

$$P_{\xi}(A) = \|A\xi\|, \qquad A \in \mathscr{A}.$$

The topology generated by the family $\{P_{\xi,\eta}(\cdot);\ \xi, \eta \in \mathfrak{D}\}$ (*resp.* $\{P_{\xi,x}(\cdot);\ \xi \in \mathfrak{D}, x \in \mathfrak{h}\}$, $\{P_{\xi}(\cdot);\ \xi \in \mathfrak{D}\}$) of the seminorms is called the *weak* topology (*resp. quasi-weak* topology, *strong* topology) and is simply denoted by t_w (*resp.* t_{qw}, t_s). It is easily seen that $(\mathscr{A};\ t_w)$ is a locally convex $*$-algebra, that is, $*$-algebra which is also a locally convex space such that the multiplication is separately continuous in both variables and the involution is continuous.

[II] σ-weak, quasi-σ-weak and σ-strong topologies

Let $\mathfrak{h}_\infty$ be the Hilbert direct sum of the Hilbert spaces $\mathfrak{h}_n$ with $\mathfrak{h}_n = \mathfrak{h}$ $(n=1, 2,\dots)$. We set

$$\mathfrak{D}_\infty(\mathscr{A}) = \{\{\xi_n\} \in \mathfrak{h}_\infty;\ \xi_n \in \mathfrak{D}\ (n=1, 2,\dots) \text{ and } \sum_{n=1}^{\infty} \|A\xi_n\|^2 < \infty \text{ for all } A \in \mathscr{A}\},$$

$$P_{\{\xi_n\},\{x_n\}}(A) = |\sum_{n=1}^{\infty} (A\xi_n \mid x_n)|$$

and

$$P_{\{\xi_n\}}(A) = [\sum_{n=1}^{\infty} \|A\xi_n\|^2]^{\frac{1}{2}}$$

where $A \in \mathscr{A}$, $\{\xi_n\} \in \mathfrak{D}_\infty(\mathscr{A})$ and $\{x_n\} \in \mathfrak{h}_\infty$. The locally convex topology generated by the family $\{P_{\{\xi_n\},\{\eta_n\}}(\cdot);\ \{\xi_n\}, \{\eta_n\} \in \mathfrak{D}_\infty(\mathscr{A})\}$ (*resp.* $\{P_{\{\xi_n\},\{x_n\}}(\cdot);\ \{\xi_n\} \in \mathfrak{D}_\infty(\mathscr{A}), \{x_n\} \in \mathfrak{h}_\infty\}$, $\{P_{\{\xi_n\}}(\cdot);\ \{\xi_n\} \in \mathfrak{D}_\infty(\mathscr{A})\}$) is called the $(\mathscr{A})$-*σ-weak topology* (*resp.* $(\mathscr{A})$-*quasi-σ-weak topology*, $(\mathscr{A})$-*σ-strong topology*) on $\mathscr{A}$ and is simply denoted by $t_{\sigma w}^{\mathscr{A}}$ (resp. $t_{q\sigma w}^{\mathscr{A}}$, $t_{\sigma s}^{\mathscr{A}}$). In particular, the $(\mathscr{L}^{\#}(\mathfrak{D}))$-$\sigma$-weak topology (*resp.* $(\mathscr{L}^{\#}(\mathfrak{D}))$-quasi-$\sigma$-weak topology, $(\mathscr{L}^{\#}(\mathfrak{D}))$-$\sigma$-strong topology) on $\mathscr{A}$ is simply called the $(\mathfrak{D})$-σ-weak topology (*resp.* $(\mathfrak{D})$-quasi-σ-weak topology, $(\mathfrak{D})$-σ-strong topology) and denoted by $t_{\sigma w}^{\mathfrak{D}}$ (*resp.* $t_{q\sigma w}^{\mathfrak{D}}$, $t_{\sigma s}^{\mathfrak{D}}$). It is easily seen that $(\mathscr{A}; t_{\sigma w}^{\mathscr{A}})$ and $(\mathscr{A}; t_{\sigma w}^{\mathfrak{D}})$ are locally convex $*$-algebras. The relations between the above topologies are as follows:

$$\begin{array}{ccccc} t_w & \leqq & t_{qw} & \leqq & t_s \\ \wedge\!\shortparallel & & \wedge\!\shortparallel & & \wedge\!\shortparallel \\ t_{\sigma w}^{\mathfrak{D}} & \leqq & t_{q\sigma w}^{\mathfrak{D}} & \leqq & t_{\sigma s}^{\mathfrak{D}} \\ \wedge\!\shortparallel & & \wedge\!\shortparallel & & \wedge\!\shortparallel \\ t_{\sigma w}^{\mathscr{A}} & \leqq & t_{q\sigma w}^{\mathscr{A}} & \leqq & t_{\sigma s}^{\mathscr{A}} \end{array} .$$

[III] Uniform and quasi-uniform topologies

G. Lassner has defined in [13] the topologies on $\mathscr{A}$ called uniform topology and quasi-uniform topology. Let $\mathscr{A}_I$ be a #-algebra on $\mathfrak{D}$ formed by adjunction of the identity operator I. A natural induced topology on $\mathfrak{D}$ is defined as follows:

Suppose that $\mathscr{S}$ is a finite subset of elements of $\mathscr{A}_I$. We define the seminorm $\|\ \|_{\mathscr{S}}$ on $\mathfrak{D}$ by

$$\|\xi\|_{\mathscr{S}} = \sum_{A \in \mathscr{S}} \|A\xi\| .$$

We define the induced topology $t_{\mathscr{A}}$ on $\mathfrak{D}$ as the topology generated by the family $\{\|\cdot\|_{\mathscr{S}};\ \mathscr{S}$ is a finite subset of $\mathscr{A}_I\}$ of the seminorms.

If $\mathfrak{D}$ is complete with respect to the induced topology $t_{\mathscr{A}}$, then $\mathscr{A}$ is said to be *closed*. A #-algebra $\mathscr{A}$ is closed if and only if $\mathfrak{D} = \bigcap_{A \in \mathscr{A}} \mathfrak{D}(\bar{A})$. A #-algebra $\mathscr{A}$ on $\mathfrak{D}$ is called *self-adjoint* if $\mathfrak{D} = \bigcap_{A \in \mathscr{A}} \mathfrak{D}(A^*)$. If $\mathscr{A}$ is self-adjoint, then it is closed. If $\mathfrak{M}$ is a bounded subset of the locally convex space $(\mathfrak{D}; t_{\mathscr{A}})$, then it is said to be *$\mathscr{A}$-bounded*. For each $\mathscr{A}$-bounded subset $\mathfrak{M}$ of $\mathfrak{D}$ we set

$$\|A\|_{\mathfrak{M}} = \sup_{\xi, \eta \in \mathfrak{M}} |(A\xi \mid \eta)|$$

and

$$P_{B,\mathfrak{M}}(A) = \sup_{\xi \in \mathfrak{M}} \|BA\xi\|$$

where $A \in \mathscr{A}$ and $B \in \mathscr{A}_I$.

DEFINITION 2.1. The locally convex topology generated by the family $\{\|\cdot\|_{\mathfrak{M}};\ \mathfrak{M}$ is an $\mathscr{A}$-bounded subset of $\mathfrak{D}\}$ (*resp.* $\{P_{B,\mathfrak{M}}(\cdot);\ B \in \mathscr{A}_I, \mathfrak{M}\}$) of the seminorms is called the *uniform topology* (*resp.* the *quasi-uniform topology*) on $\mathscr{A}$ and is simply denoted by t_u (*resp.* t_{qu}).

LEMMA 2.1. ([13] *G. Lassner*) (1) *The* $\sharp$*-algebra* $\mathscr{A}$ *is a locally convex* $*$*-algebra equipped with the topology* t_u.

(2) *The* $\sharp$*-algebra* $\mathscr{A}$ *is a locally convex algebra equipped with the topology* t_{qu}.

(3) *The topology* t_{qu} *is finer than the topology* t_u.

(4) *If* $\mathscr{A}=\mathscr{A}_b$, *then both the topology* t_u *and the topology* t_{qu} *equal the operator-norm topology.*

(5) *If there exists a norm in the* $\sharp$*-algebra* $\mathscr{A}$ *defining a finer topology than* t_u, *then* $\mathscr{A}$ *equals* $\mathscr{A}_b$.

(6) *The equality* $t_u=t_{qu}$ *if and only if the multiplication is jointly continuous with respect to the topology* t_u.

[IV] ρ-topology and λ-topology

We shall recall the ρ-topology and λ-topology defined by D. Arnal and J. P. Jurzak [2, 12].

An operator $A\in\mathscr{A}$ is called positive if

$$(A\xi\,|\,\xi)\geqq 0 \qquad \text{for all} \quad \xi\in\mathfrak{D},$$

which is denoted by $A\geqq 0$. We denote by $\mathscr{A}^+$ the set of all positive operators in $\mathscr{A}$.

For each $A\in\mathscr{A}_I^+$ we put

$$\rho_A(T)=\sup_{\xi\in\mathfrak{D}}\frac{|(T\xi|\xi)|}{(A\xi|\xi)}, \qquad T\in\mathscr{A}$$

where $\frac{\lambda}{0}=\infty$ for $\lambda>0$. This defines the normed space

$$\mathscr{N}_A=\{T\in\mathscr{A};\ \rho_A(T)<\infty\}$$

with the norm $\|\cdot\|_A\equiv\rho_A/\mathscr{N}_A$. We note that $\bigcup_{A\in\mathscr{A}_I^+}\mathscr{N}_A=\sum_{A\in\mathscr{A}_I^+}\mathscr{N}_A=\mathscr{A}$; moreover, the relation $0\leqq A\leqq B$ implies that the injection $i_{A,B}$: $(\mathscr{N}_A\colon\|\cdot\|_A)\to(\mathscr{N}_B;\|\cdot\|_B)$ is a norm-decreasing map.

For each $A\in\mathscr{A}_I$ we set

$$\lambda_A(T)=\sup_{\xi\in\mathfrak{D}}\frac{\|T\xi\|}{\|A\xi\|}, \qquad T\in\mathscr{A}$$

where $\frac{\lambda}{0}=\infty$ for $\lambda>0$. This defines the normed space

$$\mathcal{M}_A=\{T\in\mathscr{A};\ \lambda_A(T)<\infty\}$$

and the spaces $\mathcal{M}_A$ constitute a direct set.

DEFINITION 2.2. The inductive limit topology for the normed spaces $\{(\mathcal{N}_A;\ \|\cdot\|_A);\ A\in\mathscr{A}_I^\dagger\}$ (resp. $\{(\mathcal{M}_B;\ \lambda_B(\cdot));\ B\in\mathscr{A}_I\}$) is called the *$\rho$-topology* (*resp. λ-topology*) on $\mathscr{A}$ and is simply denoted by t_ρ (resp. t_λ).

LEMMA 2.2. ([12] J. P. Jurzak) The #-algebra $\mathscr{A}$ is a bornological locally convex $*$-algebra equipped with the ρ-topology.

Now we give the following

LEMMA 2.3. (1) $t_\lambda\geqq t_\rho$;

(2) $t_\rho\geqq t_{\sigma w}^{\mathscr{A}}$ *and* $t_\lambda\geqq t_{\sigma s}^{\mathscr{A}}$;

(3) $t_\rho\geqq t_u$ *and* $t_\lambda\geqq t_{qu}$.

PROOF. (1) It suffices to show that the injection $i:(\mathscr{A};\ t_\lambda)\to(\mathscr{A};\ t_\rho)$ is continuous. For each $A\in\mathscr{A}_I$ we show that $i/\mathcal{M}_A$ is continuous. Take an element T of $\mathcal{M}_A$. Then we have

$$\begin{aligned}\rho_{I+A^\# A}(T)&=\sup_{\xi\in\mathfrak{D}}\frac{|(T\xi|\xi)|}{((I+A^\# A)\xi|\xi)}\\&\leqq\sup_{\xi\in\mathfrak{D}}\frac{\|T\xi\|\,\|\xi\|}{\|\xi\|^2+\|A\xi\|^2}\\&\leqq\sup_{\xi\in\mathfrak{D}}\frac{\|T\xi\|}{\|A\xi\|}\\&=\lambda_A(T)\,.\end{aligned}$$

Hence, $i/\mathcal{M}_A$ is continuous, which implies that i is continuous.

(2) Let j be the injection of $(\mathscr{A};\ t_\rho)$ onto $(\mathscr{A};\ t_{\sigma w}^{\mathscr{A}})$. It suffices to show that j is continuous. Suppose that $\{T_n\}$ is a sequence in $\mathcal{N}_A$ $(A\in\mathscr{A}_I^\dagger)$ such that $\lim_{n\to\infty}\|T_n\|_A=0$. Namely, there is a sequence $\{\varepsilon(n)\}$ such that

$$|(T_n\xi\,|\,\eta)|\leqq\varepsilon(n)(A\xi\,|\,\xi),\qquad \xi\in\mathfrak{D}.$$

Then, for each $\xi,\eta\in\mathfrak{D}$ we have

$$|(T_n\xi\,|\,\eta)|\leqq\frac{1}{4}\{|(T_n(\xi+\eta)\,|\,\xi+\eta)|+|(T_n(\xi-\eta)\,|\,\xi-\eta)|$$

$$+|(T_n(\xi+i\eta)\,|\,\xi+i\eta)|+|(T_n(\xi-i\eta)\,|\,\xi-i\eta)|\}$$

$$\leqq\frac{\varepsilon(n)}{4}\{(A(\xi+\eta)\,|\,\xi+\eta)+(A(\xi-\eta)\,|\,\xi-\eta)$$

$$+(A(\xi+i\eta)\,|\,\xi+i\eta)+(A(\xi-i\eta)\,|\,\xi-i\eta)\}$$

$$=\varepsilon(n)\{(A\xi\,|\,\xi)+(A\eta\,|\,\eta)\}\,.$$

This inequality implies that for each $\{\xi_i\}$, $\{\eta_i\}\in\mathfrak{D}_\infty(\mathscr{A})$

$$|\sum_{i=1}^{\infty}(T_n\xi_i\,|\,\eta_i)|\leqq\varepsilon(n)\sum_{i=1}^{\infty}\{(A\xi_i\,|\,\xi_i)+(A\eta_i\,|\,\eta_i)\}$$

$$\leqq\{(\sum_{i=1}^{\infty}\|\xi_i\|^2)^{\frac{1}{2}}(\sum_{i=1}^{\infty}\|A\xi_i\|^2)^{\frac{1}{2}}+(\sum_{i=1}^{\infty}\|\eta_i\|^2)^{\frac{1}{2}}(\sum_{i=1}^{\infty}\|A\eta_i\|^2)^{\frac{1}{2}}\}\varepsilon(n)\,.$$

It hence follows that j is continuous. Similarly we can prove that $t_\lambda\geqq t_{\sigma s}^{\mathscr{A}}$.

(3) Let $\mathfrak{M}$ be each $\mathscr{A}$-bounded subset of $\mathfrak{D}$. The inequality:

$$|(T\xi\,|\,\xi)|\leqq\gamma(A\xi\,|\,\xi),\qquad \xi\in\mathfrak{D}$$

implies that

$$\sup_{\xi,\eta\in\mathfrak{M}}|(T\xi\,|\,\eta)|\leqq 2\gamma(\sup_{\xi\in\mathfrak{M}}\|\xi\|)(\sup_{\xi\in\mathfrak{M}}\|A\xi\|)\,.$$

Hence it is proved, in the same way as in (2), that $t_\rho\geqq t_u$. Similarly, we have $t_\lambda\geqq t_{qu}$.

Thus we have the following

THEOREM 2.1. *Let $\mathscr{A}$ be a #-algebra on a pre-Hilbert space $\mathfrak{D}$. Then the following diagram amon the topologies holds:*

$$\begin{array}{ccccc}
t_{qu} & \leqq & & & t_\lambda \\
\text{\rotatebox{90}{$\leqq$}} & & & & \text{\rotatebox{90}{$\leqq$}} \\
t_u & \leqq & & & t_\rho \\
\text{\rotatebox{90}{$\leqq$}} & & & & \text{\rotatebox{90}{$\leqq$}} \\
t_w & \leqq & t_{\sigma w}^{\mathfrak{D}} & \leqq & t_{\sigma w}^{\mathscr{A}} \\
\wedge\!\parallel & & & & \wedge\!\parallel \\
t_{qw} & \leqq & t_{q\sigma w}^{\mathfrak{D}} & \leqq & t_{q\sigma w}^{\mathscr{A}} \\
\wedge\!\parallel & & & & \wedge\!\parallel \\
t_s & \leqq & t_{\sigma s}^{\mathfrak{D}} & \leqq & t_{\sigma s}^{\mathscr{A}} \\
\wedge\!\parallel & & & & \wedge\!\parallel \\
t_{qu} & \leqq & & & t_\lambda
\end{array}\,.$$

REMARK. In the case of the usual bounded operator algebras we have the following connections:

$$\left.\begin{array}{ccc} t_w & \lneqq & t_s \\ \wedge\!\| & & \wedge\!\| \\ t_{\sigma w} & \lneqq & t_{\sigma s} \end{array}\right\} \lneqq t_u = t_{qu} = t_\rho = t_\lambda = \textit{operator-norm topology}.$$

For #-algebras of unbounded operators we don't know whether the relations $\leqq$ in Theorem 2.1 are strict or not.

§3. Special #-algebras

In this section we examine particular #-algebras. In the remainder of this paper we shall need some concepts of #-algebras.

DEFINITION 3.1. A #-algebra $\mathscr{A}$ is called *countably dominated* if there exists an increasing sequence of subspaces $\mathscr{N}_{A_n}$ ($A_n \in \mathscr{A}_I^\dagger$) such that $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{N}_{A_n}$.

If $\mathscr{A}$ is a countably dominated closed #-algebra on $\mathfrak{D}$, then $\mathfrak{D}$ is a Fréchet space equipped with the induced topology $t_{\mathscr{A}}$.

LEMMA 3.1. *Let* $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{N}_{A_n}$ *be a countably dominated, closed #-algebra on* $\mathfrak{D}$. *If* $S \in \mathscr{L}^{\#}(\mathfrak{D})$, *then there exists an integer n (resp. m) such that*

$$\sup_{\xi \in \mathfrak{D}} \frac{\|S\xi\|}{\|A_n\xi\|} < \infty \qquad \left(\textit{resp.} \ \sup_{\xi \in \mathfrak{D}} \frac{|(S\xi|\xi)|}{(A_m\xi|\xi)} < \infty\right)$$

PROOF. This follows from Lemma 1.1 in [2].

PROPOSITION 3.1. *If* $\mathscr{A}$ *is a closed, countably dominated #-algebra on* $\mathfrak{D}$, *then the following relations hold*:

$$t_{\sigma w}^{\mathscr{A}} = t_{\sigma w}^{\mathfrak{D}}, \ t_{q\sigma w}^{\mathscr{A}} = t_{q\sigma w}^{\mathfrak{D}} \quad \textit{and} \quad t_{\sigma s}^{\mathscr{A}} = t_{\sigma s}^{\mathfrak{D}}.$$

PROOF. It follows from Lemma 3.1 that $\mathfrak{D}_\infty(\mathscr{A}) = \mathfrak{D}_\infty(\mathscr{L}^{\#}(\mathfrak{D}))$. This implies Proposition 3.1.

PROPOSITION 3.2. (*Theorem* 1.2 *in* [2]) *If* $\mathscr{A}$ *is a closed, countably dominated #-algebra on* $\mathfrak{D}$, *then the following statements are equivalent*:

(1) *The ρ-topology* t_ρ *equals the λ-topology* t_λ;

(2) *The bounded subsets of the λ-topology coincide with the bounded subsets of the ρ-topology*;

(3) *The ρ-topology t_ρ is finer than the quasi-weak topology t_{qw};*

(4) *The ρ-topology t_ρ is finer than the $(\mathscr{A})$-quasi-σ-weak topology $t^{\mathscr{A}}_{q\sigma w}$;*

(5) *The ρ-topology t_ρ is finer than the strong topology t_s;*

(6) *The ρ-topology t_ρ is finer than the $(\mathscr{A})$-σ-strong topology $t^{\mathscr{A}}_{\sigma s}$;*

(7) *The ρ-topology t_ρ is finer than the quasi-uniform topology t_{qu};*

(8) *For every $A \in \mathscr{A}^+_I$ there exists an element $B \in \mathscr{A}_I$ such that $\mathscr{N}_A \subset \mathscr{M}_B$ and the injection: $\mathscr{N}_A \to \mathscr{M}_B$ is continuous;*

(9) *The bilinear map $(S, T) \in \mathscr{A} \times \mathscr{A} \to ST \in \mathscr{A}$ is continuous with respect to the ρ-topology.*

DEFINITION 3.2. A #-algebra $\mathscr{A}$ is called *ρ-closed* if there exists a decomposition $\mathscr{A} = \bigcup_{j \in J} \mathscr{N}_{A_j}$ such that for every $j \in J$, $\mathscr{N}_{A_j}$ is a Banach space.

PROPOSITION 3.3. (*Theorem* 1.1 *in* [2]) *If $\mathscr{A}$ is a ρ-closed, countably dominated #-algebra on $\mathfrak{D}$, then the ρ-topology equals the λ-topology.*

Examples of ρ-closed, countably dominated #-algebras have been given in [2].

EXAMPLE 3.1. Let $\mathfrak{h}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}$. By $\mathfrak{D}$ we denote the set of all finite linear combinations of the basic vectors. Every element A of $\mathscr{L}^\#(\mathfrak{D})$ is uniquely determined by a matrix $A = (a_{\mu\nu})$ defined by

$$Ae_\mu = \sum_\nu a_{\mu\nu} e_\nu.$$

The adjoint $A^\#$ of $A = (a_{\mu\nu})$ is defined by

$$A^\# = (\overline{a_{\nu\mu}}).$$

Further, $a_{\mu\nu} = 0$ for $\mu \geqq \mu_0(\nu)$ and $\nu \geqq \nu_0(\mu)$. Hence, $\mathscr{L}^\#(\mathfrak{D})$ is the set of all matrices with the property that every row and column are only a finite numbers of non-zero elements.

Now we have the following

PROPOSITION 3.4. *The maximal #-algebra $\mathscr{L}^\#(\mathfrak{D})$ is a self-adjoint, countably dominated #-algebra and the following relations are satisfied:*

$$\begin{array}{ccccc} t_w = t^{\mathfrak{D}}_{\sigma w} = t_u & \lneqq & t_{qw} = t^{\mathfrak{D}}_{q\sigma w} \lneqq & t_s = t^{\mathfrak{D}}_{\sigma s} = t_{qu}. \\ \wedge\!\parallel & & & \parallel \\ t_\rho & & \lneqq & t_\lambda \end{array}$$

PROOF. The first statement is obvious and so we shall only prove the relations between the topologies. Let $\{\gamma_n\}$ be an arbitrary sequence of positive numbers and let $A_{(\gamma_n)} \in \mathscr{L}^{\#}(\mathfrak{D})$ be the operator $(\delta_{\mu\nu}\gamma_\nu)$, where $\delta_{\mu\nu}$ is the Kronecker symbol. Then, for $\xi = \sum_{n=1}^{\infty} \alpha_n e_n \in \mathfrak{D}$ we set

$$\|\xi\|_{A_{(\gamma_n)}} \equiv \|\xi\|_{(\gamma_n)} = \Big(\sum_{n=1}^{\infty} |\alpha_n|^2 \gamma_n^2\Big)^{\frac{1}{2}}$$

We can immediately prove that the topology in $\mathfrak{D}$ generated by $\{\|\cdot\|_{(\gamma_n)};\ \{\gamma_n\}\}$ of the seminorms equals the induced topology $t_{\mathscr{L}^{\#}(\mathfrak{D})}$. This implies that every $\mathscr{L}^{\#}(\mathfrak{D})$-bounded subset of $\mathfrak{D}$ is contained in a finite-dimensional subspace of $\mathfrak{D}$. And so, $t_w = t_{\sigma w}^{\mathfrak{D}} = t_u$, $t_{qw} = t_{\sigma q w}^{\mathfrak{D}}$ and $t_s = t_{\sigma s}^{\mathfrak{D}} = t_{qu}$.

We next show that $t_\rho \lneqq t_\lambda$. We put

$$A_n e_\nu = \begin{cases} 0 & \nu \leqq n \\ e_{\nu-n} & \nu > n. \end{cases}$$

Then, $\{A_n\}$ converges to zero with respect to t_λ. In fact, for each $\xi = \alpha_1 e_1 + \cdots + \alpha_m e_m \in \mathfrak{D}$ we have

$$\|A_n \xi\| = \begin{cases} 0 \quad , & m \leqq n \\ |\alpha_{n+1}|^2 + \cdots + |\alpha_m{}^2|, & m > n. \end{cases}$$

Now we set

$$a_{\mu\nu} = \delta_{\mu\nu}\mu.$$

Then, $A \equiv (a_{\mu\nu}) \in \mathscr{L}^{\#}(\mathfrak{D})$ and if $m > n$ we have

$$\begin{aligned} \frac{\|A_n\xi\|^2}{\|A\xi\|^2} &= \frac{|\alpha_{n+1}|^2 + \cdots + |\alpha_m|^2}{|\alpha_1|^2 + 2^2|\alpha_2|^2 + \cdots + (n+1)^2|\alpha_{n+1}|^2 + \cdots + m^2|\alpha_m|^2} \\ &\leqq \frac{|\alpha_{n+1}|^2 + \cdots + |\alpha_m|^2}{(n+1)^2\{|\alpha_{n+1}|^2 + \cdots + |\alpha_m|^2\}} \\ &= \frac{1}{(n+1)^2}, \end{aligned}$$

and if $m \leqq n$, $\dfrac{\|A_n\xi\|}{\|A\xi\|} = 0$. Hence, for each $\xi \in \mathfrak{D}$

$$\frac{\|A_n\xi\|}{\|A\xi\|} \leqq \frac{1}{n+1}, \quad \text{that is,} \quad \lambda_A(A_n) \leqq \frac{1}{n+1}.$$

Namely $\{A_n\}$ converges to zero with respect to t_λ.

Since $(\mathscr{L}^\#(\mathfrak{D}); t_\rho)$ is a locally convex $*$-algebra, $\{A_n^\#\}$ converges to zero with respect to t_ρ because of the relation $t_\rho \leqq t_\lambda$.

Suppose that $t_\rho = t_\lambda$. Then, $t_\rho \geqq t_{qu}$, and so $\{A_n^\#\}$ converges to zero with respect to t_{qu}. However, we can easily prove that $\{A_n^\#\}$ does not converge to zero with respect to t_{qu}. This is a contradiction.

We now show that $t_w \lneqq t_{qw} \lneqq t_s$. If $t_w = t_{qw}$, then $t_\rho \geqq t_{qw}$. It follows from Proposition 3.2 that $t_\rho = t_\lambda$, which contradicts $t_\rho \neq t_\lambda$. The above sequence $\{A_n^\#\}$ converges to zero with respect to t_{qw}, but it does not converge to zero with respect to t_s. Hence, $t_{qw} \lneqq t_s$.

Finally we show that $t_s = t_\lambda$. Since $(\mathscr{A}; t_s)$ is metrizable, it is a bornological locally convex space. The locally convex space $(\mathscr{A}; t_\lambda)$ is bornological. Further, it follows from Proposition 1.6 in [2] that $\mathscr{S}$ is a bounded subset of $(\mathscr{A}; t_s)$ if and only if $\mathscr{S}$ is a bounded subset of $(\mathscr{A}; t_\lambda)$.

This implies $t_s = t_\lambda$, which completes the proposition.

REMARK. We don't know whether or not the topology t_ρ equals the topology t_w.

EXAMPLE 3.2. Let $\{\mathscr{A}_i\}$ be a sequence of infinite-dimensional $*$-algebras $\mathscr{A}_i$ of bounded operators on Hilbert spaces $\mathfrak{h}_i$ with the identity operators I_i. We denote the Hilbert direct sum of the Hilbert spaces $\mathfrak{h}_i$ by $\mathfrak{h}$ and we also denote by $\mathfrak{D}$ the set of all elements $\xi = (\xi_i)$, where $\xi_i \in \mathfrak{h}_i$ and $\xi_i = 0$, except for finitely many indices i. Let $\mathscr{A}$ be the Cartesian product $\prod_{i=1}^{\infty} \mathscr{A}_i$ of the $*$-algebras $\mathscr{A}_i$. Then we define

$$A\xi = (A_i\xi_i),$$

where $A = (A_i) \in \mathscr{A}$ and $\xi = (\xi_i) \in \mathfrak{D}$. Every element A of $\mathscr{A}$ is regarded as a linear operator on $\mathfrak{D}$. The algebra $\mathscr{A}$ turns out to be a $\#$-algebra on $\mathfrak{D}$ with the operations:

$$A+B = (A_i + B_i), \quad \lambda A = (\lambda A_i),$$

$$AB = (A_i B_i), \qquad A^\# = (A_i^*)$$

where $A = (A_i)$, $B = (B_i) \in \mathscr{A}$ and $\lambda \in \mathbf{C}$ (: the field of complex numbers).

Now we discuss the relations between the topologies on the $\#$-algebra $\mathscr{A} = \prod_{i=1}^{\infty} \mathscr{A}_i$. For each $A = (A_i) \in \mathscr{A}$ we set

$$\|A\|_i = \|A_i\|$$

where $\|A_i\|$ denotes the operator-norm of A_i. The locally convex topology on $\mathscr{A}$ generated by the family $\{\|\cdot\|_i;\ i=1, 2,\dots\}$ of the seminorms is called the locally uniform topology and is simply denoted by t_{lu}.

PROPOSITION 3.5. *Keeping the above notations, the relations between the topologies are as follows*:

$$\left.\begin{array}{ccc} t_w & = t_{qw} & \lneqq t_s \\ \wedge\!\!\| & & \wedge\!\!\| \\ t_{\sigma w}^{\mathscr{A}} & = t_{q\sigma w}^{\mathscr{A}} & \lneqq t_{\sigma s}^{\mathscr{A}} \end{array}\right\} \lneqq t_u = t_{qu} = t_\rho = t_\lambda = t_{lu}.$$

It is, moreover, necessary and sufficient for $t_w \lneqq t_{\sigma w}^{\mathscr{A}}$ *(resp.* $t_s \lneqq t_{\sigma s}^{\mathscr{A}}$*) on* $\mathscr{A}$ *that* $t_w \lneqq t_{\sigma w}^{\mathscr{A}}$ *(resp.* $t_s \lneqq t_{\sigma s}^{\mathscr{A}}$*) on some* $\mathscr{A}_i$.

PROOF. It suffices to show that $t_u = t_\lambda = t_{lu}$. We first show that $t_u = t_{lu}$. Let $\mathfrak{M}$ be each $\mathscr{A}$-bounded subset of $\mathfrak{D}$. Then, for some positive integer N we have

$$\mathfrak{M} \subset \{\xi = (\xi_i) \in \mathfrak{D};\ \xi_i = 0 \ \text{for} \ i > N\}.$$

In fact, if not, for each positive integer j there exists an element $\xi^{(j)} \equiv (\xi_i^{(j)})$ of $\mathfrak{M}$ with $\|\xi_i^{(j)}\| \neq 0$. We set

$$\alpha_i = \frac{i}{\|\xi_i^{(j)}\|} \quad \text{and} \quad A = (\alpha_i I_i).$$

It follows that $A \in \mathscr{A}$ and

$$\|A\xi^{(i)}\| \geqq \alpha_i \|\xi_i^{(i)}\| = i,$$

and so

$$\sup_{\xi \in \mathfrak{M}} \|A\xi\| = \infty.$$

This contradicts that $\mathfrak{M}$ is $\mathscr{A}$-bounded.

Further, it is easily seen that for each j $(1 \leqq j \leqq N)$

$$\sup_{\xi = (\xi_i) \in \mathfrak{M}} \|\xi_j\| < \infty.$$

This implies $t_u = t_{lu}$.

We next show that $t_\lambda = t_{lu}$. Since $t_{lu} = t_u \leqq t_\lambda$, we have only to show $t_\lambda \leqq t_{lu}$. Let i be the injection of $(\mathscr{A}, t_{lu})$ onto $(\mathscr{A}, t_\lambda)$. Take each bounded subset $\mathscr{B}$

of $(\mathscr{A}, t_{lu})$. Then it follows that

$$\gamma_j \equiv \sup_{B=(B_i)\in\mathscr{B}} \|B_j\| < \infty \qquad \text{for every } j.$$

Put

$$A = (\gamma_i I_i).$$

Then, we have $A \in \mathscr{A}^+$ and for each $B=(B_i)\in\mathscr{B}$ and $\xi=(\xi_i)\in\mathfrak{D}$

$$\begin{aligned}\|B\xi\| &= [\sum_{i=1}^{\infty} \|B_i\xi_i\|^2]^{\frac{1}{2}} \\ &\leqq [\sum_{i=1}^{\infty} \|B_i\|^2\|\xi_i\|^2]^{\frac{1}{2}} \\ &\leqq [\sum_{i=1}^{\infty} \gamma_i^2\|\xi_i\|^2]^{\frac{1}{2}} \\ &= \|A\xi\|.\end{aligned}$$

Hence, $\sup_{B\in\mathscr{B}} \lambda_A(B) \leqq 1$. This implies that $\mathscr{B}$ is a bounded subset of $(\mathscr{A}, t_\lambda)$, and so i maps every bounded subset of $(\mathscr{A}, t_{lu})$ into a bounded subset of $(\mathscr{A}, t_\lambda)$. Since $(\mathscr{A}, t_{lu})$ is metrizable, the injection i is continuous. This completes the proof.

REMARK. Especially if $\mathscr{A}_i$ is a standard von Neumann algebra for each i, then it is well known that the topology t_w (*resp.* t_s) on $\mathscr{A}_i$ coincides with the topology $t_{\sigma w}$ (*resp.* $t_{\sigma s}$) on $\mathscr{A}_i$. It follows from Proposition 3.5 that the topology t_w (*resp.* t_s) on $\mathscr{A}$ coincides with the topology $t_{\sigma w}^{\mathscr{A}}$ (resp. $t_{\sigma s}^{\mathscr{A}}$) on $\mathscr{A}$. There is another #-algebra $\mathscr{B}$ on which the topology t_w (*resp.* t_s) coincides with the topology $t_{\sigma w}^{\mathscr{B}}$ (*resp.* $t_{\sigma s}^{\mathscr{B}}$) ([11]).

EXAMPLE 3.3. The test function algebra $\mathscr{S}_\otimes$ is the algebraic direct sum $\mathscr{S}_\otimes = \bigoplus_{n=0}^{\infty} \mathscr{S}_n$ where $\mathscr{S}_0 = \mathbf{C}$ and $\mathscr{S}_n = \mathscr{S}(\mathbf{R}^{4n})$ is the Schwartz space of C^∞-function with rapid decrease. We denote the direct sum topology on $\mathscr{S}_\otimes$ by τ. The multiplication and the involution are defined by

$$(fg)_n(x_1,\dots, x_n) = \sum_{\mu+\nu=n} f_\mu(x_1,\dots, x_\mu)g_\nu(x_{\mu+1},\dots, x_n),$$

$$(f^*)_n(x_1,\dots, x_n) = \overline{f_n(x_n,\dots, x_1)},$$

where $f=(f_n)$, $g=(g_n)\in\mathscr{S}_\otimes$. Then, $(\mathscr{S}_\otimes, \tau)$ is a barrelled and bornological locally convex $*$-algebra [14]. Let π_ω be the G.N.S.-representation associated with a positive continuous linear functional ω on $\mathscr{S}_\otimes$ with the domain $\mathfrak{D}_\omega$ and the cyclic vector ξ_ω. The universal representation $f \to \pi(f)$ is the direct sum of all *G.N.S.-representations*

$$\pi(f) = (\pi_\omega(f)) \in \prod_\omega \pi_\omega(\mathscr{S}_\otimes)$$

defined on the algebraic direct sum $\mathfrak{D}$ of the spaces $\mathfrak{D}_\omega$. Then, $\mathscr{A} \equiv \{\pi(f); f \in \mathscr{S}_\otimes\}$ is a $\#$-algebra on $\mathfrak{D}$ with the operations:

$$\pi(f) + \pi(g) = \pi(f+g), \quad \lambda\pi(f) = \pi(\lambda f),$$

$$\pi(f)\pi(g) = \pi(fg), \quad \pi(f)^\# = \pi(f^*).$$

We give the relations between the topologies on $\mathscr{A}$.

PROPOSITION 3.6. *The following relations hold*:

$$t_w \lneqq t_u \lneqq t_\rho \lneqq t_s = t_{qu} = t_\lambda.$$

PROOF. It follows from [14] that

$$t_w \lneqq t_u \lneqq t_s = t_{qu}.$$

$(\mathscr{A}; t_u)$ is not a bornological space and $(\mathscr{A}; t_s)$ is a barrelled space. Since $(\mathscr{A}; t_\rho)$ is bornological, we have $t_u \lneqq t_\rho$.

We shall show that $t_s = t_\lambda$. Let U be a neighborhood of 0 with respect to t_λ. The set U is represented by the absolutely convex envelope of the form

$$\bigcup_{A \in \mathscr{A}_I} U_A,$$

where

$$U_A \equiv \{T \in \mathscr{N}_A; \lambda_A(T) \leqq \varepsilon_A, \varepsilon_A > 0\}.$$

Then, U_A is absolutely convex and closed with respect to t_s. Since $(\mathscr{A}; t_s)$ is a barrelled space, U_A is a neighborhood of 0 with respect to t_s. Hence, U is a neighborhood of 0 with respect to t_s, and so $t_\lambda \leqq t_s$.

Thus, we have $t_\lambda = t_s$.

EXAMPLE 3.4. Let $\mathfrak{D}$ be a pre-Hilbert space and let $\mathfrak{P}(T) = \{P(T) = \sum_{n \geqq 0} \alpha_n T^n; \alpha_n \in \mathbf{C}\}$ be the algebra of all polynomials generated by an element T

of $\mathscr{L}^\#(\mathfrak{D})$. Let Γ_∞ be the system of all positive sequences $\{\gamma_n\}$ with $1\leqq\gamma_0\leqq\gamma_1\leqq\cdots$. We introduce the topology τ_∞ defined by all seminorms

$$\|P(T)\|_{\{\gamma_n\}}=\sum_{n\geqq 0}\gamma_n|\alpha_n|,$$

where $\{\gamma_n\}\in\Gamma_\infty$ and $P(T)=\sum_n\alpha_n T^n$. Then, τ_∞ is the finest locally convex topology on the $\#$-algebra $\mathfrak{P}(T)$. The following result is obtained by K. Schmüdgen [18].

Proposition 3.7. *If T is an unbounded operator in $\mathscr{L}^\#(\mathfrak{D})$ with $T^\#=T$, then the relations between the topologies on $\mathfrak{P}(T)$ are as follows:*

$$t_u=t_{qu}=t_\rho=t_\lambda=\tau_\infty.$$

The next proposition has been established by [18], but its proof includes gaps and we shall prove it in order to give the complete reference, and our proof is based on his idea.

Proposition 3.8. ([18] *Schmüdgen*) *Let T be an unbounded operator T in $\mathscr{L}^\#(\mathfrak{D})$ with $T^\#=T$. Suppose that $\mathfrak{P}(T)$ is a closed $\#$-algebra. Then, it follows that*

$$t_s=t_{\sigma s}^{\mathfrak{D}}=t_{\sigma s}^{\mathfrak{P}(T)}=t_u=t_{qu}=t_\rho=t_\lambda=\tau_\infty.$$

To prove this we prepare the next lemma.

Lemma. Take $\{\gamma_n\}\in\Gamma_\infty$. Then there exist a sequence $\{\xi_n\}$ in $\mathfrak{D}$ and a sequence $\{\delta_n\}$ of positive numbers such that

(1) $|(T^l\xi_l\,|\,\xi_l)=\delta_l+\gamma_l+1+\sum_{\substack{i<l\\ j<l}}|(T^l\xi_i\,|\,\xi_j)|$;

(2) $|(T^l\xi_l\,|\,\xi_l)|\leqq 2\delta_l$;

(3) $|(T^l\xi_k\,|\,\xi_k)|\leqq\frac{1}{2^k}$, $\quad l<k$;

(4) $(T^l\xi_k\,|\,\xi_m)=0$, $\quad l\leqq m,\ k<m$;

(5) $$D_n\equiv\begin{vmatrix}\delta_0 & -4\delta_1 & -4\delta_2\cdots-4\delta_n\\ -4\delta_1 & \delta_2 & -4\delta_3\cdots-4\delta_{n+1}\\ -4\delta_2 & -4\delta_3 & \delta_4\cdots-4\delta_{n+2}\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ -4\delta_n & -4\delta_{n+1}\cdots\cdots & \delta_{2n}\end{vmatrix}>0.$$

PROOF. Its proof depends on induction. Let δ_0 be a positive number with $\delta_0 \geqq \gamma_0 + 1$ and $\xi_0 \in \mathfrak{D}$ with $\|\xi_0\|^2 = \delta_0 + \gamma_0 + 1$. Then, the assertion holds for $n=0$. Suppose that $\xi_0, \{\xi_1, \xi_2\}, \{\xi_3, \xi_4\}, \dots, \{\xi_{2n-3}, \xi_{2n-2}\}$ and $\delta_0, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}, \dots, \{\delta_{2n-3}, \delta_{2n-2}\}$ chosen so that (1)～(5) are satisfied. Then, we shall find elements $\{\xi_{2n-1}, \xi_{2n}\}$ of $\mathfrak{D}$ and positive numbers $\{\delta_{2n-1}, \delta_{2n}\}$ which satisfy the conditions (1)～(5). First, we can take δ_{2n-1} as

$$\delta_{2n-1} \geqq \gamma_{2n-1} + 1 + \sum_{\substack{i<2n-1 \\ j<2n-1}} |(T^{2n-1}\xi_i \mid \xi_j)| .$$

In the same way as in [18, Statement 1] we can take an element ξ_{2n-1} of $\mathfrak{D}$ satisfying

$$|(T^{2n-1}\xi_{2n-1} \mid \xi_{2n-1})| = \delta_{2n-1} + \gamma_{2n-1} + 1 + \sum_{\substack{i<2n-1 \\ j<2n-1}} |(T^{2n-1}\xi_i \mid \xi_j)| ,$$

$$|(T^{\gamma}\xi_{2n-1} \mid \xi_{2n-1})| \leqq \frac{1}{2^{2n-1}}, \qquad \gamma < 2n-1$$

and

$$(T^{\gamma}\xi_k \mid \xi_{2n-1}) = 0, \qquad \gamma \leqq 2n-1, \quad k < 2n-1.$$

Next we shall construct δ_{2n} and ξ_{2n}. The determinant D_n can be written as

$$D_n = D_{n-1}\delta_{2n} + P(\delta_0, \delta_1, \dots, \delta_{2n-1})$$

where $P(t_0, t_1, \dots, t_{2n-1})$ is a polynomial of the $(2n-1)$-variables $t_0, t_1, \dots, t_{2n-1}$. It follows from $D_{n-1} > 0$ that we can take δ_{2n} so large that $D_n > 0$ and

$$\delta_{2n} \geqq \gamma_{2n} + 1 + \sum_{\substack{i<2n \\ j<2n}} |(T^{2n}\xi_i \mid \xi_j)| .$$

We can analogously take an element ξ_n of $\mathfrak{D}$ satisfying

$$|(T^{2n}\xi_{2n} \mid \xi_{2n})| = \delta_{2n} + \gamma_{2n} + 1 + \sum_{\substack{i<2n \\ j<2n}} |(T^{2n}\xi_i \mid \xi_j)| ,$$

$$|(T^{\gamma}\xi_{2n} \mid \xi_{2n})| \leqq \frac{1}{2^{2n}}, \qquad \gamma < 2n$$

and

$$(T^{\gamma}\xi_k \mid \xi_{2n}) = 0, \qquad \gamma \leqq 2n, \quad k < 2n.$$

Thus we can take by induction $\xi_0, \xi_1, \dots, \xi_{2n}$ and $\delta_0, \delta_1, \dots, \delta_{2n}$ satisfying the conditions (1)～(5), which completes the proof of Lemma.

THE PROOF OF PROPOSITION 3.8: Take $\{\gamma_n\}\in\Gamma_\infty$. Then, there exist a sequence $\{\xi_n\}$ in $\mathfrak{D}$ and a sequence $\{\delta_n\}$ of positive numbers satisfying the conditions (1)$\sim$(5) in Lemma. We set

$$\xi=\sum_{i=1}^{\infty}\xi_i.$$

Then, it follows from the closedness of $\mathfrak{P}(T)$ that $\xi\in\mathfrak{D}$. We show that for each polynomial $P(T)=\sum_n\alpha_nT^n$

$$\|P(T)\xi\|^2=\|\sum_n\alpha_nT^n\xi\|^2\geqq\sum_n\gamma_n|\alpha_n|^2.$$

From the assumption (1)$\sim$(4) in Lemma it follows that

$$\begin{aligned}|(T^n\xi\,|\,\xi)|&\geqq|(T^n\xi_n\,|\,\xi_n)|-\sum_{\substack{i<n\\ j<n}}|(T^n\xi_i\,|\,\xi_j)|-\sum_{k<n}|(T^n\xi_k\,|\,\xi_k)|\\&\geqq|(T^n\xi_n\,|\,\xi_n)|-\sum_{\substack{i<n\\ j<n}}|(T^n\xi_i\,|\,\xi_j)|-1\\&\geqq\delta_n+\gamma_n\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad\text{(i)}\end{aligned}$$

and

$$\begin{aligned}|(T^n\xi\,|\,\xi)|&\leqq|(T^n\xi_n\,|\,\xi_n)|+\sum_{\substack{i<n\\ j<n}}|(T^n\xi_i\,|\,\xi_j)|+\sum_{k>n}|(T^n\xi_k\,|\,\xi_k)|\\&\leqq2|(T^n\xi_n\,|\,\xi_n)|\\&\leqq4\delta_n\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad\text{(ii)}.\end{aligned}$$

From the assumption (5) we have

$$\sum_n\delta_{2n}t_n\overline{t_n}-4\sum_{n\neq m}\delta_{n+m}t_n\overline{t_m}\geqq0,$$

and hence

$$\sum_n\delta_{2n}|\alpha_n|^2-4\sum_{n\neq m}\delta_{n+m}|\alpha_n|\,|\alpha_m|\geqq0.$$

From (i) and (ii) it follows that

$$\begin{aligned}0&\leqq\sum_n((T^{2n}\xi\,|\,\xi)-\gamma_{2n})\,|\alpha_n|^2-\sum_{n\neq m}|(T^{n+m}\xi\,|\,\xi)||\alpha_n|\,|\alpha_m|\\&\leqq|\sum_n(T^{2n}\xi\,|\,\xi)\alpha_n\overline{\alpha_n}-\sum_{n\neq m}(T^{n+m}\xi\,|\,\xi)\alpha_n\overline{\alpha_m}|-\sum_n\gamma_n|\alpha_n|^2\end{aligned}$$

$$=|\sum_{n,m}(T^{n+m}\xi\,|\,\xi)\alpha_n\overline{\alpha_m}|-\sum_n \gamma_n|\alpha_n|^2$$

$$=\|\sum_n \alpha_n T^n\xi\|^2-\sum_n \gamma_n|\alpha_n|^2.$$

Thus we have

$$\sum_n \gamma_n|\alpha_n|^2 \leqq \|P(T)\xi\|^2.$$

It is easily proved that the system $\{(\sum_n \gamma_n|\alpha_n|^2)^{\frac{1}{2}};\ \{\gamma_n\}\in\Gamma_\infty\}$ of the seminorms gives the same topology as τ_∞. This implies Proposition 3.8.

REMARK. It has been seen in [15] that we can't drop in Proposition 3.8 the assumption that $\mathfrak{P}(T)$ is a closed #-algebra on $\mathfrak{D}$, however, without the closedness of $\mathfrak{P}(T)$ there are #-algebras on which the topologies in Proposition 3.8 are all the same.

§4. Unbounded operator algebras as GB*-algebras

Let $\mathscr{A}$ be a #-algebra on a pre-Hilbert space $\mathfrak{D}$. It follows from Section 2 that $(\mathscr{A};\ t_w)$, $(\mathscr{A};\ t_{\sigma w}^{\mathscr{A}})$, $(\mathscr{A};\ t_{\sigma w}^{\mathfrak{D}})$, $(\mathscr{A};\ t_u)$ and $(\mathscr{A};\ t_p)$ are locally convex $*$-algebras. In this section we shall consider under what conditions the locally convex $*$-algebras $\mathscr{A}$ become GB^*-algebras defined by G. R. Allan [1] and P. G. Dixon [4].

We first recall the notations of *GB*-algebras* and *EC#-algebras*. Let $\mathbf{A}$ be a locally convex $*$-algebra with identity e. We denote by $\mathfrak{B}^*$ the collection of subsets $\mathbf{B}$ of $\mathbf{A}$ satisfying:

(1) $\mathbf{B}$ is closed, absolutely convex and bounded;

(2) $e\in\mathbf{B}$, $\mathbf{B}^2\subset\mathbf{B}$ and $\mathbf{B}^*=\mathbf{B}$.

For every $\mathbf{B}\in\mathfrak{B}^*$, the linear span of $\mathbf{B}$ forms a $*$-algebra which is normed by the Minkowski function of $\mathbf{B}$. This normed $*$-algebra is denoted by $\mathbf{A}[\mathbf{B}]$. An element x of $\mathbf{A}$ is said to be bounded if, for some non-zero complex number λ, the set $\{(\lambda x)^n;\ n=1, 2,\dots\}$ is bounded. The set of all bounded elements of $\mathbf{A}$ is denoted by $\mathbf{A}_0$. If, for every $x\in\mathbf{A}$, $(e+x^*x)^{-1}$ exists and lies in $\mathbf{A}_0$, then $\mathbf{A}$ is said to be symmetric. A locally convex $*$-algebra $\mathbf{A}$ is called a *GB*-algebra* if

(1) $\mathfrak{B}^*$ has the greatest member $\mathbf{B}_0$;

(2) $\mathbf{A}$ is symmetric;

(3) $\mathbf{A}[\mathbf{B}_0]$ is complete.

If $\mathbf{A}$ is a GB^*-algebra with identity e, then $\mathbf{A}[\mathbf{B}_0]$ is a B^*-algebra with identity e and $(e+x^*x)^{-1}\in\mathbf{A}[\mathbf{B}_0]$ for every $x\in\mathbf{A}$. For details, the reader is referred to [1, 4].

Let $\mathscr{A}$ be a $\sharp$-algebra on $\mathfrak{D}$. If $I\in\mathscr{A}$ and $(I+A^\sharp A)^{-1}\in\mathscr{A}_b$ for all $A\in\mathscr{A}$, then $\mathscr{A}$ is called *symmetric*. If $\mathscr{A}$ is a symmetric $\sharp$-algebra on $\mathfrak{D}$ and $\bar{\mathscr{A}}_b$ is a C^*-algebra (*resp.* W^*-algebra), then $\mathscr{A}$ is called an *$EC^\sharp$-algebra* (*resp.* *$EW^\sharp$-algebra*) on $\mathfrak{D}$ over $\bar{\mathscr{A}}_b$ ([5, 7]).

Let $\mathscr{A}$ be a $\sharp$-algebra with the identity operator I and let τ be a topology on $\mathscr{A}$ satisfying the condition (C):

(1) $(\mathscr{A}, \tau)$ is a locally convex $*$-algebra;

(2) $t_w\leqq\tau$;

(3) $\mathscr{A}_1\equiv\{A\in\mathscr{A}_b;\ \|\bar{A}\|\leqq 1\}$ is bounded with respect to τ.

We note that the topologies t_w, $t_{\sigma w}^{\mathscr{A}}$, t_u and t_ρ are satisfied the condition (C). We denote by $\mathfrak{B}^*(\mathscr{A}, \tau)$ the collection of subsets $\mathfrak{B}$ of $\mathscr{A}$ satisfying:

(1) $\mathfrak{B}$ is closed and bounded with respect to the topology τ and absolutely convex;

(2) $I\in\mathfrak{B}$, $\mathfrak{B}^2\subset\mathfrak{B}$ and $\mathfrak{B}^\sharp=\mathfrak{B}$.

LEMMA 4.1. *The set $\mathscr{A}_1$ is the greatest member of $\mathfrak{B}^*(\mathscr{A}, \tau)$.*

PROOF. The set $\mathscr{A}_1$ is closed with respect to the weak topology t_w. It follows from $t_w\leqq\tau$ that $\mathscr{A}_1$ is closed with respect to τ. This implies that $\mathscr{A}_1\in\mathfrak{B}^*(\mathscr{A}, \tau)$. Let $\mathfrak{B}$ be an arbitrary element of $\mathfrak{B}^*(\mathscr{A}, \tau)$. Suppose that there exists an element B of $\mathfrak{B}$ with $\|\bar{B}\|>1$. Then, there exists an element ξ of $\mathfrak{D}$ such that $\|\xi\|=1$ and $\|B\xi\|>1$. Since $\mathfrak{B}$ is bounded with respect to τ and $t_w\leqq\tau$, $\mathfrak{B}$ is bounded with respect to t_w. Now we have that

$$|((B^\sharp B)^{2^n}\xi\,|\,\xi)|\geqq\|B\xi\|^{2^{n+1}}\qquad(n=1, 2,\dots),$$

and $\lim_{n\to\infty}\|B\xi\|^{2^{n+1}}=\infty$. On the other hand, we have

$$\overline{\lim}\,((B^\sharp B)^{2^n}\xi\,|\,\xi)<\infty,$$

since $(B^{\#}B)^{2^n} \in \mathfrak{B}$ and $\mathfrak{B}$ is bounded with respect to t_w. This is a contradiction. Hence, $\mathfrak{B} \subset \mathscr{A}_1$.

LEMMA 4.2. *A $\#$-algebra $\mathscr{A}$ is an $EC^{\#}$-algebra if and only if $(\mathscr{A}; \tau)$ is a GB^*-algebra.*

PROOF. It follows from Lemma 4.1 that the normed $*$-algebra $\bar{\mathscr{A}}_b$ with operator-norm equals the normed $*$-algebra $\mathscr{A}[\mathscr{A}_1]$. This implies Lemma 4.2.

Thus we have obtained the following

THEOREM 4.1. *Let $\mathscr{A}$ be a $\#$-algebra on $\mathfrak{D}$ with the identity operator I. Then, the following conditions are equivalent:*

(1) *$\mathscr{A}$ is an $EC^{\#}$-algebra;*

(2) *$(\mathscr{A}; t_w)$ is a GB^*-algebra;*

(3) *$(\mathscr{A}; t_{\sigma w}^{\mathscr{A}})$ is a GB^*-algebra;*

(4) *$(\mathscr{A}; t_{\sigma w}^{\mathfrak{D}})$ is a GB^*-algebra;*

(5) *$(\mathscr{A}; t_u)$ is a GB^*-algebra;*

(6) *$(\mathscr{A}; t_\rho)$ is a GB^*-algebra.*

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