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WEAKLY UNBOUNDED OPERATOR ALGEBRAS

By

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§1. Introduction

In the previous paper [7] we defined a weakly (resp. strictly) unbounded $EW^{\#}$-algebra and obtained the following fact: If $\mathfrak{A}$ is an $EW^{\#}$-algebra, then there exists a projection E in $\overline{\mathfrak{A}}_b \cap \overline{\mathfrak{A}}'_b$ such that $\mathfrak{A}_E$ is a weakly unbounded $EW^{\#}$-algebra, $\mathfrak{A}_{I-E}$ is a strictly unbounded $EW^{\#}$-algebra and $\mathfrak{A}$ equals the product $\mathfrak{A}_E \times \mathfrak{A}_{I-E}$ of the $EW^{\#}$-algebras $\mathfrak{A}_E$ and $\mathfrak{A}_{I-E}$. The primary purpose of this paper is to investigate linear functionals on a weakly unbounded $EW^{\#}$-algebra.

In §3, we shall study the general theory of weakly unbounded $EW^{\#}$-algebras. First, we define the notation of a weakly unbounded $EW^{\#}$-algebra $\mathfrak{A}$ associated with a family $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ and show that the definition is equivalent to the definition of a weakly unbounded $EW^{\#}$-algebra defined in [7]. Next, we define the locally convex topologies (; weak, σ-weak, locally σ-weak, strong, σ-strong, locally σ-strong and locally uniform topologies) on $\mathfrak{A}$ and the commutants, bicommutants of $\mathfrak{A}$. Furthermore, we shall investigate the relation between the topologies and the commutants.

In §4, we shall study the dual space $\mathfrak{A}^*$ (resp. $\mathfrak{A}_*$) of $\mathfrak{A}$ with respect to the locally uniform topology (resp. σ-weak topology). Then we have that $\mathfrak{A}^*$ (resp. $\mathfrak{A}^* \cap \mathfrak{A}_*$) equals the direct sum $\sum^{\oplus}_{\lambda\in\Lambda} \mathfrak{A}^*_\lambda$ (resp. $\sum^{\oplus}_{\lambda\in\Lambda} (\mathfrak{A}_\lambda)_*$) of the dual space $\mathfrak{A}^*_\lambda$ (resp. $(\mathfrak{A}_\lambda)_*$) of the von Neumann algebra $\mathfrak{A}_\lambda$ with respect to the uniform topology (resp. σ-weak topology), (Theorem 4.1).

In §5, we shall obtain the structure of invariant subspaces of $\mathfrak{A}^* \cap \mathfrak{A}^*$: Every closed left (resp. right) invariant subspace V of $\mathfrak{A}^* \cap \mathfrak{A}_*$ is of the form;

$$\mathrm{V} = (\mathfrak{A}^* \cap \mathfrak{A}_*)E_0 \qquad (\text{resp. } \mathrm{V} = E_0(\mathfrak{A}^* \cap \mathfrak{A}_*))$$

for some projection E_0 in $\mathfrak{A}$ (Theorem 5.1).

In §6, we shall define normal and singular linear functionals on $\mathfrak{A}$ and obtain the following fact: Every element ϕ of $\mathfrak{A}^*$ is uniquely decomposed into the sun; $\phi=\phi_n+\phi_s$, where ϕ_n (resp. ϕ_s) is a normal (resp. singular) linear functional on $\mathfrak{A}$ (Theorem 6.1). Furthermore, we can characterize the singularity and normality (Theorem 6.2, 6.3).

§2. Preliminaries

We give here only the basic definitions and facts needed. For a more complete discussion of the basic properties of $EW^{\#}$-algebras the reader is referred to [6, 7].

If S and T are linear operators on a Hilbert space $\mathfrak{y}$ with domains $\mathscr{D}(S)$ and $\mathscr{D}(T)$ we say S is an extension of T, denoted by $S\supset T$, if $\mathscr{D}(S)\supset\mathscr{D}(T)$ and $S\xi=T\xi$ for all $\xi\in\mathscr{D}(T)$. If S is a closable operator we denote by $\bar{S}$ the smallest closed extension of S. Let $\mathfrak{A}$ be a set of closable operators on $\mathfrak{y}$. Then we set

$$\overline{\mathfrak{A}}=\{\bar{S};\ S\in\mathfrak{A}\}.$$

If S is a linear operator with dense domain $\mathscr{D}(S)$ we denote by S^* the hermitian adjoint of S. Let S, T be closed operators on $\mathfrak{y}$. If $S+T$ is closable, then $\overline{S+T}$ is called the strong sum of S and T, and is denoted $S+T$. The strong product is likewise defined to be $\overline{ST}$ if it exists, and is denoted by $S\cdot T$. The strong scalar multiplication $\lambda\in\mathbf{C}$ (the field of complex numbers) and S is defined by $\lambda\cdot S=\lambda S$ if $\lambda\neq 0$, and $\lambda\cdot S\cdot=0$ if $\lambda=0$.

Let $\mathfrak{D}$ be a pre-Hilbert space with an inner product $(\ |\ )$ and $\mathfrak{y}$ the completion of $\mathfrak{D}$. We denote by $\mathscr{L}(\mathfrak{D})$ the set of all linear operators on $\mathfrak{D}$. We set

$$\mathscr{L}^{\#}(\mathfrak{D})=\{A\in\mathscr{L}(\mathfrak{D});\ A^*\mathfrak{D}\subset\mathfrak{D}\}.$$

Every $A\in\mathscr{L}^{\#}(\mathfrak{D})$ is a closable operator on $\mathfrak{y}$ with domain $\mathfrak{D}$. Putting

$$A^{\#}=A^*/\mathfrak{D}\ \text{(the restriction of } A^* \text{ onto } \mathfrak{D}),$$

the map $A\to A^{\#}$ is an involution on $\mathscr{L}^{\#}(\mathfrak{D})$. It is easily showed that $\mathscr{L}^{\#}(\mathfrak{D})$ is a $*$-algebra of operators on $\mathfrak{D}$ with the involution $\#$. A $\#$-subalgebra $\mathfrak{A}$ of $\mathscr{L}^{\#}(\mathfrak{D})$ is called a $\#$-algebra on $\mathfrak{D}$. In particular, $\mathscr{L}^{\#}(\mathfrak{D})$ is called a maximal $\#$-algebra on $\mathfrak{D}$. Let $\mathfrak{A}$ be a $\#$-algebra on $\mathfrak{D}$. We set

$$\mathfrak{A}_b=\{A\in\mathfrak{A};\ \bar{A}\in\mathscr{B}(\mathfrak{y})\},$$

where $\mathscr{B}(\mathfrak{y})$ denotes the set of all bounded linear operators on $\mathfrak{y}$. If $\mathfrak{A}\neq\mathfrak{A}_b$, then $\mathfrak{A}$ is called a pure #-algebra on $\mathfrak{D}$. A #-algebra $\mathfrak{A}$ is called symmetric if it has an identity operator I and furthermore, $(I+S^\# S)^{-1}$ exists and lies in $\mathfrak{A}_b$ for all $S \in \mathfrak{A}$. A symmetric #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is called an $EC^\#$-algebra (resp. $EW^\#$-algebra) on $\mathfrak{D}$ over $\overline{\mathfrak{A}_b}$ if $\overline{\mathfrak{A}_b}$ is a C^*-algebra (resp. W^*-algebra).

A #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is said to be closed (resp. self-adjoint) if $\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathscr{D}(\bar{A})$ (resp. $\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathscr{D}(A^*)$). It is easy to show that if $\mathfrak{A}$ is a self-adjoint #-algebra on $\mathfrak{D}$ then it is closed. By ([6] Proposition 2.6) if $\mathfrak{A}$ is a closed symmetric #-algebra, then it is self-adjoint. Let $\mathfrak{A}$ be a #-algebra on $\mathfrak{D}$. We set

$$\tilde{\mathfrak{D}}(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}} \mathscr{D}(\bar{A}),\ \tilde{A}x = \bar{A}x \qquad (x \in \tilde{\mathfrak{D}}(\mathfrak{A})),$$

$$\tilde{\mathfrak{A}} = \{\tilde{A};\ A \in \mathfrak{A}\}.$$

By ([6] Proposition 2.5) we see that $\tilde{\mathfrak{A}}$ is a closed #-algebra on $\tilde{\mathfrak{D}}(\mathfrak{A})$. Furthermore, it is proved that if $\mathfrak{A}$ is a symmetric #-algebra (resp. $EC^\#$-algebra, $EW^\#$-algebra) on $\mathfrak{D}$ then $\tilde{\mathfrak{A}}$ is a closed symmetric #-algebra (resp. closed $EC^\#$-algebra, closed $EW^\#$-algebra) on $\tilde{\mathfrak{D}}(\mathfrak{A})$. $\tilde{\mathfrak{A}}$ is called the closure of $\mathfrak{A}$.

§3. General theory of weakly unbounded operator algebras

In this section we shall define a weakly unbounded $EW^\#$-algebra and show that the definition is equivalent to the definition of a weakly unbounded $EW^\#$-algebra in the previous paper [7].

Throughout this paper let Λ be an infinite set and $\{\mathfrak{y}_\lambda\}_{\lambda \in \Lambda}$ a family of Hilbert spaces $\mathfrak{y}_\lambda$. Let $\mathfrak{y}(\Lambda) = \bigoplus_{\lambda \in \Lambda} \mathfrak{y}_\lambda$, i.e., the direct sum of the Hilbert spaces $\mathfrak{y}_\lambda$ and E_λ the projection from $\mathfrak{y}(\Lambda)$ onto $\mathfrak{y}_\lambda$. Let $\mathfrak{D}(\Lambda)$ be the set $\sum^{\oplus}_{\lambda \in \Lambda} \mathfrak{y}_\lambda$ of all elements of $\mathfrak{y}(\Lambda)$ with only a finite number of non-zero coordinates. Clearly $\mathfrak{D}(\Lambda)$ is a dense subspace of $\mathfrak{y}(\Lambda)$.

Let $\mathbf{A}_\lambda$ be a $*$-algebra for every $\lambda \in \Lambda$ and $\underset{\lambda \in \Lambda}{X} \mathbf{A}_\lambda$ the Cartesian product of $\{\mathbf{A}_\lambda\}_{\lambda \in \Lambda}$. Under the operations: $\{a_\lambda\} + \{b_\lambda\} = \{a_\lambda + b_\lambda\}$, $\alpha\{a_\lambda\} = \{\alpha a_\lambda\}$, $\{a_\lambda\}\{b_\lambda\} = \{a_\lambda b_\lambda\}$ and $\{a_\lambda\}^* = \{a_\lambda^*\}$ ($\{a_\lambda\}, \{b_\lambda\} \in \underset{\lambda \in \Lambda}{X} \mathbf{A}_\lambda$, $\alpha \in \mathbf{C}$), $\underset{\lambda \in \Lambda}{X} \mathbf{A}_\lambda$ is a $*$-algebra.

Let X_λ be a linear operator on $\mathfrak{y}_\lambda$ with the domain $\mathscr{D}(X_\lambda)$ for every $\lambda \in \Lambda$. We define a linear operator (X_λ) on $\mathfrak{y}(\Lambda)$ with the domain $\mathscr{D}((X_\lambda))$ as follows:

$$\mathscr{D}((X_\lambda)) = \{\{x_\lambda\} \in \mathfrak{y}(\Lambda);\ x_\lambda \in \mathscr{D}(X_\lambda) \quad \text{for all} \quad \lambda \in \Lambda$$
$$\text{and} \ \sum_{\lambda \in \Lambda} \|X_\lambda x_\lambda\|^2 < \infty\},$$

$$(X_\lambda)\{x_\lambda\}=\{X_\lambda x_\lambda\}, \quad \{x_\lambda\}\in\mathscr{D}((X_\lambda)).$$

It is not difficult to prove the following lemma.

LEMMA 3.1. Suppose that X_λ is a densely-defined closable operator on $\mathfrak{h}_\lambda$ and $\overline{X_\lambda}=U_\lambda|\overline{X_\lambda}|$ is the polar decomposition of $\overline{X_\lambda}$ for every $\lambda\in\Lambda$. We set $X=(X_\lambda)$ and $U=(U_\lambda)$. Then:

(1) $\bar{X}=(\overline{X_\lambda})$, $X^*=(X_\lambda^*)$;

(2) $|\bar{X}|=(|\overline{X_\lambda}|)$ and $\bar{X}=U|\bar{X}|$ is the polar decomposition of $\bar{X}$.

Let $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded $*$-algebras $\mathfrak{A}_\lambda$ on $\mathfrak{h}_\lambda$. We denote by $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ the set $\{(A_\lambda);\ A_\lambda\in\mathfrak{A}_\lambda\}$ of closed operators on $\mathfrak{h}(\Lambda)$. For each $\{A_\lambda\}\in \underset{\lambda\in\Lambda}{X}\mathfrak{A}_\lambda$ and $\{\xi_\lambda\}\in\mathfrak{D}(\Lambda)$ putting

$$(A_\lambda)\{\xi_\lambda\}=\{A_\lambda\xi_\lambda\},$$

(A_λ) is a linear operator on $\mathfrak{D}(\Lambda)$. We denote by $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ the set $\{(A_\lambda);\ A_\lambda\in\mathfrak{A}_\lambda\}$ of linear operators on $\mathfrak{D}(\Lambda)$.

LEMMA 3.2. Let $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded $*$-algebras $\mathfrak{A}_\lambda$ on $\mathfrak{h}_\lambda$. Then:

(1) For each $\{A_\lambda\}\in\underset{\lambda\in\Lambda}{X}\mathfrak{A}_\lambda$ we have

$$\overline{(A_\lambda)}=(A_\lambda), \quad (A_\lambda)^*=(A_\lambda^*);$$

(2) $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is a $\#$-algebra on $\mathfrak{D}(\Lambda)$. In particular, if $\mathfrak{A}_\lambda$ is a C^*-algebra (resp. W^*-algebra) for every $\lambda\in\Lambda$ then $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is an $EC^\#$-algebra (resp. $EW^\#$-algebra) on $\mathfrak{D}(\Lambda)$ over the direct sum $\bigoplus_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ of the C^*-algebras (resp. W^*-algebras) $\mathfrak{A}_\lambda$;

(3) $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is a $*$-algebra of closed operators on $\mathfrak{h}(\Lambda)$ under the operations of strong sum, strong product, adjoint and strong scalar multiplication. In particular, if $\mathfrak{A}_\lambda$ is a C^*-algebra (resp. W^*-algebra) then $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is an EC^*-algebra (resp. EW^*-algebra) over $\bigoplus_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ defined in [2].

DEFINITION 3.1. Let $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded $*$-algebras $\mathfrak{A}_\lambda$ with identity operators on Hilbert spaces $\mathfrak{h}_\lambda$. A $\#$-algebra $\mathfrak{A}$ on $\mathfrak{D}(\Lambda)$ is called a weakly unbounded $\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ if $\mathfrak{A}$ is a $\#$-subalgebra of $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ and $\overline{\mathfrak{A}_b}=\bigoplus_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. In particular, if $\mathfrak{A}_\lambda$ is a C^*-algebra (resp. von Neumann algebra) for every $\lambda \in \Lambda$, then $\mathfrak{A}$ is called a weakly unbounded $EC^{\#}$-algebra (resp. $EW^{\#}$-algebra) associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$.

PROPOSITION 3.1. If $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$, then $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra (defined in [7]), that is, there exists a family $\{\mathfrak{B}_\gamma\}_{\gamma \in \Gamma}$ of von Neumann algebras $\mathfrak{B}_\gamma$ such that $\overline{\mathfrak{A}}$ is a $*$-subalgebra of the EW^*-algebra $\prod_{\gamma \in \Gamma} \mathfrak{B}_\gamma$ and $\overline{\mathfrak{A}}_b = \bigoplus_{\gamma \in \Gamma} \mathfrak{B}_\gamma$. Conversely if $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra, then there exists a family $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ on Hilbert spaces $\mathfrak{h}_\lambda$ such that $\overline{\mathfrak{A}}/\mathfrak{D}(\Lambda)$ is a weakly unbounded $EW^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$.

PROOF. Suppose that $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$. It is obvious that $\overline{\mathfrak{A}}$ is a $*$-subalgebra of the EW^*-algebra $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$. So, $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra.

Conversely suppose that $\mathfrak{A}$ is a weakly unbounded $EW^{\#}$-algebra, that is, there exists a family $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ on Hilbert spaces $\mathfrak{h}_\lambda$ such that $\overline{\mathfrak{A}}$ is a $*$-subalgebra $\prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ and $\overline{\mathfrak{A}_b} = \bigoplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$. For each $A \in \mathfrak{A}$, $\bar{A} = (A_\lambda) \in \prod_{\lambda \in \Lambda} \mathfrak{A}_\lambda$, and so $\mathscr{D}(\bar{A}) \supset \mathfrak{D}(\Lambda) = \sum_{\lambda \in \Lambda}^{\oplus} \mathfrak{h}_\lambda$. We therefore see that $\overline{\mathfrak{A}}/\mathfrak{D}(\Lambda)$ is an $EW^{\#}$-algebra on $\mathfrak{D}(\Lambda)$ over $\bigoplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$.

By Proposition 3.1 it is seen that for the study of weakly unbounded $EW^{\#}$-algebras we have only to study weakly unbounded $EW^{\#}$-algebras associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$.

We shall introduce locally convex topologies on a weakly unbounded $\#$-algebra $\mathfrak{A}$ associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$.

(1) Weak topology. The locally convex topology induced by seminorms:

$$P_{\xi,\eta}(A) = |(A\xi|\eta)|, \qquad \xi, \eta \in \mathfrak{D}(\Lambda),$$

is called the weak topology on $\mathfrak{A}$.

(2) Strong topology. The locally convex topology induced by seminorms

$$P_\xi(A) = \|A\xi\|, \qquad \xi \in \mathfrak{D}(\Lambda),$$

is called the strong topology on $\mathfrak{A}$.

(3) σ-weak topology. We set

$$\mathfrak{D}_\infty(\mathfrak{A}) = \{\xi_\infty = (\xi_1, \xi_2, \dots);\ \xi_i \in \mathfrak{D}(\Lambda), \qquad i = 1, 2, \dots,$$

$$\sum_{n=1}^{\infty} \|A\xi_n\|^2 < \infty \qquad \text{for all} \quad A \in \mathfrak{A}\},$$

$$P_{\xi_\infty,\eta_\infty}(A) = |\sum_{n=1}^{\infty} (A\xi_n|\eta_n)|,\ \xi_\infty = (\xi_1, \xi_2, \dots),$$

$$\eta_\infty = (\eta_1, \eta_2, \dots) \in \mathfrak{D}_\infty(\mathfrak{A}).$$

Then $P_{\xi_\infty,\eta_\infty}(\ )$ is a seminorm on $\mathfrak{A}$. The locally convex topology induced by the seminorms $\{P_{\xi_\infty,\eta_\infty}(\ );\ \xi_\infty, \eta_\infty \in \mathfrak{D}_\infty(\mathfrak{A})\}$ is called the σ-weak topology on $\mathfrak{A}$.

(4) σ-strong topology. The locally convex topology induced by seminorms

$$P_{\xi_\infty}(A) = [\sum_{n=1}^{\infty} \|A\xi_n\|^2]^{\frac{1}{2}},\ \xi_\infty = (\xi_1, \xi_2, \dots) \in \mathfrak{D}_\infty(\mathfrak{A})$$

is called the σ-strong topology on $\mathfrak{A}$.

(5) Locally σ-weak topology. We set

$$(\mathfrak{y}_\lambda)_\infty = \{x_\infty^{(\lambda)} = (x_1^{(\lambda)}, x_2^{(\lambda)}, \dots);\ x_n^{(\lambda)} \in \mathfrak{y}_\lambda, \qquad n = 1, 2, \dots,$$

$$\sum_{n=1}^{\infty} \|x_n^{(\lambda)}\|^2 < \infty\},$$

$$\mathfrak{D}_\infty(\Lambda) = \sum_{\lambda \in \Lambda}^{\oplus} (\mathfrak{y}_\lambda)_\infty,$$

$$P_{x_\infty,y_\infty}(A) = \sum_{\lambda \in \Lambda} |\sum_{n=1}^{\infty} (A_\lambda x_n^{(\lambda)}|y_n^{(\lambda)})|,\ A = (A_\lambda) \in \mathfrak{A},$$

$$x_\infty = \{x_\infty^{(\lambda)}\},\ y_\infty = \{y_\infty^{(\lambda)}\} \in \mathfrak{D}_\infty(\Lambda).$$

Then $P_{x_\infty,y_\infty}(\ )$ is a seminorm on $\mathfrak{A}$. The locally convex topology induced by the seminorms $\{P_{x_\infty,y_\infty}(\ );\ x_\infty, y_\infty \in \mathfrak{D}_\infty(\Lambda)\}$ is called the locally σ-weak topology on $\mathfrak{A}$.

(6) Locally σ-strong topology. The locally convex topology induced by seminorms

$$P_{x_\infty}(A) = \sum_{\lambda \in \Lambda} [\sum_{n=1}^{\infty} \|A_\lambda x_n^{(\lambda)}\|^2]^{\frac{1}{2}},\ A = (A_\lambda) \in \mathfrak{A},$$

$$x_\infty = \{x_\infty^{(\lambda)}\} \in \mathfrak{D}_\infty(\Lambda)$$

is called the locally σ-strong topology on $\mathfrak{A}$.

(7) Locally uniform topology. We set

$$\|A\|_\lambda=\|A_\lambda\|,\ A=(A_\lambda)\in\mathfrak{A},$$

where $\|A_\lambda\|$ means the operator norm of $A_\lambda\in\mathfrak{A}_\lambda$. Then $\|\ \|_\lambda$ is a seminorm on $\mathfrak{A}$. The locally convex topology induced by the seminorms $\{\|\ \|_\lambda;\ \lambda\in\Lambda\}$ is called the locally uniform topology on $\mathfrak{A}$.

It is easy to show that $\mathfrak{A}$ is a locally convex $*$-algebra under the involution $\#$ and weak (or, σ-weak, locally σ-weak, locally uniform) topology.

A $*$-algebra $\mathbf{A}$ is called a (complete) LMC $*$-algebra if there exists a family $\{P_i\}_{i\in I}$ of seminorms defined on $\mathbf{A}$ such that

(1) $\{P_i\}_{i\in I}$ defines a Hausdorff (complete) locally convex topology on $\mathbf{A}$;

(2) $P_i(xy)\leqq P_i(x)P_i(y)$ for each $x,\ y\in\mathbf{A}$ and $i\in I$;

(3) $P_i(x^*)=P_i(x)$ for each $x\in\mathbf{A}$ and $i\in I$.

In particular, a complete LMC $*$-algebra $\mathbf{A}$ is called a locally C^*-algebra if

(4) $P_i(x^*x)=P_i(x)^2$ for each $x\in\mathbf{A}$ and $i\in I$.

A seminorm satisfying (1)$\sim$(4) is called a C^*-seminorm on $\mathbf{A}$.

LEMMA 3.3. ([16] Prop. 10.6) A $*$-algebra $\mathbf{A}$ is a locally C^*-algebra if and only if $\mathbf{A}$ is a closed $*$-subalgebra of Cartesian product of C^*-algebras.

LEMMA 3.4. If $\mathfrak{A}$ is a weakly unbounded $\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$, then it is a LMC $*$-algebra under the involution $\#$ and locally uniform topology. In particular, if $\mathfrak{A}$ is a weakly unbounded $EC^\#$-algebra and it is closed under the locally uniform topology then it is a locally C^*-algebra.

For a more complete discussion of the basic properties of LMC $*$-algebras the reader is referred to [1, 5, 12, 16].

We shall introduce commutants and bicommutants of a weakly unbounded $\#$-algebra $\mathfrak{A}$ associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ as follows:

$$\mathfrak{A}'=\{C\in\mathscr{B}(\mathfrak{H}(\Lambda));\ (CA\xi|\eta)=(AC\xi|\eta)\ \text{for all}\ A\in\mathfrak{A}\ \text{and}\ \xi,\eta\in\mathfrak{D}(\Lambda)\},$$

$$\mathfrak{A}^c=\{S\in\mathscr{L}^\#(\mathfrak{D}(\Lambda));\ SA=AS\quad\text{for all}\ A\in\mathfrak{A}\},$$

$$\mathfrak{A}^{cc}=\{A\in\mathscr{L}^\#(\mathfrak{D}(\Lambda));\ SA=AS\quad\text{for all}\ S\in\mathfrak{A}^c\}.$$

PROPOSITION 3.2. Let $\mathfrak{A}$ be a weakly unbounded $\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Then:

(1) $\mathfrak{A}'=\bigoplus_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)'$, $\mathfrak{A}''=\bigoplus_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)''$;

(2) $\mathfrak{A}^c = \prod_{\lambda \in \Lambda} (\mathfrak{A}_\lambda)'$, $\mathfrak{A}^{cc} = \prod_{\lambda \in \Lambda} (\mathfrak{A}_\lambda)''$.

DEFINITION 3.2. $\mathfrak{A}'$ (resp. $\mathfrak{A}''$) is called the bounded commutant (resp. bounded bicommutant) of $\mathfrak{A}$. $\mathfrak{A}^c$ (resp. $\mathfrak{A}^{cc}$) is called the commutant (resp. bicommutant) of $\mathfrak{A}$.

PROPOSITION 3.3. Let $\mathfrak{A}$ be a weakly unbounded $\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$. Then the following algebras (1)~(8) equal:

(1) $\mathfrak{A}^{cc}$;
(2) $\prod_{\lambda \in \Lambda} (\mathfrak{A}_\lambda)''$;
(3) the weak closure $[\mathfrak{A}]^{\varpi}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$;
(4) the locally σ-weak closure $[\mathfrak{A}]^{l\sigma\varpi}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$;
(5) the σ-weak closure $[\mathfrak{A}]^{\sigma\varpi}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$;
(6) the strong closure $[\mathfrak{A}]^{s}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$;
(7) the locally σ-strong closure $[\mathfrak{A}]^{l\sigma s}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$;
(8) the σ-strong closure $[\mathfrak{A}]^{\sigma s}$ of $\mathfrak{A}$ in $\mathscr{L}^{\#}(\mathfrak{D}(\Lambda))$.

PROOF. The following inclusions are obvious:

$$\begin{array}{ccccc} [\mathfrak{A}]^{\sigma\varpi} & \subset & [\mathfrak{A}]^{l\alpha\varpi} & \subset & [\mathfrak{A}]^{\varpi} \\ \cup & & \cup & & \cup \\ [\mathfrak{A}]^{\sigma s} & \subset & [\mathfrak{A}]^{l\sigma\varpi} & \subset & [\mathfrak{A}]^{s} \end{array}$$

We have only to show $\mathfrak{A}^{cc} \subset [\mathfrak{A}]^{\sigma s}$ and $[\mathfrak{A}]^{\varpi} \subset \mathfrak{A}^{cc}$. These are proved after a slight modification of ([9] Theorem 3).

§4. Dual spaces of a weakly unbounded EC$^{\#}$-algebra

In this section we shall study the dual spaces of a weakly unbounded $EC^{\#}$-algebra.

In the Cartesian product $\mathop{X}_{\gamma \in \Gamma} \mathbf{X}_\gamma$ of vector spaces $\mathbf{X}_\gamma$, the vector space spanned by $\bigcup_{\gamma \in \Gamma} \mathbf{X}_\gamma$ (or, more precisely, by $\bigcup_{\gamma \in \Gamma} l_\gamma(\mathbf{X}_\gamma)$, where l_γ is the injection mapping of $\mathbf{X}_\gamma$ in the product) is called the direct sum of $\mathbf{X}_\gamma$, and denoted by $\sum_{\gamma \in \Gamma}^{\oplus} \mathbf{X}_\gamma$. It is the set of those elements of $\mathop{X}_{\gamma \in \Gamma} \mathbf{X}_\gamma$ with only a finite number of non-zero co-ordinates. If each $\mathbf{X}_\gamma$ is a locally convex space, then the direct sum $\mathbf{X} = \sum_{\gamma \in \Gamma}^{\oplus} \mathbf{X}_\gamma$ can be given the topology by considering $\mathbf{X}$ as the inductive limit of the locally convex spaces $\mathbf{X}_\gamma$ by l_γ. This topology is the finest locally convex space topology such that induce the original topology on each $\mathbf{X}_\gamma$. This topology is called the direct sum topology for $\mathbf{X}$, and under it $\mathbf{X}$ is called the topological direct sum of $\mathbf{X}_\gamma$. Then the following facts are well known. The dual of the topological direct sum $\sum_{\gamma=\Gamma}^{\oplus} \mathbf{X}_\gamma$ is the product $\underset{\gamma\in\Gamma}{X} \mathbf{X}_\gamma^*$ of the duals (, where $\mathbf{X}_\gamma^*$ denotes the dual space of the locally convex space $\mathbf{X}_\gamma$). The dual of the topological product $\underset{\gamma\in\Gamma}{X} \mathbf{X}_\gamma$ is the direct sum $\sum_{\gamma\in\Gamma}^{\oplus} \mathbf{X}_\gamma^*$ of the duals.

In this section let $\mathfrak{A}$ be a weakly unbounded $EC^\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Let $\mathfrak{A}^*$ (resp. $\mathfrak{A}_*$, $\mathfrak{A}_\sim$) denote the set of all locally uniformly (resp. σ-weakly, locally σ-weakly) continuous linear functionals on $\mathfrak{A}$ and $\mathfrak{A}_+^*$ (resp. $\mathfrak{A}_*^+$, $\mathfrak{A}_\sim^+$) the set of all positive elements of $\mathfrak{A}^*$ (resp. $\mathfrak{A}_*$, $\mathfrak{A}_\sim$). For each $\lambda\in\Lambda$ $\mathfrak{A}_\lambda^*$ (resp. $(\mathfrak{A}_\lambda)_*$) the set of all uniformly (resp. σ-weakly) continuous linear functionals on the C^*-algebra $\mathfrak{A}_\lambda$.

THEOREM 4.1.

(1) $\mathfrak{A}^* = \sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda^*$;

(2) $\mathfrak{A}_+^* = \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_+^*$.

Suppose that $\mathfrak{A}$ is a weakly unbounded $EW^\#$-algebra. Then:

(3) $\mathfrak{A}^* \cap \mathfrak{A}_* = \mathfrak{A}_\sim = \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*$;

(4) $\mathfrak{A}^* \cap \mathfrak{A}_*^+ = \mathfrak{A}_\sim^+ = \sum_{\lambda\in\Gamma}^{\oplus} (\mathfrak{A}_\lambda)_*^+$.

PROOF. (1) Suppose that $\sum_{\lambda\in\Delta} f_\lambda \in \sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda^*$ (Δ; finite subset of Λ). For each $A=(A_\lambda)\in\mathfrak{A}$,

$$\Big(\sum_{\lambda\in\Delta} f_\lambda\Big)(A) = \sum_{\lambda\in\Delta} f_\lambda(A_\lambda).$$

Hence it is easily showed that $\sum_{\lambda\in\Delta} f_\lambda \in \mathfrak{A}^*$. Conversely suppose that $f\in\mathfrak{A}^*$. Then there exist a finite subset Δ of Λ and a positive number γ such that

$$|f(A)| \leqq \gamma \sum_{\lambda\in\Delta} \|A\|_\lambda$$

for all $A\in\mathfrak{A}$. For each $\lambda_0\in\Lambda$ and $A_{\lambda_0}\in\mathfrak{A}_{\lambda_0}$ we set

$$l_{\lambda_0}(A_{\lambda_0}) = (B_\lambda);\ (B_{\lambda_0} = A_{\lambda_0} \quad \text{and} \quad B_\lambda = 0 \qquad \text{for} \quad \lambda \neq \lambda_0).$$

Then, since $\overline{\mathfrak{A}_b} = \bigoplus_{\lambda\in\Lambda} \mathfrak{A}_\lambda$, $l_{\lambda_0}(A_{\lambda_0}) \in \mathfrak{A}$. Thus, for each $\lambda\in\Lambda$ it is seen that l_λ is a map of $\mathfrak{A}_\lambda$ into $\mathfrak{A}$. Suppose $\lambda_0 \not\in \Delta$. For each $A_{\lambda_0} \in \mathfrak{A}_{\lambda_0}$,

$$|f(l_{\lambda_0}(A_{\lambda_0}))| \leqq \gamma \sum_{\lambda \in \Delta} \|l_{\lambda_0}(A_{\lambda_0})\|_\lambda = 0.$$

That is, if $\lambda \not\in \Delta$ then f vanishes on $l_\lambda(\mathfrak{A}_\lambda)$. We set

$$f_\lambda = f \circ l_\lambda.$$

Then it is easily showed that $f_\lambda \in \mathfrak{A}_\lambda^\dagger$ and $f = \sum_{\lambda \in \Delta} f_\lambda$. Thus, $f \in \sum_{\lambda \in \Lambda}^{\oplus} \mathfrak{A}_\lambda^*$.

(2); Suppose $f \in \mathfrak{A}_+^*$. By (1), $f = \sum_{\lambda \in \Delta} f_\lambda \in \sum_{\lambda \in \Lambda}^{\oplus} \mathfrak{A}_\lambda^*$ for some finite number Δ of Λ. For each $\lambda_0 \in \Delta$ and $A_{\lambda_0} \in \mathfrak{A}_{\lambda_0}$ we have

$$0 \leqq f(l_{\lambda_0}(A_{\lambda_0})^* l_{\lambda_0}(A_{\lambda_0})) = f_{\lambda_0}(A_{\lambda 0}^* A_{\lambda_0}).$$

Hence, $f_\lambda \geqq 0$ for all $\lambda \in \Delta$. The converse is obvious.

(3); We can prove $\mathfrak{A}_\sim = \sum_{\lambda \in \Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*$ in the same way as (1). The inclusion $\mathfrak{A}_\sim \subset \mathfrak{A}^* \cap \mathfrak{A}_*$ follows from the definitions of locally uniform, locally σ-weak and σ-weak topologies. Suppose $f \in \mathfrak{A}^* \cap \mathfrak{A}_*^+$. Let $\tilde{\mathfrak{A}}$ be the closure of the EW^*-algebra $\mathfrak{A}$, that is,

$$\widetilde{\mathfrak{D}(A)}(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}} \mathscr{D}(\bar{A}),$$

$$\tilde{A}x = \bar{A}x, \quad A \in \mathfrak{A}, \quad x \in \widetilde{\mathfrak{D}(A)}(\mathfrak{A}),$$

$$\tilde{\mathfrak{A}} = \{\tilde{A};\ A \in \mathfrak{A}\}.$$

For each $A \in \mathfrak{A}$ we set

$$\tilde{f}(\tilde{A}) = f(A).$$

Then we can easily show $\tilde{f} \in \tilde{\mathfrak{A}}_*^+ \cap \tilde{\mathfrak{A}}^*$. By ([6] Theorem 4.8) there exists an element $\xi_\infty = (\xi_1, \xi_2, \dots)$ of $\mathfrak{D}_\infty(\tilde{\mathfrak{A}})$ such that $\xi_n = \{\xi_n^{(\lambda)}\} \in \widetilde{\mathfrak{D}(A)}(\mathfrak{A})$ $(n = 1, 2, \dots)$ and $\tilde{f} = \sum_{n=1}^{\infty} \omega_{\xi_n}$ (, where $\omega_\xi(A) = (A\xi|\xi)$). Since $f \in \mathfrak{A}^*$, there are a finite subset Δ of Λ and a positive number γ such that

$$|f(A)| \leqq \gamma \sum_{\lambda \in \Delta} \|A\|_\lambda$$

for all $A \in \mathfrak{A}$. Then we have

$$f(A) = \sum_{n=1}^{\infty} (\tilde{A}\xi_n|\xi_n) = \sum_{n=1} \sum_{\lambda \in \Lambda} (A_\lambda \xi_n^{(\lambda)}|\xi_n^{(\lambda)}).$$

We can now show that $\xi_n^{(\lambda_0)}=0$ for each $\lambda_0 \notin \Delta$. In fact, suppose that $\xi_n^{(\lambda_0)} \neq 0$ for some $\lambda_0 \notin \Delta$. Putting

$$A=(A_\lambda),\ (A_\lambda=0 \text{ for all } \lambda \in \Delta \text{ and } A_\lambda=I \text{ for all } \lambda \notin \Delta),$$

$A \in \bigoplus_{\lambda\in\Lambda} \mathfrak{A}_\lambda = \mathfrak{A}_b$ and $|f(A)| \leqq \gamma \sum_{\lambda\in\Delta} \|A_\lambda\| = 0$. Hence, $f(A)=0$. On the other hand, we have

$$0<\|\xi_n^{(\lambda_0)}\|^2 \leqq \sum_{n=1}^{\infty} \sum_{\lambda\in\Lambda-\Delta} \|\xi_n^{(\lambda)}\|^2 = f(A).$$

This is a contradiction. Therefore we get that $\xi_n^{(\lambda)}=0$ for all $\lambda \notin \Delta$. Hence,

$$f(A)=\sum_{n=1}\sum_{\lambda\in\Delta}(A_\lambda\xi_n^{(\lambda)}|\xi_n^{(\lambda)})=\sum_{\lambda\in\Delta}\sum_{n=1}^{\infty}\omega_{\xi_n^{(\lambda)}}(A_\lambda).$$

By ([2] Ch. I, §4, Th. I), $f_\lambda := \sum_{n=1}^{\infty} \omega_{\xi_n^{(\lambda)}} \in (\mathfrak{A}_\lambda)_*^+$. Therefore, $f=\sum_{\lambda\in\Delta} f_\lambda \in \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*^+$. Generally suppose $f \in \mathfrak{A}^* \cap \mathfrak{A}_*$. Let $\tilde{f}=|\tilde{f}|U$ be the polar decomposition of $\tilde{f} \in \tilde{\mathfrak{A}}_*$ ([6] Proposition 4.6). Then, $|\tilde{f}| \in \tilde{\mathfrak{A}}_*^+$, $U=(U_\lambda) \in \bigoplus_{\lambda\in\Lambda} \mathfrak{A}_\lambda$ and $|\tilde{f}|=\tilde{f}U^*$. Furthermore, we have

$$\begin{aligned}||\tilde{f}|(\tilde{A})| = |\tilde{f}(U^*\tilde{A})| &\leqq \gamma \sum_{\lambda\in\Delta} \|U_\lambda^* A_\lambda\| \\ &\leqq \gamma \|\sum_{\lambda\in\Delta}\|A_\lambda\|\end{aligned}$$

for all $A \in \mathfrak{A}$. Hence, $|\tilde{f}| \in \tilde{\mathfrak{A}}_*^+ \cap \tilde{\mathfrak{A}}^*$. By the above argument, $|\tilde{f}|=\sum_{\lambda\in\Delta} f_\lambda \in \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*^+$. Therefore, $f=\sum_{\lambda\in\Delta} f_\lambda U_\lambda \in \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*$.

(4); This follows from the proof of (3).

We give the direct sum topology the dual space $\mathfrak{A}^*=\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda^*$ of a weakly unbounded $EC^\#$-algebra $\mathfrak{A}$ associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Then the topological direct sum $\mathfrak{A}^*=\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{A}_\lambda^*$ is complete and $\mathfrak{A}^* \cap \mathfrak{A}_* = \sum_{\lambda\in\Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*$ is a closed subspace of $\mathfrak{A}^*$.

Corollary. If $\mathfrak{A}$ is a weakly unbounded $EW^\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$, then the dual $(\mathfrak{A}^* \cap \mathfrak{A}_*)^*$ of the topological direct sum $\mathfrak{A}^* \cap \mathfrak{A}_*$ equals the Cartesian product $\mathop{X}_{\lambda\in\Lambda} \mathfrak{A}_\lambda$ of the C^*-algebras $\mathfrak{A}_\lambda$.

§5. Invariant subspaces of the dual space

Let $\mathfrak{A}$ be a weakly unbounded $EC^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. For $A\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ and $f\in\mathfrak{A}^*$, we define actions of A on f by:

$$<X, Af>=f(XA),$$

$$<X, fA>=f(AX), \quad X\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda.$$

A subspace $\mathbf{V}$ of $\mathfrak{A}^*$ is called left (resp. right) invariant if $A\mathbf{V}\subset\mathbf{V}$ (resp. $\mathbf{V}A\subset\mathbf{V}$) for all $A\in\mathfrak{A}$. A both side invariant subspace is merely called invariant.

LEMMA 5.1. Let $\mathfrak{A}$ be a weakly unbounded $EC^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. If $\mathbf{V}$ is a left (resp. right) invariant subspace of $\mathfrak{A}^*$, then $(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)\mathbf{V}\subset\mathbf{V}$ (resp. $\mathbf{V}(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)\subset\mathbf{V}$).

PROOF. Suppose that $A=(A_\lambda)\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ and $\phi\in\mathbf{V}$. Since $\mathbf{V}\subset\mathfrak{A}^*=\sum_{\lambda\in\Lambda}^{\oplus}\mathfrak{A}_\lambda^*$, there exists a finite subset Δ of Λ such that $\phi=\sum_{\lambda\in\Delta}\phi_\lambda$ $(\phi_\lambda\in\mathfrak{A}_\lambda^*)$. Then we set

$$A_\Delta=(B_\lambda) \quad (B_\lambda=A_\lambda \text{ for all } \lambda\in\Delta \text{ and } B_\lambda=0 \text{ for all } \lambda\in\Lambda-\Delta).$$

Then we have that $A_\Delta\in\mathfrak{A}_b$ and

$$A\phi=\sum_{\lambda\in\Delta}A_\lambda\phi_\lambda=A_\Delta\phi.$$

Hence, $A\phi\in\mathfrak{A}\mathbf{V}=\mathbf{V}$.

LEMMA 5.2. If $\mathfrak{A}$ is an $EW^{\#}$-algebra, then every σ-weakly closed left (resp. right) ideal $\mathfrak{J}$ of $\mathfrak{A}$ contains a unique projection E such that $\mathfrak{J}=\mathfrak{A}E$ (resp. $\mathfrak{J}=E\mathfrak{A}$). If $\mathfrak{J}$ is a 2-sided ideal, then E belongs to the center $\mathfrak{A}'\cap\mathfrak{A}''$.

PROOF. Suppose that $\mathfrak{J}$ is a σ-weakly closed left ideal of $\mathfrak{A}$. It is easily showed that $\overline{\mathfrak{J}_b}$ is a σ-weakly closed left ideal of the von Neumann algebra $\overline{\mathfrak{A}_b}$. By ([2] Ch. I, § 3, Cor. 3) there is a unique projection E in $\mathfrak{J}_b$ such that $\overline{\mathfrak{J}_b}=\overline{\mathfrak{A}_b}E$. We shall show that $\mathfrak{J}=\mathfrak{A}E$. The inclusion $\mathfrak{A}E\subset\mathfrak{J}$ follows from $E\in\mathfrak{J}_b$. Conversely take an arbitrary element A of $\mathfrak{J}$. Let $A=U|A|$ be the polar decomposition and $|\bar{A}|=\int_0^\infty \lambda dE(\lambda)$ the spectral resolution of $|\bar{A}|$. Then, $|A|=U^\# A\in\mathfrak{J}$, and so $|\bar{A}|_n:\int_0^n \lambda dE(\lambda)=|\bar{A}|E(n)\in\bar{\mathfrak{J}_b}$. Hence, $|\bar{A}|_n=|\bar{A}|_n\bar{E}$. Since $|A|_n$ converges weakly to $|A|$, we have $|A|=|A|E$. Thus, $\mathfrak{J}=\mathfrak{A}E$.

THEOREM 5.1. Let $\mathfrak{A}$ be a weakly unbounded $EW^\#$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Then:

(1) There exists a one-to-one correspondence; $\mathbf{V}\Leftrightarrow\mathfrak{J}$ between the closed left (resp. right) invariant subspaces $\mathbf{V}$ of $\mathfrak{A}^*\cap\mathfrak{A}_*$ and the σ-weakly closed right (resp. left) ideals $\mathfrak{J}$ of the $EW^\#$-algebra $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ determined by;

$$\mathbf{V}^0=\mathfrak{J} \quad\text{and}\quad \mathfrak{J}^0=\mathbf{V},$$

where $\mathbf{V}^0$ and $\mathfrak{J}^0$ mean the polars of $\mathbf{V}$ and $\mathfrak{J}$ in $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ and in $\mathfrak{A}^*\cap\mathfrak{A}_*$ respectively.

(2) Every closed left (resp. right) invariant subspace $\mathbf{V}$ of $\mathfrak{A}^*\cap\mathfrak{A}_*$ is of the form;

$$\mathbf{V}=(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0 \qquad (\text{resp. } \mathbf{V}=E_0(\mathfrak{A}^*\cap\mathfrak{A}_*))$$

by some projection E_0 in $\mathfrak{A}$.

(3) $\mathbf{V}$ is invariant if and only if E_0 is central.

PROOF. (1); Suppose that $\mathfrak{J}$ is a σ-weakly closed right ideal of $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. By Lemma 5.1 there is a projection E_0 in $\mathfrak{A}$ with $\mathfrak{J}=(I-E_0)(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)$. We shall show that $\mathfrak{J}^0=(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0$. If $\phi\in\mathfrak{J}^0$ and $A\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$, then

$$0=<(I-E_0)A,\ \phi>=<A,\ \phi(I-E_0)>.$$

Hence, $\phi(I-E_0)=0$, i.e., $\phi=\phi E_0$. Thus, $\mathfrak{J}^0\subset(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0$. Conversely suppose $\phi\in\mathfrak{A}^*\cap\mathfrak{A}_*$. Then,

$$<\mathfrak{J},\ \phi E_0>=<E_0\mathfrak{J},\ \phi>=0.$$

Hence, $\phi E_0\in\mathfrak{J}^0$, and so $(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0\subset\mathfrak{J}^0$. Thus, $\mathfrak{J}^0=(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0$. Putting $E_0=(E_\lambda^{(0)})$, $(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0=\sum_{\lambda\in\Lambda}^{\oplus}(\mathfrak{A}_\lambda)_*E_\lambda^{(0)}$. Therefore $\mathbf{V}:=(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0$ is a closed left invariant subspace of $\mathfrak{A}^*\cap\mathfrak{A}_*$.

Suppose that $\mathbf{V}$ is a closed left invariant subspace of $\mathfrak{A}^*\cap\mathfrak{A}_*$. Then we shall show that $\mathfrak{J}:=\mathbf{V}^0$ is a σ-weakly closed right ideal of $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. For each $A\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ we have

$$\begin{aligned} <\mathfrak{J}A, \mathbf{V}> &= <\mathfrak{J}, A\mathbf{V}> \\ &= <\mathfrak{J}, \mathbf{V}> \qquad \text{(Lemma 5.1)} \\ &= 0. \end{aligned}$$

Therefore, $\mathfrak{J}(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)\subset\mathfrak{J}$. That is, $\mathfrak{J}$ is a right ideal of $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. Let $\{A_\alpha\}$ be a net in $\mathfrak{J}$ that converges σ-weakly to $A\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. Since $\mathbf{V}\subset\mathfrak{A}^*\cap\mathfrak{A}_*$, we have

$$0=<A_\alpha, \mathbf{V}> \longrightarrow <A, \mathbf{V}>.$$

Hence, $<A, \mathbf{V}>=0$, and so $A\in\mathfrak{J}$. Therefore $\mathfrak{J}$ is σ-weakly closed.

From the general theory of locally convex space, it follows that $\mathbf{V}^{oo}$ is a closed absolutely convex enveloping of $\mathbf{V}$ in $\mathfrak{A}^*\cap\mathfrak{A}_*$. Therefore, $\mathbf{V}^{oo}=\mathbf{V}$. Similarly, $\mathfrak{J}^{oo}=\mathfrak{J}$. Hence, we can prove that $\mathbf{V}\Leftrightarrow\mathfrak{J}$ is a one-to-one correspondence.

(2); Suppose that $\mathbf{V}$ is a closed left invariant subspace of $\mathfrak{A}^*\cap\mathfrak{A}_*$. By (1), $\mathbf{V}^o$ is a σ-weakly closed right ideal of $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$, and so $\mathbf{V}^o=(I-E_0)(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)$ for some projection E_0 in $\mathfrak{A}$ (Lemma 5.2). Hence,

$$\mathbf{V}=\mathbf{V}^{oo}=((I-E_0)(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda))^o=(\mathfrak{A}^*\cap\mathfrak{A}_*)E_0.$$

(3); This is now almost obvious.

DEFINITION 5.1. The projection E_0 in Theorem 5.1 is called the support projection of $\mathbf{V}$ in $\mathfrak{A}$.

LEMMA 5.3. ([15] Theorem 7.3) *If* $\mathbf{A}$ *is a* C^*-*algebra, then* $\mathbf{A}$ *admits the universal enveloping von Neumann algebra* $\mathfrak{U}(\mathbf{A})$. *Furthermore, there is a unique isometry of the second dual space* $\mathbf{A}^{**}$ *of* $\mathbf{A}$ *onto* $\mathfrak{U}(\mathbf{A})$ *which is a homeomorphic with respect to* $\sigma(\mathbf{A}^{**}, \mathbf{A}^*)$-*topology and the* σ-*weak topology on* $\mathfrak{U}(\mathbf{A})$.

By Lemma 5.3 we see that the second dual $\mathbf{A}^{**}$ of a C^*-algebra $\mathbf{A}$ is a von Neumann algebra, and the dual space $\mathbf{A}^*$ is the Banach space of all σ-weakly continuous linear functionals on $\mathbf{A}^{**}$.

Let $\mathfrak{A}$ be a weakly unbounded EC^*-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Let $\mathfrak{U}(\mathfrak{A}_\lambda)$ be a universal enveloping von Neumann algebra of the C^*-algebra $\mathfrak{A}_\lambda$ for each $\lambda\in\Lambda$. We set

$$\mathfrak{U}(\mathfrak{A})=\prod_{\lambda\in\Lambda}\mathfrak{U}(\mathfrak{A}_\lambda).$$

$\mathfrak{U}(\mathfrak{A})$ is called a universal enveloping $EW^{\#}$-algebra of $\mathfrak{A}$. From Lemma 5.3, the following fact is easily proved.

THEOREM 5.2. Let $\mathfrak{A}$ be a weakly unbounded $EC^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Then there is an isomorphism of the second dual $\mathfrak{A}^{**}$ of $\mathfrak{A}$ onto $\mathfrak{U}(\mathfrak{A})$ which is a homeomorphic with respect to $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-topology and the locally σ-weak topology on $\mathfrak{U}(\mathfrak{A})$.

COROLLARY 5.1. Let $\mathfrak{A}$ be a weakly unbounded $EC^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$. Then:

(1) There exists a one-to-one correspondence; $\mathbf{V}\Leftrightarrow\mathfrak{J}$ between the closed left (resp. right) invariant subspaces $\mathbf{V}$ of $\mathfrak{A}^*$ and the σ-weakly closed right (resp. left) ideals of $\mathfrak{U}(\mathfrak{A})$ determined by;

$$\mathbf{V}^0=\mathfrak{J} \quad \text{and} \quad \mathfrak{J}^0=\mathbf{V},$$

where $\mathbf{V}^0$ and $\mathfrak{J}^0$ mean the polars of $\mathbf{V}$ and $\mathfrak{J}$ in $\mathfrak{U}(\mathfrak{A})$ and in $\mathfrak{A}^*$ respectively.

(2) Every closed left (resp. right) invariant subspace $\mathbf{V}$ of $\mathfrak{A}^*$ is of the form;

$$\mathbf{V}=\mathfrak{A}^*E_0 \qquad (\text{resp. } \mathbf{V}=E_0\mathfrak{A}^*)$$

for some projection E_0 in $\mathfrak{U}(\mathfrak{A})$.

(3) $\mathbf{V}$ is invariant if and only if E_0 is central.

PROOF. This follows from Theorem 5.1 and Theorem 5.2.

§6. Normal and singular functionals

In this section let $\mathfrak{A}$ be a weakly unbounded $EW^{\#}$-algebra associated with $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{U}(\mathfrak{A}_\lambda)$ a universal enveloping von Neumann algebra of the C^*-algebra $\mathfrak{A}_\lambda$. By Corollary 5.1 for each $\lambda\in\Lambda$ there exists a projection $E_\lambda^{(0)}$ in $\mathfrak{U}(\mathfrak{A}_\lambda)\cap\mathfrak{U}(\mathfrak{A}_\lambda)'$ such that $(\mathfrak{A}_\lambda)_*=\mathfrak{A}_\lambda^*E_\lambda^{(0)}$. We set

$$E_0=(E_\lambda^{(0)}).$$

Then it is easily showed that E_0 is a projection in $\mathfrak{U}(\mathfrak{A})$ such that $\overline{E_0}\in\bigoplus_{\lambda\in\Lambda}\mathfrak{U}(\mathfrak{A}_\lambda)'$ and $\mathfrak{A}^*\cap\mathfrak{A}_*=\mathfrak{A}^*E_0$.

DEFINITION 6.1. The functionals in $\mathfrak{A}^*\cap\mathfrak{A}_*$ are called normal and $\mathfrak{A}^*\cap\mathfrak{A}_*$ itself is called the predual of $\mathfrak{A}$. On the contrary, the functionals in $\mathfrak{A}^*(I-E_0)$ are called singular and $\mathfrak{A}^*(I-E_0)$ is denoted by $(\mathfrak{A}^*\cap\mathfrak{A}_*)^\perp$.

THEOREM 6.1. (1) Every element ϕ of $\mathfrak{A}^*$ is uniquely decomposed into the sum

$$\phi=\phi_n+\phi_s;\ \phi_n\in\mathfrak{A}^*\cap\mathfrak{A}_*,\ \phi_s\in(\mathfrak{A}^*\cap\mathfrak{A}_*)^\perp .$$

ϕ_n and ϕ_s are called the normal part and the singular part of ϕ respectively.

(2) Suppose that $\mathbf{V}$ is a closed right (resp. left) invariant subspace of $\mathfrak{A}^*$. Then,

$$\mathbf{V}\cap(\mathfrak{A}^*\cap\mathfrak{A}_*)=\mathbf{V}E_0,\quad \mathbf{V}\cap(\mathfrak{A}^*\cap\mathfrak{A}_*)^\perp=\mathbf{V}(I-E_0).$$

LEMMA 6.1.

(1) $\mathfrak{A}^*\cap\mathfrak{A}_*=\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_*$.

(2) $(\mathfrak{A}^*\cap\mathfrak{A}_*)^\perp=\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_*^\perp$

PROOF. (1); This follows from Theorem 4.1

(2); This follows from

$$(\mathfrak{A}^*\cap\mathfrak{A}_*)^\perp=\mathfrak{A}^*(I-E_0)=\sum^{\oplus}_{\lambda\in\Lambda}\mathfrak{A}_\lambda^*(I-E_\lambda^{(0)})$$

$$=\sum^{\oplus}_{\lambda\in\Lambda}(\mathfrak{A}_\lambda)_*^\perp.$$

THEOREM 6.2. Suppose that ϕ is a non-zero element of $\mathfrak{A}_+^*$. Then the following conditions are equivalent:

(1) ϕ is singular;

(2) There exists a finite subset Δ of Λ such that

$$\phi=\sum_{\lambda\in\Delta}\phi_\lambda,\quad \phi_\lambda\in(\mathfrak{A}_\lambda)_*^\perp\ (\lambda\in\Delta);$$

(3) For each non-zero projection E in $\mathfrak{A}$ there exists a non-zero projection F in $\mathfrak{A}$ such that $E\geqq F$ and $<F,\phi>=0$.

PROOF. (1)⇔(2); This follows from Lemma 6.1.

(2)⇒(3); Suppose that $E:=(E_\lambda)$ is a non-zero projection in $\mathfrak{A}$. We set

$$\Delta_1=\{\lambda\in\Delta;\ E_\lambda\neq0\}.$$

If $\lambda\in\Delta_1$, then $\phi_\lambda\in(\mathfrak{A}_\lambda)_*$ and $E_\lambda\neq0$. By ([15] Theorem 8.5) there exists a non-zero projection G_λ in $\mathfrak{A}_\lambda$ such that

$$E_\lambda \geqq G_\lambda \quad \text{and} \quad <G_\lambda, \phi_\lambda> = 0.$$

We set

$$F_\lambda = \begin{cases} G_\lambda, & \lambda \in \Delta_1 \\ E_\lambda, & \lambda \notin \Delta_1 . \end{cases}$$

Then it is easily showed that $F: =(F_\lambda)$ is a non-zero projection in $\mathfrak{A}$ such that $E \geqq F$ and

$$<F, \phi> = \sum_{\lambda \in \Delta} <F_\lambda, \phi_\lambda> = \sum_{\lambda \in \Delta_1} <G_\lambda, \phi_\lambda> = 0.$$

(3)⇒(2); Since $\phi \in \mathfrak{A}_+^*$, there is a finite subset Δ of Λ such that

$$\phi = \sum_{\lambda \in \Delta} \phi_\lambda, \quad \phi_\lambda \in (\mathfrak{A}_\lambda)_+^* .$$

By the assumption (3) and ([15] Theorem 8.5), $\phi_\lambda \in (\mathfrak{A}_\lambda)_*^\perp$ for all $\lambda \in \Delta$. Hence, $\phi \in \sum_{\lambda \in \Lambda}^{\oplus} (\mathfrak{A}_\lambda)_*^\perp = (\mathfrak{A}^* \cap \mathfrak{A}_*)^\perp$.

THEOREM 6.3. Suppose that ϕ is a non-zero element of $\mathfrak{A}^*$. Then the following conditions are equivalent:

(1) ϕ is normal;

(2) There exists a finite subset Δ of Λ such that

$$\phi = \sum_{\lambda \in \Delta} \phi_\lambda, \quad \phi_\lambda \in (\mathfrak{A}_\lambda)_*;$$

(3) For every orthogonal family $\{E^{(i)}\}_{i \in I}$ of projections in $\mathfrak{A}$,

$$\phi(\sum_{i \in I} E^{(i)}) = \sum_{i \in I} \phi(E^{(i)}).$$

PROOF. This follows from Theorem 4.1 and ([15] Corollary 8.8).

References

[1] R. M. Brooks, *On locally m-convex *-algebras,* Pacific J. Math., **23** (1967), 5–23.

[2] J. Dixmier, *Les algebres d'operateurs dans l'espace hilbertian,* Gausthier-Villars, Paris, 2é edition (1969).

[3] P. G. Dixon, *Unbounded operator algebras,* Proc. London Math. Soc., (3) **23** (1971), 53–69.

[4] N. Dunford and J. Schwartz, *Linear operators, Vol. II,* New York; Interscience Pub. (1963).

[5] A. INOUE, *Locally C*-algebras,* Mem. Fac. Sci. Kyushu Univ., **25** (2) (1971), 197–235.
[6] A. INOUE, *On a class of unbounded operator algebras,* Pacific J. Math., **65** (1976), 77–95.
[7] A. INOUE, *On a class of unbounded operator algebras IV,* J. Math. Anal. Appl., (to appear).
[8] A. INOUE, *Unbounded representations of symmetric *-algebras,* J. Math. Soc. Japan, **29** (1977), (to appear).
[9] A. INOUE, *A commutant of an unbounded operator algebra,* (to appear).
[10] J. P. JURZAK, *Simple facts about algebras of unbounded operators,* J. Functional Anal., **21** (1976), 469–482.
[11] G. LASSNER, *Topological algebras of operators,* Reports on Math. Phys., **3** (1972), 279–293.
[12] E. A. MICHAEL, *Locally multiplicatively convex topological algebras,* Mem. Amer. Soc., **11** (1952).
[13] R. T. POWERS, *Self-adjoint algebras of unbounded operators,* Commun. Math. Phys., **21** (1971), 85–124.
[14] A. P. ROBERTSON and W. ROBERTSON, *Topological vector spaces,* Cambridge, 1966.
[15] M. TAKESAKI, *The theory of operator algebras.*
[16] W. ZELAZKO, *Metric generalization of Banach algebras,* Dissertations Math. **47** (1965).