Generalized Wigner-Yanase-Dyson Skew Information and Uncertainty Relation

Kenjiro Yanagi
Applied Mathematical Science,
Graduate School of Science and Engineering,
Yamaguchi University,
2-16-1, Tokiwadai, Ube, 755-8611, Japan.
Email: yanagi@yamaguchi-u.ac.jp

***Abstract*—We give a trace inequality related to the uncertainty relation of generalized Wigner-Yanase-Dyson skew information which includes our result in [16].**

I. Introduction

It is known that the relation between quantum covariances and quantum Fisher informations is studied and the study is applied to generalize a recently proved uncertainty relation based on quantum Fisher information. For example see [1], [6], [7]. Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [13]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson to

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H.Lieb in [10]. And also this quantity was generalized by Chen and Luo in [4] to

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^\beta, H])\rho^{1-\alpha-\beta}] \\ = & \frac{1}{2}\{Tr[\rho H^2] + Tr[\rho^{\alpha+\beta}H\rho^{1-\alpha-\beta}H] \\ & -Tr[\rho^\alpha H\rho^{1-\alpha}H] - Tr[\rho^\beta H\rho^{1-\beta}H]\}, \end{aligned}$$

where $\alpha, \beta \geq 0, \alpha + \beta \leq 1$. The convexity of $I_{\rho,\alpha,\beta}(H)$ with respect to ρ was proven by Cai and Luo in [3] under some restrictive condition. From the physical point of view, an observable H is generally considered to be an unbounded opetrator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$ (the set of all bounded linear operators on the Hilbert space $\mathcal{H}$) as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [12]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [9], [14]. In our paper [14] and [16], we defined a generalized skew information and then derived a kind of uncertainty relation. In section 2, we discuss various properties of Wigner-Yanase-Dyson skew information. In section 3, we give an uncertainty relation of generalized Wigner-Yanase-Dyson skew information.

II. Trace inequality of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \quad (1)$$

for a quantum state ρ and two observables A and B. The further strong results was given by Schrodinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A, B)|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by $Cov_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [12], [9], [14])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (2)$$

then he derived the uncertainty relation on $U_\rho(H)$ in [11]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2. \quad (3)$$

Note that we have the following relation

$$0 \le I_\rho(H) \le U_\rho(H) \le V_\rho(H). \quad (4)$$

The inequality (3) is a refinement of the inequality (1) in the sense of (4). In [16], we studied one-parameter extended inequality for the inequality (3).

Definition 2.1: For $0 \le \alpha \le 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \quad (5) \end{aligned}$$

and we also define

$$\begin{aligned} J_{\rho,\alpha}(H) &= \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \quad (6) \end{aligned}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \ne \frac{1}{2}Tr[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \le I_\rho(H) \le J_\rho(H) \le J_{\rho,\alpha}(H), \quad (7)$$

since we have $Tr[\rho^{1/2} H \rho^{1/2} H] \le Tr[\rho^\alpha H \rho^{1-\alpha} H]$. (See [2], [5] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \quad (8)$$

as a direct generalization of Eq.(2), then we have

$$0 \le I_{\rho,\alpha}(H) \le U_{\rho,\alpha}(H) \le U_\rho(H) \quad (9)$$

due to the first inequality of (7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (4),(8),(9), our situation is that we have

$$0 \le I_{\rho,\alpha}(H) \le I_\rho(H) \le U_\rho(H)$$

and

$$0 \le I_{\rho,\alpha}(H) \le U_{\rho,\alpha}(H) \le U_\rho(H).$$

We gave the following uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (3).

Theorem 2.1 ([16]): For $0 \le \alpha \le 1$, a quantum state ρ and observablea A, B,

$$U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \ge \alpha(1-\alpha)|Tr[\rho[A, B]]|^2. \quad (10)$$

Now we define the two parameter extensions of Wigner-Yanase skew information and give an uncertainty relation under some conditions in the next section.

Definition 2.2: For $\alpha, \beta \ge 0$, a quantum state ρ and an observable H, we define the generalized Wigner-Yanase-Dyson skew information

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}Tr\left[(i[\rho^\alpha, H_0])(i[\rho^\beta, H_0])\rho^{1-\alpha-\beta}\right] \\ = & \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] \\ & -Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] - Tr[\rho^\beta H_0 \rho^{1-\beta} H_0]\} \end{aligned}$$

and we define

$$\begin{aligned} & J_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}Tr\left[(i\{\rho^\alpha, H_0\})(i\{\rho^\beta, H_0\})\rho^{1-\alpha-\beta}\right] \\ = & \frac{1}{2}\{Tr[\rho H_0^2] + Tr[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] \\ & +Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] + Tr[\rho^\beta H_0 \rho^{1-\beta} H_0]\}, \end{aligned}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$. We remark that $\alpha + \beta = 1$ implies $I_{\rho,\alpha}(H) = I_{\rho,\alpha,1-\alpha}(H)$ and $J_{\rho,\alpha}(H) = J_{\rho,\alpha,1-\alpha}(H)$. We also define

$$U_{\rho,\alpha,\beta}(H) = \sqrt{I_{\rho,\alpha,\beta}(H) J_{\rho,\alpha,\beta}(H)}.$$

III. Main Theorem

In this section we assume that ρ is an invertible density matrix and A, B are Hermitian matrices. We also assume that $\alpha, \beta \ge 0$ do not necessarily satisfy the condition $\alpha + \beta \le 1$. We give the main theorem as follows;

Theorem 3.1: For $\alpha, \beta \ge 0$ and $\alpha + \beta \ge 1$ or $\alpha + \beta \le \frac{1}{2}$,

$$U_{\rho,\alpha,\beta}(A) U_{\rho,\alpha,\beta}(B) \ge \alpha\beta |Tr[\rho[A, B]]|^2. \quad (11)$$

In order to prove Theorem 3.1, we use the several lemmas. By spectral decomposition, there exists an orthonormal basis $\{\phi_1, \phi_2, \ldots, \phi_n\}$ consisting of eigenvectors of ρ. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$. Thus, ρ has a spectral representation

$$\rho = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i|. \quad (12)$$

We use the notation $f_\alpha(i, j) = \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha$. And we also use $h_{ij} = \langle\phi_i|H_0|\phi_j\rangle$, $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ij} = \langle\phi_i|B_0|\phi_j\rangle$. Then we have the following lemmas.

Lemma 3.1:

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}\sum_{i<j}\{\lambda_i + \lambda_j + f_{\alpha+\beta}(i, j) - f_\alpha(i, j) - f_\beta(i, j)\}|h_{ij}|^2. \end{aligned}$$

Proof of Lemma 3.1. By (12),

$$\rho H_0^2 = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i| H_0^2.$$

Then

$$Tr[\rho H_0^2] = \sum_{i=1}^{n} \lambda_i \langle\phi_i|H_0^2|\phi_i\rangle = \sum_{i=1}^{n} \lambda_i \|H_0|\phi_i\rangle\|^2. \quad (13)$$

Since

$$\rho^{\alpha} H_0 = \sum_{i=1}^{n} \lambda_i^{\alpha} |\phi_i\rangle\langle\phi_i| H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_{i=1}^{n} \lambda_i^{1-\alpha} |\phi_i\rangle\langle\phi_i| H_0,$$

we have

$$\rho^{\alpha} H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{n} \lambda_i^{\alpha}\lambda_j^{1-\alpha} |\phi_i\rangle\langle\phi_i| H_0 |\phi_j\rangle\langle\phi_j| H_0.$$

Thus

$$\begin{aligned} & Tr[\rho^{\alpha} H_0 \rho^{1-\alpha} H_0] \\ = & \sum_{i,j=1}^{n} \lambda_i^{\alpha}\lambda_j^{1-\alpha} h_{ij}h_{ji} \\ = & \sum_{i,j=1}^{n} \lambda_i^{\alpha}\lambda_j^{1-\alpha} |h_{ij}|^2. \end{aligned} \quad (14)$$

By the similar calculations we have

$$\begin{aligned} & Tr[\rho^{\beta} H_0 \rho^{1-\beta} H_0] \\ = & \sum_{i,j=1}^{n} \lambda_i^{\beta}\lambda_j^{1-\beta} h_{ij}h_{ji} \\ = & \sum_{i,j=1}^{n} \lambda_i^{\beta}\lambda_j^{1-\beta} |h_{ij}|^2. \end{aligned} \quad (15)$$

$$\begin{aligned} & Tr[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] \\ = & \sum_{i,j=1}^{n} \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} h_{ij}h_{ji} \\ = & \sum_{i,j=1}^{n} \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} |h_{ij}|^2. \end{aligned} \quad (16)$$

From (5), (13), (14), (15), (16),

$$\begin{aligned} & I_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{\beta}\lambda_j^{1-\beta})|h_{ij}|^2 \\ = & \frac{1}{2}\sum_{i}(\lambda_i + \lambda_i - \lambda_i - \lambda_i)|h_{ii}|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{\beta}\lambda_j^{1-\beta})|h_{ij}|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_j + \lambda_j^{\alpha+\beta}\lambda_i^{1-\alpha-\beta} - \lambda_j^{\alpha}\lambda_i^{1-\alpha} - \lambda_j^{\beta}\lambda_i^{1-\beta})|h_{ji}|^2 \\ = & \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) - f_{\alpha}(i,j) - f_{\beta}(i,j))|h_{ij}|^2. \end{aligned}$$

□

Lemma 3.2:

$$J_{\rho,\alpha,\beta}(H) \geq \sum_{i<j}(\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) + f_{\alpha}(i,j) + f_{\beta}(i,j))|h_{ij}|^2.$$

Proof of Lemma 3.2. By (6), (13), (14), (15), (16), we have

$$\begin{aligned} & J_{\rho,\alpha,\beta}(H) \\ = & \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{\beta}\lambda_j^{1-\beta})|h_{ij}|^2 \\ = & \frac{1}{2}\sum_{i}(\lambda_i + \lambda_i + \lambda_i + \lambda_i)|h_{ii}|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_i^{\alpha+\beta}\lambda_j^{1-\alpha-\beta} + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{\beta}\lambda_j^{1-\beta})|h_{ij}|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_j + \lambda_j^{\alpha+\beta}\lambda_i^{1-\alpha-\beta} + \lambda_j^{\alpha}\lambda_i^{1-\alpha} + \lambda_j^{\beta}\lambda_i^{1-\beta})|h_{ji}|^2 \\ = & 2\sum_{i}\lambda_i|h_{ii}|^2 \\ & + \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) + f_{\alpha}(i,j) + f_{\beta}(i,j)|h_{ij}|^2 \\ \geq & \frac{1}{2}\sum_{i<j}(\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) + f_{\alpha}(i,j) + f_{\beta}(i,j)|h_{ij}|^2. \end{aligned}$$

□

Lemma 3.3: For any $t > 0$ and $\alpha, \beta \geq 0, \alpha + \beta \geq 1$ or $\alpha + \beta \leq \frac{1}{2}$, the following inequality holds;

$$(t^{1-\alpha-\beta} + 1)^2(t^{2\alpha} - 1)(t^{2\beta} - 1) \geq 16\alpha\beta(t-1)^2. \quad (17)$$

Proof of Lemma 3.3. It is sufficient to prove (17) for $t \geq 1$ and $\alpha, \beta \geq 0, \alpha + \beta \geq 1$ or $\alpha + \beta \leq \frac{1}{2}$. By Lemma 3.3 in [16] we have for $0 \leq p \leq 1$ and $s \geq 1$,

$$(1-2p)^2(s-1)^2 - (s^p - s^{1-p})^2 \geq 0.$$

Then we can rewrite as follows;

$$(s^{2p} - 1)(s^{2(1-p)} - 1) \geq 4p(1-p)(s-1)^2.$$

We assume that $\alpha, \beta \geq 0$. We put $p = \alpha/(\alpha+\beta)$ and $s^{1/(\alpha+\beta)} = t$. Then

$$(t^{2\alpha}-1)(t^{2\beta}-1) \geq \frac{4\alpha\beta}{(\alpha+\beta)^2}(t^{\alpha+\beta}-1)^2.$$

Then we have

$$\begin{aligned} & (t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \\ \geq\ & \frac{4\alpha\beta}{(\alpha+\beta)^2}(t^{1-\alpha-\beta}+1)^2(t^{\alpha+\beta}-1)^2. \end{aligned} \quad (18)$$

We put $\alpha+\beta = k$ and $f(t) = (t^{1-k}+1)(t^k-1) - 2k(t-1)$. Then

$$\begin{aligned} f'(t) &= (1-k)t^{-k}(t^k-1) + k(t^{1-k}+1)t^{k-1} - 2k \\ &= (1-k)(1-t^{-k}) + k(1+t^{k-1}) - 2k. \end{aligned}$$

and

$$\begin{aligned} f''(t) &= (1-k)kt^{-k-1} + k(k-1)t^{k-2} \\ &= k(k-1)(t^{k-2} - t^{-k-1}). \end{aligned}$$

When $k = \alpha+\beta \geq 1$ or $k = \alpha+\beta \leq \frac{1}{2}$, it is easy to show that $f''(t) \geq 0$ for $t \geq 1$. Since $f'(1) = 0$, we have $f'(t) \geq 0$ for $t \geq 1$. And since $f(1) = 0$, we have $f(t) \geq 0$ for $t \geq 1$. Hence we have for $\alpha+\beta \geq 1$ or $\alpha+\beta \leq \frac{1}{2}$,

$$(t^{1-\alpha-\beta}+1)(t^{\alpha+\beta}-1) \geq 2(\alpha+\beta)(t-1).$$

It follows from (18) that we get

$$(t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \geq 16\alpha\beta(t-1)^2.$$

□

Proof of Theorem 3.1. Since

$$\begin{aligned} & (t^{1-\alpha-\beta}+1)^2(t^{2\alpha}-1)(t^{2\beta}-1) \\ =\ & (t+1+t^{\alpha+\beta}+t^{1-\alpha-\beta})^2 - (t^\alpha + t^{1-\alpha} + t^\beta + t^{1-\beta})^2, \end{aligned}$$

we put $t = \frac{\lambda_i}{\lambda_j}$ in (17). Then we have

$$\left\{\frac{\lambda_i}{\lambda_j} + 1 + \left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha+\beta} + \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha-\beta}\right\}^2$$

$$-\left\{\left(\frac{\lambda_i}{\lambda_j}\right)^{\alpha} + \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha} + \left(\frac{\lambda_i}{\lambda_j}\right)^{\beta} + \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\beta}\right\}^2$$

$$\geq 16\alpha\beta\left(\frac{\lambda_i}{\lambda_j} - 1\right)^2.$$

Then we have

$$\begin{aligned} & \{\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) - f_\alpha(i,j) - f_\beta(i,j)\} \\ & \times\{\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j) + f_\alpha(i,j) + f_\beta(i,j)\} \\ =\ & (\lambda_i + \lambda_j + f_{\alpha+\beta}(i,j))^2 - (f_\alpha(i,j) + f_\beta(i,j))^2 \\ \geq\ & 16\alpha\beta(\lambda_i - \lambda_j)^2. \end{aligned} \quad (19)$$

Since

$$\begin{aligned} Tr[\rho[A,B]] &= Tr[\rho[A_0,B_0]] \\ &= 2iImTr[\rho A_0 B_0] \\ &= 2iIm\sum_{i<j}(\lambda_i - \lambda_j)a_{ij}b_{ji} \\ &= 2i\sum_{i<j}(\lambda_i - \lambda_j)Im a_{ij}b_{ji}, \end{aligned}$$

$$\begin{aligned} |Tr[\rho[A,B]]| &= 2|\sum_{i<j}(\lambda_i - \lambda_j)Im a_{ij}b_{ji}| \\ &\leq 2\sum_{i<j}|\lambda_i - \lambda_j||Im a_{ij}b_{ji}|. \end{aligned}$$

Then we have

$$|Tr[\rho[A,B]]|^2 \leq 4\left\{\sum_{i<j}|\lambda_i - \lambda_j||Im a_{ij}b_{ji}|\right\}^2.$$

By (19) and Schwarz inequality,

$$\begin{aligned} & \alpha\beta|Tr[\rho[A,B]]|^2 \\ \leq\ & 4\alpha\beta\left\{\sum_{i<j}|\lambda_i - \lambda_j||Im a_{ij}b_{ji}|\right\}^2 \\ =\ & \frac{1}{4}\left\{\sum_{i<j}4\sqrt{\alpha\beta}|\lambda_i - \lambda_j||Im a_{ij}b_{ji}|\right\}^2 \\ \leq\ & \frac{1}{4}\left\{\sum_{i<j}4\sqrt{\alpha\beta}|\lambda_i - \lambda_j||a_{ij}||b_{ji}|\right\}^2 \\ \leq\ & \frac{1}{4}\left\{\sum_{i<j}\{K^2 - L^2\}^{1/2}|a_{ij}||b_{ji}|\right\}^2 \\ \leq\ & \frac{1}{2}\sum_{i<j}(K-L)|a_{ij}|^2 \times \frac{1}{2}\sum_{i<j}(K+L)|b_{ij}|^2, \end{aligned}$$

where $K = \lambda_i + \lambda_j + f_{\alpha+\beta}(i,j), L = f_\alpha(i,j) + f_\beta(i,j)$. Then we have

$$I_{\rho,\alpha,\beta}(A)J_{\rho,\alpha,\beta}(B) \geq \alpha\beta|Tr[\rho[A,B]]|^2.$$

We also have

$$I_{\rho,\alpha,\beta}(B)J_{\rho,\alpha,\beta}(A) \geq \alpha\beta|Tr[\rho[A,B]]|^2.$$

Hence we have the final result (11). □

Remark 3.1: We remark that (10) is derived by putting $\beta = 1-\alpha$ in (11). Then Theorem 3.1 is a generalization of Theorem 2.1 given in [16].

Remark 3.2: When $\alpha, \beta \geq 0$ and $\frac{1}{2} < \alpha+\beta < 1$, we can show an example which Theorem 3.1 does not hold as follows; Let $\alpha = 1/2, \beta = 1/4$ and

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$U_{\rho,\alpha,\beta}(A)U_{\rho,\alpha,\beta}(B) = 0.004487,$$
$$\alpha\beta|Tr[\rho[A,B]]|^2 = 0.125.$$

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