

Classical and Quantum Cosmology of Multigravity

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Recently, a multigraviton theory on a simple closed circuit graph corresponding to the discretization of S^1 compactification of the Kaluza-Klein (KK) theory has been considered. In the present paper, we extend this theory to that on a general graph and study what modes of particles are included. Furthermore, we generalize it in a possible nonlinear theory based on the vierbein formalism and study classical and quantum cosmological solutions in the theory. We found that scale factors in a solution for this theory repeat acceleration and deceleration.

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I. INTRODUCTION

Both astronomical and cosmological data seem to require the presence of yet directly undetected dark matter and dark energy in the universe. The necessity for these mysterious components occurs at distances where the gravitational interaction is not understood sufficiently. This suspicious coincidence inspires a search for modifications of the general relativity at large distances. It is important to study massive and multigraviton theory for understanding cosmology and unification. In the linear-field theory, gravitons have the Fierz-Pauli (FP) type masses [1]. But there is an ambiguity in its nonlinear generalization. We studied thus far the linear multigraviton theory on a circle corresponding to S^1 compactification of the KK theory with dimensional deconstruction [2]. This model is an extended version of Hamamoto's model [3] for a massive graviton.

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In this paper, we construct the FP Lagrangian for multigravitons associated with a general graph and investigate what modes of particles are included. Furthermore, we extend it to nonlinear theory based on the vierbein formalism [4, 5]. Nonlinear extensions of multigraviton theory have been studied many authors [6]. In the present paper we focus on the semiclassical sector of the theory which governs the evolution of the universe; in other words, we will not consider nonlocal contributions and terms with higher derivatives in the possible complete theory here.

The features of our model are following: (i) Gravitons as the fluctuation from Minkowski space-time have the FP type masses [1]. (ii) This model is based on a generalized dimensional deconstruction method. So, the mass spectrum in the model can be tuned more easily than in the KK theory. (iii) The mass term has a reflection symmetry assigned at each vertex and an exchange symmetry assigned at each edge of a graph.

In this paper, beginning with graph theoretical description, we introduce the dimensional deconstruction [7, 8] and description of the linear theory of multigravity as the basis of our model in Sec. II. A nonlinear extension of the model is proposed in Sec. III. In Sec. IV, we consider the vacuum cosmological solutions of the case associated with the four-site star graph and the four-site path graph. The study on the quantum cosmological model is exhibited in Sec. V. Finally, we summarize our work and give remarks about the outlook in Sec. VI.

II. MULTIGRAVITON THEORY ON A GENERAL GRAPH

A. FP on a graph

We consider the matrix representation of the graph theory.¹ A graph G is a pair of V and E , where V is a set of vertices (sites) while E is a set of edges (links). An edge connects two vertices; two vertices located at the ends of an edge e are denoted as $o(e)$ and $t(e)$. Then, we introduce two matrices, an incidence matrix and a graph Laplacian, associated with a specific graph. The incidence matrix E represents the condition of connection or structure of a graph, and the graph Laplacian Δ can be obtained by EE^T , where E^T is the transposed

¹ Please see [9] for a brief review of application of graph theory to field theory, and textbooks [10, 11] for algebraic graph theory.

matrix of E . By use of these matrices, a quadratic form of vectors $a^T \Delta a (= a^T E E^T a)$ can be written as a sum of $(a_{t(e)} - a_{o(e)})^2$. If all a_i ($i = 1, 2, \dots, \#V$), the components of a , take the same value, $E^T a = 0$ and then $\Delta a = 0$.

So, we consider the Lagrangian for massive gravitons $h_{\mu\nu}^v$ on each vertex with the Stückelberg vector fields A_μ^e on each edge and a scalar field ϕ^v on each vertex:

$$L_m = L_0 - \frac{m^2}{2} \sum_{v \in V} [h^{v\mu\nu} (E E^T h_{\mu\nu})^v - h^v (E E^T h)^v] - 2 \sum_{v \in V} [m (E A_\mu)^v + \partial_\mu \phi^v] (\partial_\nu h^{v\mu\nu} - \partial^\mu h^v) - \frac{1}{2} \sum_{e \in E} (\partial_\mu A_\nu^e - \partial_\nu A_\mu^e)^2, \quad (2.1)$$

where L_0 is the linearized Einstein-Hilbert Lagrangian:

$$L_0 = \sum_{v \in V} \left[-\frac{1}{2} \partial_\lambda h_{\mu\nu}^v \partial^\lambda h^{v\mu\nu} + \partial_\lambda h^{v\lambda}{}_\mu \partial_\nu h^{v\mu\nu} - \partial_\mu h^{v\mu\nu} \partial_\nu h^v + \frac{1}{2} \partial_\lambda h^v \partial^\lambda h^v \right], \quad (2.2)$$

and $h^v \equiv \eta^{\mu\nu} h_{\mu\nu}^v$.

This action is invariant under the following transformations:

$$h_{\mu\nu}^v \rightarrow h_{\mu\nu}^v + \partial_\mu \xi_\nu^v + \partial_\nu \xi_\mu^v, \quad A_\mu^e \rightarrow A_\mu^e + m (E^T \xi_\mu)^e - \partial_\mu \zeta^e, \quad \phi^v \rightarrow \phi^v + m (E \zeta)^v, \quad (2.3)$$

where ξ^v and ζ^e are parameters on each vertex and each edge respectively. The massive modes of vector and scalar fields are absorbed by the massive modes of graviton fields due to the symmetry à la Stückelberg.

Now we examine the gauge fixing of the Lagrangian. Suppose the following gauge fixing terms:

$$L_{gf} = - \sum_{v \in V} \left[\partial_\nu h^{v\mu\nu} - \frac{1}{2} \partial^\mu h^v - m (E A^\mu)^v - \partial^\mu \phi^v \right]^2 - \sum_{e \in E} \left[\partial_\mu A^{e\mu} - \frac{m}{2} (E^T h)^e + m (E^T \phi)^e \right]^2, \quad (2.4)$$

then, the gauge-fixed Lagrangian becomes

$$L_m + L_{gf} = \frac{1}{2} H^{\mu\nu} (\partial^2 - m^2 E E^T) \left(H_{\mu\nu} - \frac{1}{2} H \eta_{\mu\nu} \right) + A^\mu (\partial^2 - m^2 E^T E) A_\mu + 3 \phi (\partial^2 - m^2 E E^T) \phi, \quad (2.5)$$

where $H_{\mu\nu} = h_{\mu\nu} + \phi \eta_{\mu\nu}$. Here the indices v and e , and the notion of sum over them are omitted.

In the next section, we will see that the mass spectra of fields in the Lagrangian for specific graphs with large number of vertices are similar to those of a five-dimensional model with a compactified extra space.

B. Dimensional deconstruction

It is assumed that we put fields on vertices or edges. An idea that there are four dimensional fields on the sites (vertices) and links (edges), dubbed as dimensional deconstruction, is introduced by Arkani-Hamed *et al.* [7, 8]. In this scheme, the square of mass matrix is proportional to the Laplacian of the associated graph.

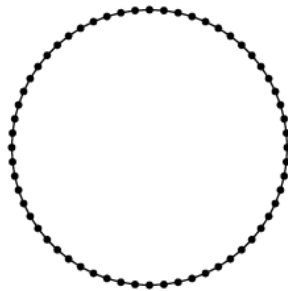


FIG. 1: The cycle graph C_{60} .

In the case of a cycle graph (a ‘closed circuit’) with N sites (denoted as C_N , and C_{60} is shown in Fig. 1 for example), when N becomes large, the model on the graph coincides with the five-dimensional theory with S^1 (circle) compactification. In other words, the mass scale of the model f over N corresponds to the inverse of the circumference of the circle:

$$M_\ell^2 = 4f^2(\sin \pi\ell/N)^2 \quad \rightarrow \quad M_\ell^2 = (2\pi\ell/L)^2, \quad (f/N \rightarrow 1/L). \quad (2.6)$$

The mass spectrum is given by the eigenvalues of the graph Laplacian of C_N , which can be expressed as

$$\Delta = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{pmatrix}. \quad (2.7)$$

For a cycle graph, the linear graviton model presented in the previous subsection coincides with the model proposed in Ref. [2]. The model is a most general linear multigraviton theory on a generic graph.

C. Particle content in the multigraviton theory on a graph

For this model, we investigate what modes of particles are contained. Although any graph is available for the model, here we consider two types, a cycle graph C_N and a path graph P_N . The path graph has a simple structure like a chain, and has two ends ($i = 1$ and N) and the i -th vertex are adjacent to $(i - 1)$ -th and $(i + 1)$ -th vertices ($1 < i < N$). For example, we show C_4 and P_4 in Fig. 2. The incidence matrix for C_4 is defined as

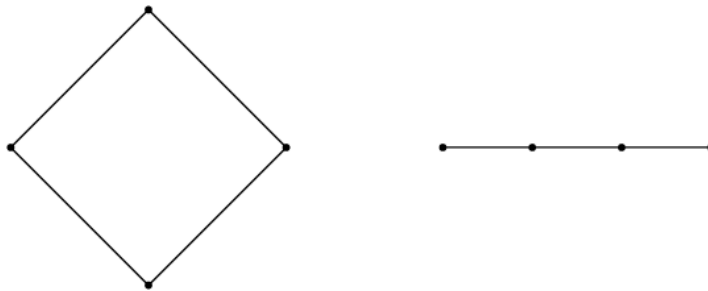


FIG. 2: The cycle graph C_4 and the path graph P_4 .

$$E(C_4) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad (2.8)$$

and then

$$E(C_4)E(C_4)^T = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} = E(C_4)^T E(C_4). \quad (2.9)$$

The eigenvalues of EE^T are $\{0, 2, 2, 4\}$ for C_4 .

On the other hand, the incidence matrix for P_4 is given by

$$E(P_4) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \quad (2.10)$$

Thus

$$E(P_4)E(P_4)^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad (2.11)$$

and

$$E(P_4)^T E(P_4) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad (2.12)$$

are different in their sizes. The eigenvalues of EE^T are $\{0, 2 - \sqrt{2}, 2, 2 + \sqrt{2}\}$ and those of $E^T E$ are $\{2 - \sqrt{2}, 2, 2 + \sqrt{2}\}$ for P_4 . For P_N , it is known that the Laplacian eigenvalues are $4 \sin^2 \frac{k\pi}{2N}$ ($k = 0, 1, \dots, N-1$). If we introduce a mass scale f and consider the large N limit as in (2.6), we find $4f^2 \sin^2 \frac{k\pi}{2N} \rightarrow \left(\frac{\pi k}{L}\right)^2$ where $f/N = 1/L$. This spectrum corresponds to that of the compactification on S^1/Z_2 , where the circumference of S^1 is $2L$.

In the multigraviton theory associated with the cycle graph C_N ($\#V = N$, $\#E = N$), $N-1$ massive spin-two's, a massless spin-two, $N-1$ massive vectors, a massless vector, $N-1$ massive scalars, and a massless scalar seem to be included, as seen from the gauge-fixed Lagrangian (2.5). The mass spectra of different spin fields are the same, except for zero modes. This is due to the fact that eigenvalues of EE^T and ones of $E^T E$ are the same except for zero eigenvalues.

However, $N-1$ massive spin two, a massless spin two, a massless vector, and a massless scalar are left physically, because massive vectors and massive scalars are absorbed by massive spin two fields to form massive gravitons with five degrees of freedom each.

Similarly, in the model associated with the path graph P_N ($\#V = N$, $\#E = N-1$), $N-1$ massive spin two's, a massless spin two, and a massless scalar is left physically, the massless vector mode is absent.

The limits of N to infinity in the cases of C_N and P_N realize the KK theory with S^1 and S^1/Z_2 compactification, respectively.

III. NONLINEAR EXTENSION OF A MULTIGRAVITON THEORY ON A TREE GRAPH

Now we will consider a nonlinear extension of the linear theory. Following Nibbelink *et al.* [4, 5], we introduce a useful ‘tool’:

$$\langle ABCD \rangle \equiv -\varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma}A_{\mu}^aB_{\nu}^bC_{\rho}^cD_{\sigma}^d, \quad (3.1)$$

where ε is the totally antisymmetric tensor. Using this expression, we have the Einstein-Hilbert term replacing A and B by vierbeins and C and D by the curvature 2-form. In addition, because the fourth power of vierbein in the angle bracket is equal to the determinant of vierbeins ($\langle eeee \rangle = \langle e^4 \rangle = 24|e|$), this expression means that the Einstein-Hilbert term and the cosmological term have the similar structure.

We now assume that the following term is assigned for each edge of a graph:

$$\langle (e_1e_1 - e_2e_2)^2 \rangle, \quad (3.2)$$

where e_1 and e_2 are vierbeins at two ends of one edge. Note that this term has a reflection symmetry $e \leftrightarrow -e$ at each vertex and an exchange symmetry $e_1 \leftrightarrow e_2$ at each edge.

In the weak field limit, *i.e.* $e_1 = \eta + f_1$, $e_2 = \eta + f_2$,

$$\langle (e_1e_1 - e_2e_2)^2 \rangle = 8 \left(([f_1] - [f_2])^2 - [(f_1 - f_2)^2] \right) + O(f^3), \quad (3.3)$$

where η is the Minkowski metric, and $[f] = \text{tr} f$ for notational simplicity. This quadratic term corresponds to the FP mass term.²

On the other hand, the Einstein-Hilbert term $\frac{1}{2}|e|R$ contains the kinetic terms of a graviton in the lowest order up to the total derivative:

$$\frac{1}{2}|e|R = -\frac{1}{2}\partial_{\lambda}f_{\mu\nu}\partial^{\lambda}f^{\mu\nu} + \partial_{\lambda}f^{\lambda}_{\mu}\partial_{\nu}f^{\nu\mu} - \partial_{\mu}f^{\mu\nu}\partial_{\nu}f - \frac{1}{2}\partial_{\lambda}f\partial^{\lambda}f + O(f^3), \quad (3.4)$$

and $\frac{1}{2}R$ contains the following terms in the first order:

$$\frac{1}{2}R = -\partial^{\lambda}\partial_{\lambda}f + \partial_{\mu}\partial_{\nu}f^{\mu\nu} + O(f^2). \quad (3.5)$$

In the case of a tree graph (a graph with no closed circuit—the path graph P_N is a tree graph, for example), we have the nonlinear Lagrangian of multigraviton theory without

² It is known that the asymmetric part of f can be omitted [12].

higher derivative and nonlocal terms,

$$L_m = \frac{1}{2} \exp \Phi \sum_{v \in V} |e_v| R_v + \frac{M^2}{24} \sum_{e \in E} \left\langle (e_{o(e)} e_{o(e)} - e_{t(e)} e_{t(e)})^2 \right\rangle, \quad (3.6)$$

where R_v is the scalar curvature associated with e_v and $M^2 \equiv 3m^2/2$. The scalar zero-mode field Φ can be identified as $\phi_1 = \phi_2 = \dots = \Phi$.

IV. CLASSICAL COSMOLOGY OF THE MULTIGRAVITON THEORY

Now we consider two vacuum cosmological models, associated with a four-site star graph and a four-site path graph respectively. Both the star graph and the line graph are tree graphs. The star graph consists of one central vertex and the other vertices adjacent to the central one. The star graph $K_{1,3}$ is shown in Fig. 3. The incidence matrix for $K_{1,3}$ is

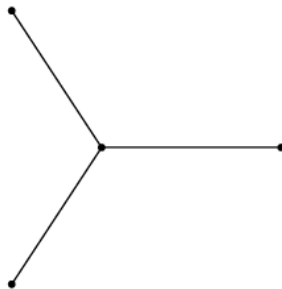


FIG. 3: The star graph $K_{1,3}$.

$$E(K_{1,3}) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.1)$$

Thus

$$E(K_{1,3})E(K_{1,3})^T = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad (4.2)$$

and

$$E(K_{1,3})^T E(K_{1,3}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \quad (4.3)$$

One can see that the eigenvalues of EE^T are $\{0, 1, 1, 4\}$ and those of $E^T E$ are $\{1, 1, 4\}$ for the star graph $K_{1,3}$. For $K_{1,N-1}$, N eigenvalues of the Laplacian are $\{0, 1, \dots, 1, N\}$. The degeneracy of $N - 2$ eigenvalues ($= 1$) is apparently due to the symmetry of the star graph.

In the case of the star graph, the associated Lagrangian for multigravitons is the following;

$$L_{star} = \frac{1}{2} \exp \Phi \sum_{i=1}^4 |e_i| R_i + \frac{M^2}{24} \sum_{i=2}^4 \langle (e_1 e_i - e_i e_i)^2 \rangle, \quad (4.4)$$

where, e_1 is on the center of the graph. On the other hand, the Lagrangian of the case of the path graph is

$$L_{path} = \frac{1}{2} \exp \Phi \sum_{i=1}^4 |e_i| R_i + \frac{M^2}{24} \sum_{i=1}^3 \langle (e_i e_i - e_{i+1} e_{i+1})^2 \rangle, \quad (4.5)$$

where, e_1 and e_4 are on each end of the graph.

Now let us introduce the setting for cosmology. We assume the homogeneous universe with a spatially-constant scalar field $\Phi(t)$ and the following metric;

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{-\Phi(t)} dt^2 + e^{-\Phi(t)} A_i^2(t) (dr^2 + r^2 d\Omega^2), \quad (4.6)$$

where $A_i(t)$ ($i = 1, \dots, 4$) are scale factors. Then,

$$\langle (e_i e_i - e_j e_j)^2 \rangle = e^{-2\Phi(t)} (e^{a_i(t)} - e^{a_j(t)}) (e^{2a_i(t)} - e^{2a_j(t)}), \quad (4.7)$$

where $a_i(t) \equiv \ln A_i(t)$.

We show the results of numerical calculations for the two models on the same appropriate initial conditions in Fig. 4 and Fig. 5. In both cases the scalar field Φ behaves similarly and in each case scale factors repeat the increase and the decrease. The oscillation of the scale factors in the path graph case include more different modes than that of the scale factors in the case of the star graph where the degeneracy of eigenvalues exists.

The star graph model has more symmetries than the path graph model. Therefore a lot of modes in the star graph are degenerate, while there is no degeneracy in the spectrum of the line graph. In the path graph case, increase of the number of sites gives the more

complicated behaviors of the scale factors. On the other hand, in the star graph case, the symmetries are preserved even if the number of sites increases. Therefore, the behaviors of scale factors are much similar to those in the four-site model, essentially.

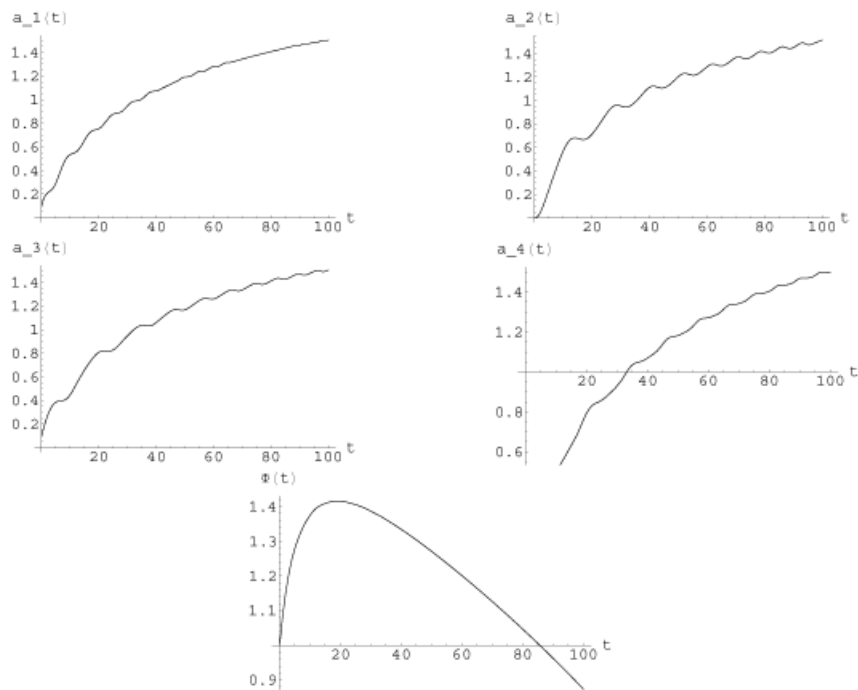


FIG. 4: Numerical solutions of a 's and Φ in the case of the four-site star graph.

V. QUANTUM COSMOLOGY OF THE MULTIGRAVITON THEORY

A. The Wheeler-DeWitt equation

In the previous section, we have seen the oscillatory behavior in the evolution of scale factors. As a qualitative analysis, we only show the characteristic solutions. In fact, oscillations must be dependent on the initial conditions. What are the natural conditions? To study the initial state, we have to consider quantum behavior of cosmology. Note that quantum cosmology of multigraviton theory has never been studied yet as far as we know.

In this section we consider a minimal model based on a graph P_2 , which is shown in Fig. 6 This model has two gravitons,³ or two scale factors. The Lagrangian density is given by

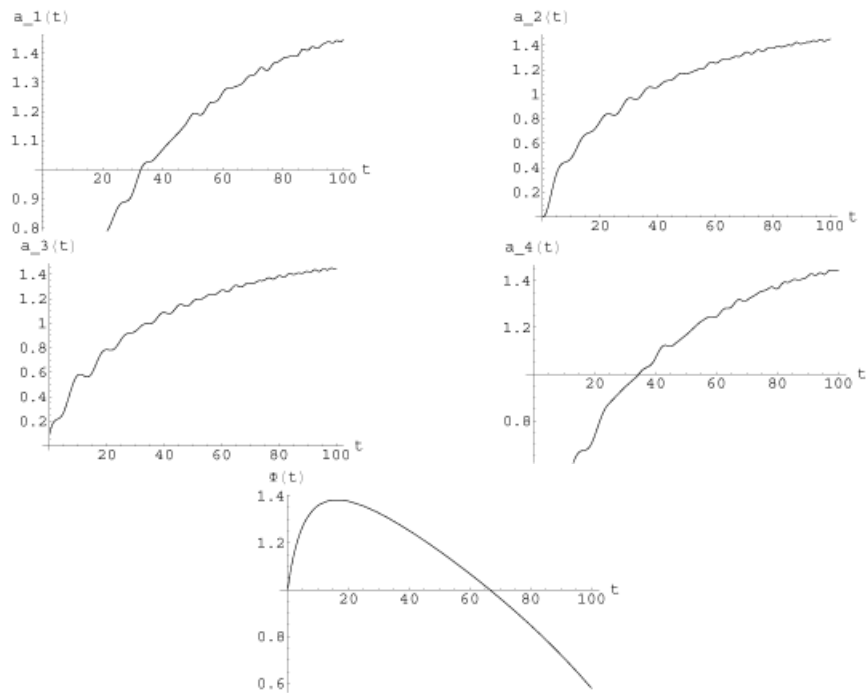


FIG. 5: Numerical solutions of a 's and Φ in the case of the four-site path graph.



FIG. 6: The path graph P_2 .

$$\mathcal{L} = \frac{1}{2\kappa^2} \exp \Phi [|e_g| R_g + |e_f| R_f] + \frac{M^2}{24} \langle (e_g e_g - e_f e_f)^2 \rangle + (\text{surface terms}), \quad (5.1)$$

where two graviton fields are labeled by g and f . This model in this case is very similar to f - g gravity [17] or bigravity [18], but our model also contains a massless scalar field.

We take the metric ansatz as follows:

$$ds_f^2 = e^{-\Phi} (-N^2 dt^2 + A^2 d\Omega_3^2), \quad (5.2)$$

$$ds_g^2 = e^{-\Phi} (-N^2 dt^2 + B^2 d\Omega_3^2). \quad (5.3)$$

³ In this case, the eigenvalues of mass are 0 and $2M/\sqrt{3}$.

These choices are equivalent to field redefinitions $g_{\mu\nu} = e^{-\Phi} g_{\mu\nu}^{(E)}$ so that

$$\mathcal{L} = \frac{1}{2\kappa^2} \left[|e_g^{(E)}| R_g^{(E)} + |e_f^{(E)}| R_f^{(E)} \right] + \dots$$

and often quoted as the choice of the Einstein frame. Here we assume that A , B and Φ depend only on t , and $d\Omega_3^2 = dx^2 + dy^2 + dz^2$. The lapse function will be set as $N = 1$ after the calculation (by the redefinition of t). Each metric is homogeneous, isotropic, and flat in the Einstein frame, in the present analysis. Then the action reads

$$\int L dt = \int dt \left[\frac{3}{2N\kappa^2} \left\{ e^{3\alpha} (-4\dot{\alpha}^2 + \dot{\Phi}^2) + e^{3\beta} (-4\dot{\beta}^2 + \dot{\Phi}^2) \right\} + NM^2 e^{-2\Phi} (e^\alpha + e^\beta)(e^\alpha - e^\beta)^2 \right], \quad (5.4)$$

where $\alpha = \ln A$, and $\beta = \ln B$. The dot indicates the derivative with respect to t . The conjugate variables are

$$\pi_\alpha = -\frac{12e^{3\alpha}\dot{\alpha}}{N\kappa^2}, \quad \pi_\beta = -\frac{12e^{3\beta}\dot{\beta}}{N\kappa^2}, \quad \pi_\Phi = \frac{3(e^{3\beta} + e^{3\alpha})\dot{\Phi}}{N\kappa^2}, \quad (5.5)$$

thus we obtain the Hamiltonian of the universe as

$$H = N \left[-\frac{\kappa^2}{24} \left\{ e^{-3\alpha} \pi_\alpha^2 + e^{-3\beta} \pi_\beta^2 \right\} + \frac{\kappa^2}{6(e^{3\alpha} + e^{3\beta})} \pi_\Phi^2 - M^2 e^{-2\Phi} (e^\alpha + e^\beta)(e^\alpha - e^\beta)^2 \right]. \quad (5.6)$$

From the Hamiltonian, we obtain the Wheeler-DeWitt (WDW) equation for the wave function of the universe $\Psi(\alpha, \beta, \Phi)$ [13]. Although there are ambiguities in the ordering, we adopt the simple replacement of conjugate variables by the derivatives with respect to the corresponding dynamical variables.⁴ The WDW equation for the present model is

$$\left[-\frac{\kappa^2}{24} \left\{ e^{-3\alpha} \frac{\partial^2}{\partial \alpha^2} + e^{-3\beta} \frac{\partial^2}{\partial \beta^2} \right\} + \frac{\kappa^2}{6(e^{3\alpha} + e^{3\beta})} \frac{\partial^2}{\partial \Phi^2} + M^2 e^{-2\Phi} (e^\alpha + e^\beta)(e^\alpha - e^\beta)^2 \right] \Psi(\alpha, \beta, \Phi) = 0. \quad (5.7)$$

Now we introduce new variables x and y . They are defined as

$$x = \frac{\alpha + \beta}{2}, \quad y = \frac{\alpha - \beta}{2}. \quad (5.8)$$

Since

$$\frac{\partial}{\partial \alpha} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \beta} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \quad (5.9)$$

⁴ Another plausible choice is adoption of the Laplacian in the minisuperspace. The qualitative behavior is not changed by the choice of the operator orderings.

the WDW equation (5.7) is rewritten as

$$\left[-\frac{\kappa^2}{6} \left\{ \cosh 3y \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2 \sinh 3y \frac{\partial^2}{\partial x \partial y} \right\} + \frac{\kappa^2}{6 \cosh 3y} \frac{\partial^2}{\partial \phi^2} + 64M^2 e^{-4\phi} e^{6x} \cosh y \sinh^2 y \right] \Psi(x, y, \phi) = 0, \quad (5.10)$$

where we have also introduced $\phi = \Phi/2$ for simplicity.

B. Wave-packet solutions

To analyze the WDW equation (5.10), we assume the wave packet ansatz. The wave packet in quantum cosmology was originally introduced in the references [14, 15], and is utilized recently for various models such as in Ref. [16]. The use of the wave packet is crucial for the case with no special ‘initial’ state served as in the case with the positively curved homogeneous space.

The general form of the wave function is written by

$$\Psi(x, y, \phi) = \sum_n C_n(x, \phi) \psi_n(x, y, \phi), \quad (5.11)$$

where

$$\left[-\frac{\kappa^2}{6} \frac{\partial^2}{\partial y^2} + 64M^2 e^{-4\phi} e^{6x} \frac{\cosh y \sinh^2 y}{\cosh 3y} \right] \psi_n(x, y, \phi) = E_n(x, \phi) \psi_n(x, y, \phi). \quad (5.12)$$

We assume that x and ϕ are slowly evolving variables, while y is a rapidly changing variable.⁵ In other words, we assume $\partial_x \ln C_n \ll 1$ and $\partial_\phi \ln C_n \ll 1$.

Further we approximate the equation if y has a small amplitude. Then

$$\left[-\frac{\kappa^2}{6} \frac{\partial^2}{\partial y^2} + 64M^2 e^{6x-4\phi} y^2 \right] \psi_n(x, y, \phi) = E_n(x, \phi) \psi_n(x, y, \phi). \quad (5.13)$$

If x and ϕ are slowly-developing variables, this is no other than the equation for a harmonic oscillator. The differential equation

$$\psi''(y) - by^2\psi(y) + c\psi(y) = 0 \quad (5.14)$$

has the solution

$$\psi_n(y) = \exp\left(-\frac{1}{2} [\sqrt{b}] y^2\right) H_n([b]^{1/4} y), \quad (5.15)$$

⁵ This assumption leads to a universe with the increasing mean size, which looks like our present universe. Some violent evolutions can occur in the very early universe, but we do not consider the possibility here.

where H_n is Hermite polynomial in the definition of *Mathematica* and ψ is normalizable if

$$c = c_n = 2\sqrt{b} \left(n + \frac{1}{2} \right), \quad n = \text{integer} \quad (5.16)$$

Therefore the approximation gives the solution of (5.13) which leads to

$$E_n(x, \phi) = \frac{\kappa^2}{6} c_n, \quad (5.17)$$

where c_n is given by (5.16) with

$$b = \frac{6}{\kappa^2} 64M^2 e^{6x-4\phi}. \quad (5.18)$$

Now the differential equation for C_n becomes

$$\left[-\frac{\kappa^2}{6} \frac{\partial^2}{\partial x^2} + \frac{\kappa^2}{6} \frac{\partial^2}{\partial \phi^2} + E_n(x, \phi) \right] C_n(x, \phi) = 0, \quad (5.19)$$

and can be approximated as

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \phi^2} - 16M\sqrt{\frac{6}{\kappa^2}} \left(n + \frac{1}{2} \right) e^{3x-2\phi} \right] C_n(x, \phi) = 0. \quad (5.20)$$

Further rewriting variables as

$$X \equiv \frac{3x - 2\phi}{\sqrt{5}}, \quad Z \equiv \frac{3\phi - 2x}{\sqrt{5}}, \quad (5.21)$$

leads to

$$\left[\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Z^2} - 16M\sqrt{\frac{6}{\kappa^2}} \left(n + \frac{1}{2} \right) e^{\sqrt{5}X} \right] C_n(X, Z) = 0. \quad (5.22)$$

Finally, separating variables as $C_n(X, Z) = f_k(Z)\varphi_{kn}(X)$ according to

$$\left[\frac{\partial^2}{\partial Z^2} + k^2 \right] f_k(Z) = 0, \quad (5.23)$$

tells us the solution

$$f_k(Z) = e^{-ikZ}, \quad \varphi_{kn}(X) = K_{2ik/\sqrt{5}} \left[\frac{2\sqrt{a_n} e^{\sqrt{5}X/2}}{\sqrt{5}} \right], \quad (5.24)$$

where K_ν is the modified Bessel function of the second kind with

$$a_n = 16M\sqrt{\frac{6}{\kappa^2}} \left(n + \frac{1}{2} \right). \quad (5.25)$$

The wave packet can be written in the form

$$\Psi = \sum_{n=0}^{\infty} \psi_n(y) \int_{-\infty}^{\infty} dk A_n(k) K_{2ik/\sqrt{5}} \left[\frac{2\sqrt{a_n} e^{\sqrt{5}X/2}}{\sqrt{5}} \right] e^{-ikZ}. \quad (5.26)$$

The wave function behaves oscillatory in the region $X < 0$ and exponentially damps in the region $X > 0$. This is because the exponential potential ‘wall’ in (5.22). The amplitude with respect to X has a maximum peak at $X \sim 0$ independently to k . Therefore the general wave packet, in which $A(k)$ is taken to be a Gaussian, has a peak at $X \sim 0$, because other peaks are destructively superposed.

The universe with $X \sim 0$ is preferred in general. Even in classical solution, oscillatory y leads to $x \sim 2/3\phi$ can be confirmed.

C. Comparison to the case with no oscillation

If we assume ‘classically’ $y \sim 0$, *i.e.*, assume $\alpha = \beta$, WDW equation reads

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \phi^2} \right] \psi = 0, \quad (5.27)$$

or

$$\left[\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Z^2} \right] \psi = 0. \quad (5.28)$$

The solution of this differential equation is:

$$\psi = f_1(x - \phi) + f_2(x + \phi) = g_1(X - Z) + g_2(X + Z). \quad (5.29)$$

This shows much different behaviors from the ‘correct’ solution of the WDW equation. No typical peak can be expected. This is rather trivial, but this comparison reminds us the fact that there is at least zero-point oscillation in any oscillatory quantum system.

VI. CONCLUSION AND OUTLOOK

We have studied the simple and Lorentz-invariant theory of multigraviton, and have shown typical cosmological solutions. We focused our attention on the models associated with the four-site star graph and the path graph and found that vacuum cosmological solutions with the scale factors show the repeated accelerating and decelerating expansions.

The differences between these two models were discussed from a viewpoint about symmetries. By using a simplest model, we also qualitatively showed that the oscillatory behavior is considered as necessary in quantum universe. We should investigate more plausible and applicable solutions for classical as well as quantum cosmology, including usual matter.

To this end, we should study how the gravitons and the scalar field couples to various matter fields. To consider various coupled fields, incorporation of supersymmetry or supergravity is also of much interest. Permitting higher derivative terms and nonlocal terms in the action will bring more possibilities to the completion of nonlinearity and be worth studying still.

As the future works, from the mathematical point of view, it is interesting to construct models with the use of generic graphs, such as weighted graphs, fractals, and so on.

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- [1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. **A173** (1939) 211.
 - [2] N. Kan and K. Shiraishi, Class. Quant. Grav. **20** (2003) 4965 [arXiv:gr-qc/0212113].
 - [3] S. Hamamoto, Prog. Theor. Phys. **97** (1997) 327 [arXiv:hep-th/9611141].
 - [4] S. G. Nibbelink, M. Peloso and M. Sexton, Eur. Phys. J. **C51** (2007) 741 [arXiv:hep-th/0610196].
 - [5] S. G. Nibbelink and M. Peloso, Class. Quant. Grav. **22** (2005) 1313 [arXiv:hep-th/0411184].
 - [6] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Ann. Phys. **305** (2003) 96 [arXiv:hep-th/0210184]; N. Arkani-Hamed and M. D. Schwartz, Phys. Rev. **D69** (2004) 104001 [arXiv:hep-th/0302110]; M. D. Schwartz, Phys. Rev. **D68** (2003) 024029 [arXiv:hep-th/0303114]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Mod. Phys. Lett. **A19** (2004) 1435 [arXiv:hep-th/0312269]; S. Nojiri and S. D. Odintsov, Phys. Lett. **B590** (2004) 295 [arXiv:hep-th/0403162]; F. Bauer, T. Hallgren and G. Seidl, Nucl. Phys.

- B781** (2007) 32 [arXiv:hep-th/0608176]; G. Seidl, e-Print: arXiv:0901.4304 [hep-th].
- [7] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. **86** (2001) 4757 [arXiv:hep-th/0104005].
- [8] C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. **D64** (2001) 1050050 [arXiv:hep-th/0104035].
- [9] N. Kan and K. Shiraishi, J. Math. Phys. **46** (2005) 112301 [arXiv:hep-th/0409268].
- [10] C. Godsil and G. Royle, *Algebraic Graph Theory* (Springer, New York, 2001).
- [11] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra* (London Mathematical Society Student Texts 75) (Cambridge University Press, Cambridge, UK, 2010).
- [12] C. Bizdadea et al., JHEP **02** (2005) 016; C. Bizdadea et al., Eur. Phys. J. **C48** (2006) 265.
- [13] For a concise review, J. J. Halliwell, “Introductory Lectures on Quantum Cosmology”, in *Proceedings of the Jerusalem Winter School on Quantum Cosmology and Baby Universe* (edited by T. Piran, World Scientific, Singapore, 1991), arXiv:0909.2566[gr-qc].
- [14] Y. Kazama and R. Nakayama, Phys. Rev. **D32** (1985) 2500.
- [15] C. Kiefer, Phys. Rev. **D38** (1988) 1761.
- [16] A. Y. Kamenshchik, C. Kiefer and B. Sandhöfer, Phys. Rev. **D76** (2007) 064032.
- [17] C. J. Isham, A. Salam and J. Strathdee, Phys. Rev. **D3** (1971) 867; A. Salam and J. Strathdee, Phys. Rev. **D16** (1977) 2668; A. Salam and J. Strathdee, Phys. Lett. **B67** (1977) 429; C. J. Isham and D. Storey, Phys. Rev. **D18** (1978) 1047.
- [18] T. Damour and I. I. Kogan, Phys. Rev. **D66** (2002) 104024; D. Blas, C. Deffayet and J. Garriga, Class. Quant. Grav. **23** (2006) 1697; D. Blas, C. Deffayet and J. Garriga, Phys. Rev. **D76** (2007) 104036; D. Blas, Int. J. Theor. Phys. **46** (2007) 2258; Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. Lett. **99** (2007) 131101.