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REMARKS ON LEFT HILBERT ALGEBRAS WITH RESPECT TO MINKOWSKY FORMS

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In this paper we shall give an example induced by standard left Hilbert algebras and show the properties of the algebra.

EXAMPLE 1. Let $\mathfrak{A}$ be a left Hilbert algebra and $\mathfrak{H}$ the closure of $\mathfrak{A}$. Put $\tilde{\mathfrak{A}}=\{(\xi, \eta)\in\mathfrak{H}\oplus\mathfrak{H};\ \xi, \eta\in\mathfrak{A}\}$. If we introduce the multiplication-operation, $\sharp$-operation and the Minkowsky form as follows:

$$(\xi_1, \eta_1)(\xi_2, \eta_2)=(\xi_1\xi_2,\ \xi_1\eta_2+\eta_1\xi_2);$$

$$(\xi, \eta)^{\tilde{\sharp}}=(\xi^\sharp, \eta^\sharp);$$

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)]=(\xi_1|\eta_2)+(\eta_1|\xi_2);$$

then $\tilde{\mathfrak{A}}$ is a left Hilbert algebra with respect to the Minkowsky form [,].

PROOF. It is evident that $\tilde{\mathfrak{A}}$ is a involutive algebra. Take (ξ_1, η_1), (ξ_2, η_2) and (ξ_3, η_3) in $\tilde{\mathfrak{A}}$. We have

$$\begin{aligned}[(\xi_1, \eta_1)(\xi_2, \eta_2), (\xi_3, \eta_3)]&=(\xi_1\xi_2|\eta_3)+(\xi_1\eta_2+\eta_1\xi_2|\xi_3)\\ &=[(\xi_2, \eta_2), (\xi_1, \eta_1)^{\tilde{\sharp}}(\xi_3, \eta_3)].\end{aligned}$$

For $(\xi_1, \eta_1)\in\tilde{\mathfrak{A}}$ and $(\xi_2, \eta_2)\in\mathfrak{D}^b\times\mathfrak{D}^b$, we have

$$\begin{aligned}[(\xi_1, \eta_1)^{\tilde{\sharp}}, (\xi_2, \eta_2)]&=(\xi^\sharp|\eta_2)+(\eta^\sharp|\xi_2)\\ &=[(\xi_b^2, \eta_b^2), (\xi_1, \eta_1)].\end{aligned}$$

Hence the map: $\tilde{\mathfrak{A}}\ni(\xi, \eta)\rightarrow(\xi, \eta)^{\tilde{\sharp}}$ is closable. Take $(\xi_0, \eta_0)\in\tilde{\mathfrak{A}}$. Since the norm $\|\xi_0\xi\|+\|\xi_0\eta+\eta_0\xi\|$ is equivalent with the norm $\|(\xi_0\xi,\ \xi_0\eta+\eta_0\xi)\|$, the map: $\tilde{\mathfrak{A}}\ni(\xi, \eta)\rightarrow(\xi_0, \eta_0)(\xi, \eta)$ is continuous. This completes the proof. We can easily prove the following proposition.

PROPOSITION 2. (1) For $(\xi, \eta) \in \tilde{\mathfrak{A}}$ we have

$$\pi(\xi, \eta)=\begin{pmatrix} \pi(\xi) & 0 \\ \pi(\eta) & \pi(\xi) \end{pmatrix} \text{ with respect to } \mathfrak{H}\oplus\mathfrak{H}.$$

(2) $\mathfrak{D}^{\tilde{b}}=\mathfrak{D}^b \times \mathfrak{D}^b$.

(3) $\tilde{\mathfrak{A}}'=\mathfrak{A}' \times \mathfrak{A}'$. For $(\xi, \eta) \in \tilde{\mathfrak{A}}'$ we have

$$\pi'(\xi, \eta)=\begin{pmatrix} \pi'(\xi) & 0 \\ \pi'(\eta) & \pi'(\xi) \end{pmatrix}.$$

PROPOSITION 3.

(1) $\mathfrak{L}(\tilde{\mathfrak{A}})=\left\{\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}; x, y \in \mathfrak{L}(\mathfrak{A})\right\}$ where $\mathfrak{L}(\tilde{\mathfrak{A}})=\pi(\tilde{\mathfrak{A}})''$.

(2) $\mathfrak{L}(\tilde{\mathfrak{A}})'=\left\{\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}; x, y \in \mathfrak{L}(\mathfrak{A})'\right\}$.

PROOF. Since $\pi(\tilde{\mathfrak{A}})=\left\{\begin{pmatrix} \pi(\xi) & 0 \\ \pi(\eta) & \pi(\xi) \end{pmatrix}; \pi(\xi), \pi(\eta) \in \pi(\mathfrak{A})\right\}$ and $\pi(\mathfrak{A})$ is a nondegenerate *-algebra,

$$\pi(\tilde{\mathfrak{A}})'=\left\{\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}; x, y \in \pi(\mathfrak{A})'\right\}.$$

This completes the proof.

Let $S=J\Delta^{1/2}$ be the polar decomposition. By Tomita's theory we have $J\mathfrak{L}(\mathfrak{A})J=\mathfrak{L}(\mathfrak{A})'$ and $\{\sigma_t\}$ is a one parameter group of automorphisms of $\mathfrak{L}(\mathfrak{A})$, where $\sigma_t(x)=\Delta^{it}x\Delta^{-it}$ for $x \in \mathfrak{L}(\mathfrak{A})$. Put $\tilde{J}=\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ and $\widetilde{\Delta^{it}}=\begin{pmatrix} \Delta^{it} & 0 \\ 0 & \Delta^{it} \end{pmatrix}$ with respect to $\mathfrak{H}\oplus\mathfrak{H}$.

THEOREM 4. (1) *$\tilde{J}$ is a anti-Bogoluibov operator and $\widetilde{\Delta^{it}}$ is a Bogoluibov operator.*

(2) *$\tilde{J}\mathfrak{L}(\tilde{\mathfrak{A}})\tilde{J}=\mathfrak{L}(\tilde{\mathfrak{A}})'$. Hence $\mathfrak{L}(\tilde{\mathfrak{A}})$ is anti-U-involutive isomorphic to $\mathfrak{L}(\tilde{\mathfrak{A}})'$.*

(3) *$\{\tilde{\sigma}_t\}$ is a one-parameter group of U-involutive automorphisms of $\mathfrak{L}(\tilde{\mathfrak{A}})$, where $\tilde{\sigma}_t(x)=\widetilde{\Delta^{it}}x\widetilde{\Delta^{-it}}$ for $x \in \mathfrak{L}(\tilde{\mathfrak{A}})$.*

PROOF. From $\tilde{J}U=U\tilde{J}$, $\tilde{J}$ is a anti-Bogoluibov operator. Similarly $\widetilde{\Delta^{it}}$ is a Bogoluibov operator. Take$\begin{pmatrix} x & 0 \\ y & x \end{pmatrix} \in \mathfrak{L}(\tilde{\mathfrak{A}})$. From Proposition 3 and $\tilde{J}\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}\tilde{J}=\begin{pmatrix} JxJ & 0 \\ JyJ & JxJ \end{pmatrix}$, we have $\tilde{J}\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}\tilde{J}$ belongs to $\mathfrak{L}(\tilde{\mathfrak{A}})'$. Since $\tilde{\sigma}_t\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$ $=\begin{pmatrix} \Delta^{it}x\Delta^{-it} & 0 \\ \Delta^{it}y\Delta^{-it} & \Delta^{it}x\Delta^{-it} \end{pmatrix} \in \mathfrak{L}(\tilde{\mathfrak{A}})$, $\tilde{\sigma}_t$ is a U-involutive automorphism of $\mathfrak{L}(\tilde{\mathfrak{A}})$. This completes the proof.

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