Memoirs of the Faculty of Science, Kyushu University
Ser. A, Vol. 31, No. 2, 1977

CROSSED PRODUCTS OF LEFT HILBERT ALGEBRAS WITH RESPECT TO MINKOWSKY FORMS

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(Received October 18, 1976)

In the previous paper [3] the present author studied left Hilbert algebras with respect to Minkowsky forms. In the present paper we study the crossed products of those algebras by groups of Bogoluibov operators.

1. Bogoluibov operators

Let $\mathfrak{H}$ be a Hilbert space with a Minkowsky form $[\ ,\ ]$ and U be the unitary hermitian operator associated with $[\ ,\ ]$.

DEFINITION 1.1. A unitary operator Γ is said to be a Bogoluibov operator if Γ commutes with U.

A unitary operator Γ is a Bogoluibov operator if and only if Γ is a U-unitary operator.

PROPOSITION 1.2. Let $\mathfrak{A}$ be a left Hilbert algebra with respect to the Minkowsky form $[\ ,\ ]$ and Γ be a Bogoluibov operator satisfying the following conditions:

(1) $\Gamma\mathfrak{A}=\mathfrak{A}$;

(2) $\Gamma\xi^{\#}=(\Gamma\xi)^{\#}$ for $\xi\in\mathfrak{A}$;

(3) $\Gamma(\xi\eta)=(\Gamma\xi)(\Gamma\eta)$ for $\xi, \eta\in\mathfrak{A}$.

Then we have $\Gamma\pi(\xi)\Gamma^{-1}=\pi(\Gamma\xi)$ for $\xi\in\mathfrak{A}$.
Therefore the map α_Γ; $\pi(\mathfrak{A})\ni x\rightarrow\alpha_\Gamma(x)=\Gamma x\Gamma^{-1}\in\pi(\mathfrak{A})$ is a U-automorphism of $\pi(\mathfrak{A})$.

PROOF. For ξ and η in $\mathfrak{A}$, we have

$$\Gamma\pi(\xi)\Gamma^{-1}\eta=\Gamma(\xi\Gamma^{-1}\eta)=\pi(\Gamma\xi)\eta.$$

Hence $\Gamma\pi(\xi)\Gamma^{-1}=\pi(\Gamma\xi)$.

PROPOSITION 1.3. Let $\mathfrak{A}$ and Γ be as in Proposition 1.2. Then we have

(i) $\Gamma\mathfrak{D}^{\#}=\mathfrak{D}^{\#}$, $\Gamma\xi^{\#}=(\Gamma\xi)^{\#}$ for $\xi\in\mathfrak{D}^{\#}$;

(ii) $\Gamma\mathfrak{D}^{b}=\mathfrak{D}^{b}$, $\Gamma\xi^{b}=(\Gamma\xi)^{b}$ for $\xi\in\mathfrak{D}^{b}$;

(iii) $\Gamma\mathfrak{A}'=\mathfrak{A}'$, $\Gamma(\xi\eta)=(\Gamma\xi)(\Gamma\eta)$ for $\xi,\eta\in\mathfrak{A}'$.

PROOF. For any $\xi\in\mathfrak{D}^{\#}$, there exists a sequence $\{\xi_n\}$ in $\mathfrak{A}$ such that $\lim\xi_n=\xi$, $\lim\xi_n^{\#}=\xi^{\#}$. Since we have

$$\lim\Gamma\xi_n=\Gamma\xi;$$

$$\lim(\Gamma\xi_n)^{\#}=\lim\Gamma\xi_n^{\#}=\Gamma\xi^{\#},$$

we get $\Gamma\xi\in\mathfrak{D}^{\#}$ and $(\Gamma\xi)^{\#}=\Gamma\xi^{\#}$.

(ii) For any η in $\mathfrak{D}^{b}$, we have

$$[\xi^{\#},\Gamma\eta]=[(\Gamma^{-1}\xi)^{\#},\eta]=[\eta^{b},\Gamma^{-1}\xi]=[\Gamma\eta^{b},\xi]$$

for all $\xi\in\mathfrak{A}$. Hence $\Gamma\eta$ belongs to $\mathfrak{D}^{b}$ and $(\Gamma\eta)^{b}=\Gamma\eta^{b}$.

(iii) Take η in $\mathfrak{A}'$. For any ξ in $\mathfrak{A}$, we have

$$\pi(\xi)\Gamma\eta=\Gamma\pi(\Gamma^{-1}\xi)\eta=\Gamma\pi'(\eta)\Gamma^{-1}\xi.$$

Hence $\Gamma\eta$ belongs to $\mathfrak{A}'$ and $\pi'(\Gamma\eta)=\Gamma\pi'(\eta)\Gamma^{-1}$. Take η and ζ in $\mathfrak{A}'$. Then we have

$$\Gamma(\eta\zeta)=\Gamma\pi'(\zeta)\eta=\pi'(\Gamma\zeta)\Gamma\eta=(\Gamma\eta)(\Gamma\zeta).$$

This completes the proof.

2. Crossed products of U-involutive algebras

Let M be a U-involutive algebra acting on a Hilbert space $\mathfrak{H}$ with a Minkowsky form [,] and Aut(M) be the group of all U-automorphisms of M. For a discrete group G, we introduce a Minkowsky form in a Hilbert space $G\otimes\mathfrak{H}$ by;

$$[\sum h\otimes\xi_h,\ \sum h\otimes\eta_h]=\sum[\xi_h,\ \eta_h].$$

Let a map α; $G\ni g\to\alpha_g$ be a homomorphism of G on Aut(M).

For $g\in G$ and $A\in M$, the bounded operator $g\otimes A$ on the Hilbert space $G\otimes\mathfrak{H}$ is defined by;

$$g\otimes A(\sum h\otimes\xi_h)=\sum gh\otimes\alpha_h^{-1}(A)\xi_h.$$

We get easily $(g\otimes A)(h\otimes B)=gh\otimes\alpha_h^{-1}(A)B$.

PROPOSITION 2.1. For $g\in G$ and $A\in M$, we get

$$(g\otimes A)^U=g^{-1}\otimes\alpha_g(A^U).$$

PROOF. Take any two elements $\sum h\otimes\xi_h$ and $\sum h\otimes\eta_h$ in $G\otimes\mathfrak{H}$. Since we have

$$\begin{aligned}[g\otimes A(\sum h\otimes\xi_h),\ \sum h\otimes\eta_h]&=\sum[\alpha_{g^{-1}h}{}^{-1}(A)\xi_{g^{-1}h},\ \eta_h]\\&=\sum[\xi_{g^{-1}h},\ \alpha_{g^{-1}h}{}^{-1}(A^U)\eta_h]=[\sum h\otimes\xi_h,\ \sum h\otimes\alpha_h^{-1}(A^U)\eta_{gh}]\\&=[\sum h\otimes\xi_h,\ (g^{-1}\otimes\alpha_g(A^U))\sum h\otimes\eta_h],\end{aligned}$$

we get $(g\otimes A)^U=g^{-1}\otimes\alpha_g(A^U)$.

DEFINITION 2.2. The U-involutive algebra generated by $\{g\otimes A;\ g\in G,\ A\in M\}$ is called the crossed product of M by G and we denote it by $G\otimes M$.

3. Crossed products of left Hilbert algebras with respect to Minkowsky forms

Let $\mathfrak{A}$ be a left Hilbert algebra with respect to a Minkowsky form and $\mathfrak{H}$ be the closure of $\mathfrak{A}$. Let G be a group of $\sharp$-automorphisms of $\mathfrak{A}$. We denote the linear subspace $\{\sum h\otimes\xi_h;\ \xi_h\in\mathfrak{A}\}$ of $G\otimes\mathfrak{H}$, where the summation is finite, by $G\otimes\mathfrak{A}$. We introduce the multiplication-operation and $\sharp$-operation in $G\otimes\mathfrak{A}$ as follows:

$$(\sum_h h\otimes\xi_h)(\sum_k k\otimes\eta_k)=\sum_{h,k}hk\otimes k^{-1}(\xi_h)\eta_k;$$

$$(\sum h\otimes\xi_h)^\sharp=\sum h^{-1}\otimes h(\xi_h^\sharp).$$

It is evident that $G\otimes\mathfrak{A}$ is a involutive algebra. The involutive algebra $G\otimes\mathfrak{A}$ is called the crossed product of $\mathfrak{A}$ by G. The following theorem is obtained analogously with Theorem 2 in [7].

THEOREM 3.1. *Let $\mathfrak{A}$ be a left Hilbert algebra with respect to a Minkowsky form. If G is a group of Bogoluibov operators satisfying the following conditions:*
(1) $g\mathfrak{A}=\mathfrak{A}$;
(2) $g\xi^{\#}=(g\xi)^{\#}$;
(3) $g(\xi\eta)=(g\xi)(g\eta)$,
then we have
(i) *the crossed product* $G\otimes\mathfrak{A}$ *is a left Hilbert algebra with respect to the Minkowsky form*

$$[\sum h\otimes\xi_h, \sum h\otimes\eta_h]=\sum[\xi_h, \eta_h] \quad in \quad G\otimes\mathfrak{H};$$

(ii) $\pi(G\otimes\mathfrak{A})=G\otimes\pi(\mathfrak{A})$, *where* $\alpha_g(x)=gxg^{-1}$ *for* $g\in G$, $x\in\pi(\mathfrak{A})$.

PROOF. (i) It is evident that $G\otimes\mathfrak{A}$ is dense in $G\otimes\mathfrak{H}$. Take $\sum h\otimes\xi_h$, $\sum k\otimes\eta_k$ and $\sum l\otimes\zeta_l$ in $G\otimes\mathfrak{A}$. We have

$$[(\sum h\otimes\xi_h)(\sum k\otimes\eta_k), \sum l\otimes\zeta_l]=\sum\delta_{hk,l}[k^{-1}(\xi_h)\eta_k, \zeta_l]$$
$$=\sum\delta_{k,h^{-1}l}[\eta_k, k^{-1}(\xi_h^{\#})\zeta_l]=[\sum k\otimes\eta_k, (\sum h\otimes\xi_h)^{\#}(\sum l\otimes\zeta_l)].$$

Hence (1) of Definition 3.1 in [3] holds.

For any $\sum h\otimes\xi_h$ and $\sum k\otimes\eta_k$ in $G\otimes\mathfrak{A}$, we have

$$\|(\sum h\otimes\xi_h)(\sum k\otimes\eta_k)\|^2\leqq\sum\|\pi(k^{-1}(\xi_h))\|^2\|\eta_k\|^2$$
$$\leqq\gamma\|\sum k\otimes\eta_k\|^2.$$

Thus the map: $G\otimes\mathfrak{A}\ni\sum k\otimes\eta_k\to(\sum h\otimes\xi_h)(\sum k\otimes\eta_k)$ is continuous. Take $\sum h\otimes\xi_h$ in $G\otimes\mathfrak{A}$. For any $\sum k\otimes\eta_k$, $\eta_k\in\mathfrak{D}^b$, where the summation is finite, we get

$$[(\sum h\otimes\xi_h)^{\#}, \sum k\otimes\eta_k]=\sum\delta_{h^{-1},k}[h(\xi_h^{\#}), \eta_k]$$
$$=\sum\delta_{h^{-1},k}[\eta_k^b, h(\xi_h)]=[\sum k^{-1}\otimes k^{-1}(\eta_k^b), \sum h\otimes\xi_h].$$

Since the set $\{\sum k\otimes\eta_k; \eta_k\in\mathfrak{D}^b\}$ is dense in $G\otimes\mathfrak{H}$, the map: $G\otimes\mathfrak{A}\ni\sum h\otimes\xi_h\to(\sum h\otimes\xi_h)^{\#}$ is closable. It is obvious that $(G\otimes\mathfrak{A})^2$ is dense in $G\otimes\mathfrak{H}$. Therefore $G\otimes\mathfrak{A}$ is a left Hilbert algebra with respect to the Minkowsky form.

(ii) Take $g\in G$ and $\xi\in\mathfrak{A}$. Since we have

$$g\otimes\pi(\xi)\sum h\otimes\xi_h=\sum gh\otimes\pi(h^{-1}\xi)\xi_h$$

$$=(g\otimes\xi)(\sum h\otimes\xi_h) \quad \text{for} \quad \sum h\otimes\xi_h \quad \text{in} \quad G\otimes\mathfrak{A},$$

$$\pi(g\otimes\xi)=g\otimes\pi(\xi).$$

Therefore $\pi(G\otimes\mathfrak{A})=G\otimes\pi(\mathfrak{A})$. This completes the proof.

EXAMPLE. Let $\mathfrak{A}$ be an achieved left Hilbert algebra and $\mathfrak{H}$ the closure of $\mathfrak{A}$. Put $\tilde{\mathfrak{A}}=\{(\xi, \eta)\in\mathfrak{H}\oplus\mathfrak{H};\ \xi, \eta\in\mathfrak{A}\}$. If we introduce the multiplication-operation, $\sharp$-operation and the Minkowsky form as in [4], then $\tilde{\mathfrak{A}}$ is a left Hilbert algebra with respect to the Minkowsky form. Furthermore $\{\Delta^{it}\}$ is a one-parameter automorphism group of $\mathfrak{A}$ by Corollary 9.1 in [5]. Therefore $\{\widetilde{\Delta^{it}}\}$ is a one-parameter group of Bogoluibov operators and forms a one-parameter automorphism group of $\tilde{\mathfrak{A}}$, where $\widetilde{\Delta^{it}}=\Delta^{it}\oplus\Delta^{it}$ with respect to $\mathfrak{H}\oplus\mathfrak{H}$.

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