Relative Extending Modules, and Indecomposable Decompositions of Lifting Modules

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Abstract

In the study on the ring and module theory, projective modules and injective modules are very important. A projective module can be characterized from the homological situation, from which an injective module can be defined as the dual.

In 1953, Eckmann-Schopf showed the existence theorem of an injective hull: Any module is essentially embeded in an injective module. On the other hand, in 1960, Bass introduced the projective cover of a module as the dual of the injective hull. In general, projective covers do not always exist. Thus, Bass treated the existence of projective covers, and he introduced a (semi-)perfect ring as a ring for which every (finitely generated) module has the projective cover. As a generalization of artinian rings, these rings are important.

An injective module has a property that any its submodule is essentially extended in a direct summand. The module with such a property is called an extending module or a CS-module. As a dual of the property, a projective supplemented module has a property that any its submodule is co-essentially lifted to a direct summand. The module having this property is called a lifting module. In the history of ring theory, extending and lifting modules are introduced in papers of Utumi and Bass, respectively. Actually, the right continuous ring R by Utumi is a von Neumann regular ring which is extending as a right R-module. On the other hand, the perfect ring of Bass means a ring for which every projective module is lifting. Injective modules are extending modules, but in general projective modules are not lifting modules.

From the beginning of 1980, these modules were extensively studied by Harada, Oshiro, Mohamed, Müller, Smith, Wisbauer and other many ring theorists.

After Utumi's works, continuous rings were generalized to continuous modules and quasi-continuous modules. As generalizations of these modules,

relative continuous modules and relative quasi-continuous modules were introduced by López-Permouth, Oshiro, and Rizvi, and recently, as duals, relative discrete modules and relative (quasi-)discrete modules were considered by Keskin-Harmanci.

From 1958 to 1959, Matlis and Papp studied injective modules over right noetherian rings and they showed the following result: A ring R is right noetherian if and only if every injective R-module has an indecomposable decomposition.

As an improved version of this result, in 1982, the following was shown by Müller-Rizvi: A ring R is right noetherian if and only if every continuous R-module has an indecomposable decomposition.

Futhermore, in 1984, Okado showed the following result: A ring R is right noetherian if and only if every extending R-module has an indecomposable decomposition. By the way, a problem of " When is a direct sum of injective (continuous, quasi-continuous, extending) modules injective (continuous, quasi-continuous, extending)? " was studied by many researchers. In addition, by Oshiro, projective modules over perfect (semiperfect) rings was studied by using a lifting property.

In 1972, the result of 「Projective modules over right perfect rings have an indecomposable decomposition」 was shown by Anderson-Fuller. Also, the result of 「Projective modules over right perfect rings have the exchange property」 was given by Yamagata, Harada-Ishii.

In spite of such situations, the following fundamental problems are unsolved as the biggest problems now in this field.

Problem (1): Does any lifting module have an indecomposable decomposition ?

Problem (2): Does any lifting module have the (finite) internal exchange property ?

Problem (3): Which ring R has the property that every lifting R-module has an indecomposable decomposition ?

Problem (4): When is a direct sum of lifting (CS-) modules necessarily lifting (CS-) module ?

In this paper, we treat the above problems.

In Chapter 1, we describe known results as preliminaries. In particular, by using a lifting property, we give some characterizations of perfect rings and semiperfect rings including artinian rings. We note that these are implicitly due to Bass and are explicitly shown by Oshiro.

In Chapter 2, we study Problem (1), (3) for right perfect rings and semiperfect rings. And we introduce " dual relative ojectivity " and give a sufficient condition for lifting modules over right perfect rings to satisfy the exchange property. This is a result for Problem (2).

In Chapter 3, we give a characterization for a direct sum of relative (quasi-)continuous modules to be relative (quasi-)continuous modules. This is a result about Problem (4).

要旨

環及び加群の研究に重要な概念であり、非常に良い性質を持った加群として、射影加群と入射加群がある。これらの加群は、ホモロジー代数的に見れば双対の関係にある。

1953 年、Eckmann-Schopf によって、任意の加群は入射加群の中に稠密に埋蔵されるという移入包絡 (injective hull) の存在定理が示された。一方、1960 年、Bass により、移入包絡の双対として射影被覆 (projective cover) が考察された。移入包絡と射影被覆は、存在すれば同型の違いを除いて一意的に定まる。移入包絡は常に存在するが、射影被覆は移入包絡と異なり常に存在するとは限らない。そこで Bass は射影被覆を考察し、すべての有限生成加群が射影被覆を持つ準完全環と、すべての加群が射影被覆を持つ完全環を研究した。これらの環は、環論及び加群論の研究において重要な役割をはたす環となった。

入射加群は「任意の部分加群は直和因子に essential に extend される」という性質を持つ。この性質を持つ加群を extending 加群、或いは CS-加群という。その性質の双対として、supplemented 射影加群は「任意の部分加群は直和因子に co-essential に lift される」という性質を持つ。この性質を持つ加群が lifting 加群である。これらの性質はそれぞれ歴史的には Utumi と上述の Bass の論文で考察されているとみてよい。実際、Utumi の右連続環 R は、von Neumann regular ring で、右 R-加群として extending 加群である環のことである。一方、Bass の完全環とは、すべての射影加群が lifting になる環を意味する。一般に、任意の入射加群は extending 加群であるが、その双対の性質としての、任意の射影加群は lifting 加群であることは成立しない。1980 年の始めから、これらの加群は、Harada、Oshiro、Mohamed、Müller、Smith、Wisbauer 等多くの環論研究者によって活発に研究されている。

Utumi の連続環はその後連続加群、準連続加群へと一般化された。更に、1998 年、López-Oshiro-Rizvi により、連続加群と準連続加群が新しい比較概念を用いて、比較連続加群と比較準連続加群へと一般化された。最近、Keskin-Harmanci によって、比較 (準) 連続加群の双対として比較 (準) 離散加群が考

察されている。

1958年から1959年にかけて、Matlis と Papp によって、右 Noether 環上の入射加群が研究され、右ネーター環が「すべての入射加群は直既約分解を持つ」ことで特徴づけられた。

この結果の拡張として、1982年に Müller-Rizvi により、右ネーター環であることと,「すべての連続加群は直既約分解を持つ」ことは同値であることが示された。

更に、1984年に Okado によって、右ネーター環であることと、「すべての extending 加群は直既約分解を持つ」ことは同値であるという結果が示された。

又、“入射加群 (連続加群, 準連続加群, CS 加群) の直和はいつ入射加群 (連続加群, 準連続加群, CS 加群) になるか？”という問題に関しても、多くの研究者により研究された。

1972年に、Anderson-Fuller により、「右完全環上の射影加群は直既約分解を持つ」という結果が示されている。そして、Yamagata、Harada-Ishii によって、「右完全環上の射影加群は exchange property を満たす」という結果が与えられている。

このような状況の下で、現在、この分野で最も大きな課題は、次の基本的な問題を解決することである。

Problem (1): lifting 加群は直既約分解を持つか？

Problem (2): lifting 加群は (finite) internal exchange property を満たすか？

Problem (3): すべての lifting 加群が直既約分解を持つ環は何か？

Problem (4): lifting(CS-) 加群の直和はいつ lifting(CS-) 加群になるか？

本論文では、これらの問題に関連した研究を行った。各章の内容は以下の通りである。

第一章では、準備として既知の結果を述べる。特に、Artin 環を含む完全環及び準完全環の lifting 性を用いた特徴づけを与える。これらの結果は実際には上述の Bass によるものであるが、Oshiro によって明確な形で与えられた。

第二章では、projective cover の存在が保証できる (Artin 環を含む) 完全

環及び準完全環に対して上記 Problem (1), (3) に関する結果を述べる。主定理で、右完全環 (準完全環) 上の lifting 加群 (有限生成 lifting 加群) は直既約分解を持つという基本的な結果を示す。更に、“ dual relative ojective 性 ”を導入し、右完全環上の lifting 加群が exchange property を満たすための十分条件を与える。これは上記 Problem (2) に関する結果である。

第三章では、上記 Problem (4) に関連して比較連続加群 (比較準連続加群) の直和が比較連続加群 (比較準連続加群) になるための特徴づけを与える。

目次

Introduction

Rings are algebras that are abstract mathematical objects with the binary operations of $+$ and $\times$, and modules mean vector spaces over rings. In the study on the ring and module theory, projective modules and injective modules are very important. A projective module is defined as a direct summand of a free module. This module can be characterized from the homological situation, from which an injective module can be defined as the dual.

In 1953, Eckmann-Schopf([9]) showed the existence theorem of an injective hull: Any module is essentially embeded in an injective module. On the other hand, in 1960, Bass([5]) introduced the projective cover of a module as the dual of the injective hull. In general, projective covers do not always exist. Thus, Bass treated the existence of projective covers, and he introduced a (semi-)perfect ring as a ring for which every (finitely generated) module has the projective cover. As a generalization of artinian rings, these rings are important.

An injective module has a property that any its submodule is essentially extended in a direct summand. The module with such a property is called an extending module or a CS-module. As a dual of the property, a projective supplemented module has a property that any its submodule is co-essentially lifted to a direct summand. The module having this property is called a lifting module. In the history of ring theory, extending and lifting modules are introduced in papers of Utumi([39]) and Bass([5]), respectively. Actually, the right continuous ring R by Utumi is a von Neumann regular ring which is extending as a right R-module. On the other hand, the perfect ring of Bass means a ring for which every projective module is lifting. Injective modules are extending modules, but in general projective modules are not lifting modules.

From the beginning of 1980, these modules were extensively studied by

Harada, Oshiro, Mohamed, Müller, Smith, Wisbauer and other many ring theorists. These situations are witnessed in the following books on this field.

1. M. Harada: *Factor categories with applications to direct decomposition of modules*, Lect. Notes Pure Appl. Math. **88**, Marcel Dekker, New York (1983).

2. S.H. Mohamed and B.J. Müller: *Continuous and Discrete modules*, London Math. Soc. Lect. Notes **147**, Cambrige Univ. Press, (1990).

3. N.V. Dung, N.V. Huynh, P.F. Smith and R. Wisbauer: *Extending modules*, Pitman Research Notes in Mathematics Series **313**, Longman Group Limited (1994).

4. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer: *Lifting modules*, Birkhauser Boston, Boston (2007).

After Utumi's works, continuous rings were generalized to continuous modules and quasi-continuous modules. As generalizations of these modules, relative continuous modules and relative quasi-continuous modules were introduced by López-Permouth, Oshiro, and Rizvi([25]), and recently, as duals, relative discrete modules and relative (quasi-)discrete modules were considered by Keskin-Harmanci ([21]).

For more details of backgrounds and results with regard to relative (quasi-)continuous modules and relative (quasi-)discrete modules, we can refer to the following papers:

(1) K. Oshiro: *Continuous modules and quasi-continuous modules*, Osaka J. Math. **20**(1983), 681-694.

(2) S.R. López-Permouth, K. Oshiro and S.T. Rizvi: *On the relative (quasi-)continuity of modules*, Comm. Algebra **26**(1998), 3497-3510.

(3) D. Keskin and A. Harmanci: *A relative version of the lifting property of modules*, Algebra Colloquium **11**(3)(2004), 361-370.

(4) N. Orhan and D. Keskin: *Characterization of lifting modules in terms of cojective modules and the class of $\mathcal{B}(M,X)$*, Int. J. Math. **16**(6)(2005), 647-660.

From 1958 to 1959, Matlis([27]) and Papp([38]) studied injective modules over right noetherian rings and they showed the following result: A ring R

is right noetherian if and only if every injective R-module has an indecomposable decomposition.

As an improved version of this result, in 1982, the following was shown by Müller-Rizvi([30]): A ring R is right noetherian if and only if every continuous R-module has an indecomposable decomposition.

Futhermore, in 1984, Okado([31]) showed the following result: A ring R is right noetherian if and only if every extending R-module has an indecomposable decomposition. By the way, a problem of " When is a direct sum of injective (continuous, quasi-continuous, extending) modules injective (continuous, quasi-continuous, extending)? " was studied by many researchers(see [8], [28]). In addition, by Oshiro([33]), projective modules over perfect (semiperfect) rings was studied by using a lifting property.

In 1972, the result of 「Projective modules over right perfect rings have an indecomposable decomposition」 was shown by Anderson-Fuller([1]). Also, the result of 「Projective modules over right perfect rings have the exchange property」 was given by Yamagata([43]), Harada-Ishii([14]).

In spite of such situations, the following fundamental problems are unsolved as the biggest problems now in this field.

Problem (1): Does any lifting module have an indecomposable decomposition ?

Problem (2): Does any lifting module have the (finite) internal exchange property ?

Problem (3): Which ring R has the property that every lifting R-module has an indecomposable decomposition ?

Problem (4): When is a direct sum of lifting (CS-) modules necessarily lifting (CS-) module ?

Now, this paper is a summary based on the following two papers concerning these problems;

(1) Y. Kuratomi and C. Chang: *Lifting modules over right perfect rings*, Communications in Algebra, to appear.

(2) C. Chang and K. Oshiro: *Direct sums of relative (quasi-)continuous*

modules, East-West J. of Mathematics **6**(2)(2004), 125-130.

In Chapter 1, we describe known results as preliminaries. In particular, by using a lifting property, we give some characterizations of perfect rings and semiperfect rings including artinian rings. We note that these are implicitly due to Bass and are explicitly shown by Oshiro([33], cf., [45]).

Theorem A. *Let R be a ring. The following conditions are equivalent:*

(1) R is semiperfect;

(2) Every finitely generated projective right R-module is lifting.

Theorem B. *Let R be a ring. The following conditions are equivalent:*

(1) R is right perfect;

(2) Every projective right R-module is lifting.

In Chapter 2, we study Problem (1), (3) for right perfect rings and semiperfect rings. And we introduce " dual relative ojectivity " and give a sufficient condition for lifting modules over right perfect rings to satisfy the exchange property. This is a result for Problem (2). The following results are shown:

Theorem C. *Let R be a right perfect (semiperfect) ring and let M be a (finitely generated) lifting module. Then M has an indecomposable decomposition.*

Theorem C is a dual of the result of Okado mentioned above. In addition, it expands the result of 「Projective modules over right perfect rings have an indecomposable decomposition」 due to Anderson-Fuller([1]). In Anderson-Fuller([1]), they also showed that projective modules over semiperfect rings have an indecomposable decomposition. Therefore, it is a problem whether Theorem C holds or not on semiperfect rings. We remain this as an open problem.

By the way, the converse of Theorem C does not true, that is, the condition " every lifting R-module has an indecomposable decomposition." does not characterize R to be a right perfect ring. In fact, there is a non-right perfect ring for which every lifting module is semisimple.

Theorem D. *Let R be a right perfect ring and let M be a lifting module. If M is dual M-ojective, then M has the exchange property.*

The result of 「Projective modules over right perfect rings have the exchange property」 was shown by Yamagata, Harada-Ishii as stated above. We notice that Theorem D implies the above result of Yamagata, Harada-Ishii as a corollary. However, there is a non-right perfect ring for which every projective module has the exchange property(see, Kutami-Oshiro([24])), that is, the converse of the result of Yamagata and Harada-Ishii is not true.

In Chapter 3, we give a characterization for a direct sum of relative (quasi-)continuous modules to be relative (quasi-)continuous modules. This is a result about Problem (4).

Theorem E. *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then the following are equivalent:*

(1) $P = \sum \oplus_{i\in I} M_i$ is N-(quasi-)continuous;

(2) (a) Each M_i is N-(quasi-)continuous.

(b) $\sum \oplus_{j\in I-\{i\}} M_j$ is A_i-injective, for any $i \in I$ and any $A_i \in \mathcal{A}(N, M_i)$;

(3) (a) Each M_i is N-(quasi-)continuous.

(b) For any distinct $i, j \in I$ and $A_i \in \mathcal{A}(N, M_i)$, M_j is A_i-injective.

(c) For any $i \in I$ and $A_i \in \mathcal{A}(N, M_i)$, the condition (B) holds for $(A_i, \sum \oplus_{j\in I-\{i\}} M_j)$.

A sufficient condition for a direct sum of (quasi-)continuous modules to be (quasi-)continuous modules was given by Mohamed-Müller([28]). Theorem E generalizes the above result of Mohamed-Müller. As a dual of the result mentioned above, relative (quasi-)discrete modules was recently studied by Orhan-Keskin([32]).

序 論

環とは＋、－、×の演算を持った数学的対象を抽象化した代数であり、加群とはその上のベクトル空間のことである。環及び加群の研究に重要な概念であり、非常に良い性質を持った加群として、射影加群と入射加群がある。これらの加群は、ホモロジー代数的に見れば双対の関係にある。

1953年、Eckmann-Schopf([9])によって、任意の加群は入射加群の中に稠密に埋蔵されるという移入包絡(injective hull)の存在定理が示された。一方、1960年、Bass([5])により、移入包絡の双対として射影被覆(projective cover)が考察された。移入包絡と射影被覆は、存在すれば同型の違いを除いて一意的に定まる。移入包絡は常に存在するが、射影被覆は移入包絡と異なり常に存在するとは限らない。そこでBassは射影被覆を考察し、すべての有限生成加群が射影被覆を持つ準完全環と、すべての加群が射影被覆を持つ完全環を研究した。これらの環は、環論及び加群論の研究において重要な役割をはたす環となった。

入射加群は「任意の部分加群は直和因子にessentialにextendされる」という性質を持つ。この性質を持つ加群をextending加群、或いはCS-加群という。その双対として、supplemented射影加群は「任意の部分加群は直和因子にco-essentialにliftされる」という性質を持つ。この性質を持つ加群がlifting加群である。これらの性質はそれぞれ歴史的にはUtumi([39])と上述のBassの論文で考察されているとみてよい。実際、Utumiの右連続環Rは、von Neumann regular ringで、右R-加群としてextending加群である環のことである。一方、Bassの完全環とは、すべての射影加群がliftingになる環を意味する。一般に、任意の入射加群はextending加群であるが、その双対な性質としての、任意の射影加群はlifting加群であることは成立しない。1980年の始めから、これらの加群は、Harada、Oshiro、Mohamed、Müller、Smith、Wisbauer等多くの環論研究者によって活発に研究されている。この状況は次の本が出版されていることからよく分かる；

1. M. Harada: *Factor categories with applications to direct decomposition of modules*, Lect. Notes Pure Appl. Math. **88**, Marcel Dekker, New York

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2. S.H. Mohamed and B.J. Müller: *Continuous and Discrete modules*, London Math. Soc. Lect. Notes **147**, Cambrige Univ. Press (1990).

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Utumi の連続環はその後連続加群、準連続加群へと一般化された。更に、連続加群と準連続加群が新しい比較概念を用いて、比較連続加群と比較準連続加群へと一般化された ([25])。最近、Keskin-Harmanci([21]) によって、比較 (準) 連続加群の双対として比較 (準) 離散加群が考察されている。

比較 (準) 連続加群と比較 (準) 離散加群についての背景、諸結果については次の論文を参照することができる。

(1) K. Oshiro: *Continuous modules and quasi-continuous modules*, Osaka J. Math. **20**(1983), 681-694.

(2) S.R. López-Permouth, K. Oshiro and S.T. Rizvi: *On the relative (quasi-)continuity of modules*, Comm. Algebra **26**(1998), 3497-3510.

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1958 年から 1959 年にかけて、Matlis([27]), Papp([38]) によって、右 Noether 環上の入射加群が研究され、右ネーター環が「すべての入射加群は直既約分解を持つ」ことで特徴づけられた。

この結果の拡張として、1982 年に Müller-Rizvi([30]) により、右ネーター環であることと,「すべての連続加群は直既約分解を持つ」ことは同値であることが示された。

更に、1984 年に Okado([31]) によって、右ネーター環であることと、「すべての extending 加群は直既約分解を持つ」ことは同値であるという結果が示された。

又、“入射加群 (連続加群, 準連続加群, CS 加群) の直和はいつ入射加群 (連続加群, 準連続加群, CS 加群) になるか？”という問題に関しても、多くの研究者により研究された ([8], [28] を参照)。

1972 年には、Anderson-Fuller([1]) により、「右完全環上の射影加群は直既約分解を持つ」という結果が示されている。そして、Yamagata([43])、Harada-Ishii([14]) によって、「右完全環上の射影加群は exchange property を満たす」という結果が与えられている。

このような状況の下で、現在、この分野で最も大きな課題は、次の基本的な問題を解決することである。

Problem (1): lifting 加群は直既約分解を持つか？

Problem (2): lifting 加群は (finite) internal exchange property を満たすか？

Problem (3): すべての lifting 加群が直既約分解を持つ環は何か？

Problem (4): lifting(CS-) 加群の直和はいつ lifting(CS-) 加群になるか？

本論文は、これらの問題に関する次の二つの論文をもとにしてまとめたものである；

(1) Y. Kuratomi and C. Chang: *Lifting modules over right perfect rings*, Communications in Algebra, to appear.

(2) C. Chang and K. Oshiro: *Direct sums of relative (quasi-)continuous modules*, East-West J. of Mathematics **6**(2)(2004), 125-130.

各章の内容を以下に述べる。

第一章では、準備として既知の結果を述べる。特に、Artin 環を含む完全環及び準完全環の lifting 性を用いた特徴づけを与える。これらの結果は実際には上述の Bass によるものであるが、Oshiro によって明確な形で次のように与えられた。([33], cf., [45])

Theorem A. R を環とする。 このとき、 次は同値である：

(1) R は準完全環である；

(2) 有限生成射影右 R-加群は lifting 加群である。

Theorem B. R を環とする。 このとき、 次は同値である：

(1) R は右完全環である；

(2) すべての射影右 R-加群は lifting 加群である。

第二章では、projective cover の存在が保証できる (Artin 環を含む) 完全環及び準完全環に対して上記 Problem (1), (3) に関する結果 (Theorem C) を述べる。更に、“ dual relative ojective 性 ”を導入し、右完全環上の lifting 加群が exchange property を満たすための十分条件 (Theorem D) を与える。これは上記 Problem (2) に関する結果である。

Theorem C. R を右完全環 (準完全環) とし、M を (有限生成)lifting 加群とする。このとき、M は直既約分解を持つ。

Theorem C は、先に述べた Okado の結果の双対であり、この結果は、Anderson-Fuller([1]) による「右完全環上の射影加群は直既約分解を持つ」という結果の拡張になっている。Anderson-Fuller([1]) では、準完全環上の射影加群は直既約分解を持つことが述べられている。この観点から見れば、Theorem C は準完全環上でいえるかどうかが問題となるが、現在のところ未解決である。

ところで、“ 右完全環であることと, すべての lifting 加群は直既約分解を持つこととは同値でない. ” 実際、lifting 加群が直既約分解を持つ右完全環でない環が存在することが知られている。(Example 2.2.6.)

Theorem D. R を右完全環とし、M を lifting 加群とする。このとき、M が dual M-ojective ならば、M は exchange property を満たす。

Yamagata, Harada-Ishii によって、「右完全環上の射影加群は exchange property を満たす」という結果が示されている。Theorem D を使えば、上に述べた Yamagata、Harada-Ishii の結果が簡単に証明できる。しかし、射影加群が exchange property を満たす右完全環でない環が存在することが知られており、その例は Kutami-Oshiro([24]) に紹介されている。このことは Yamagata, Harada-Ishii の結果の逆が成立しないことを示している。

第三章では、上記 Problem(4) に関連して比較連続加群 (比較準連続加群) の直和が比較連続加群 (比較準連続加群) になるための特徴づけを与える。

Theorem E. $\{M_i\}_{i\in I}$ と N を加群とし、$P = \sum \oplus_{i\in I} M_i$ とする。 このとき、 次は同値である：

(1) $P = \sum \oplus_{i\in I} M_i$ は N-(quasi-)continuous である；

(2) (i) 各 M_i が N-(quasi-)continuous である

(ii) $\sum \oplus_{j \in I-\{i\}} M_j$ が A_i-injective である (ただし、$i \in I$, $A_i \in \mathcal{A}(N, M_i)$);

(3) (i) 各 M_i が N-(quasi-)continuous である

(ii) 相異なる $i, j \in I$ と $A_i \in \mathcal{A}(N, M_i)$ に対して M_j が A_i-injective である

(iii) 各 $i \in I$, $A_i \in \mathcal{A}(N, M_i)$ と $\sum \oplus_{j \in I-\{i\}} M_j$ に対して (B) が成り立つ。

1990 年、Mohamed-Müller([28]) により、連続加群 (準連続加群) の直和が連続加群 (準連続加群) であるための十分条件が与えられた。Theorem E は、Mohamed-Müller の結果を一般化したものである。最近、上記の結果の双対的な結果として Orhan-Keskin([32]) により、比較連続加群 (比較準連続加群) の双対である比較離散加群 (比較準離散加群) が研究されている。

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Chapter 1
Preliminaries

In this chapter, we state notations, definitions and known facts for which we can refer Anderson-Fuller [2], Mohamed-Müller [32], Dung-Huynh-Smith-Wisbauer [12] and Baba-Oshiro [5]. While proofs are often provided, the reader can refer standard texts for the background details for the more common concepts.

Throughout this paper, all rings R considered are associative rings with identity and all R-modules are unital.

§1.1 Notations

The notation M_R is used to stress that M is a right R-module. Let M be a right R-module. The notation $N \leq M$ means that N is a submodule of M, and the notation $N \leq_{\oplus} M$ means that N is a direct summand of M.

For a module M and an index set A, we denote by $M^{(A)}$ the direct sum of A copies of M.

Let M be a right R-module and K a submodule of M. K is called an *essential* submodule of M (or M is an *essential* extension of K) if $K \cap L \neq 0$ for any non-zero submodule L of M. In this case we denote $K \leq_e M$. Dually, a submodule K of M is called a *small* submodule (or *superfluous* submodule) of M, abbreviated $K \ll M$, in the case when, for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

For an element $m \in M$, we denote by $(0 : m)$, the *right annihilator* of m which is the set $\{r \in R \mid mr = 0\}$. By $Z(M)$, we denote the singular submodule of M, i.e., $Z(M) = \{m \in M \mid (0 : m) \leq_e R_R\}$. For R-modules M and N, $\mathrm{Hom}_R(M, N)$ means the set of all R-homomorphisms from M to N. In particular, we put $\mathrm{Hom}_R(M, M) = \mathrm{End}_R(M)$.

§1.2 Closed submodules and extending modules

Proposition 1.1 ([17, Proposition 1.1]). *(i) Let M be a module with submodules $K \leq L \leq M$. Then $K \leq_e M$ if and only if $K \leq_e L$ and $L \leq_e M$.*

(ii) Let I be any set and let $\{M_i \mid i \in I\}$ be a family of submodules of M with $M = \oplus_{i \in I} M_i$. If $N_i \leq_e M_i$ for each $i \in I$, then $\oplus_{i \in I} N_i \leq_e M$.

Proposition 1.2 (cf., [2, Proposition 5.17]). *Let M be a module with submodules $K \leq L \leq M$.*

(i) If $L \ll M$, then $K \ll M$.

(ii) If $L \ll M$ and $f : M \to N$ is a homomorphism, then $f(L) \ll N$.

(iii) If $K \ll M$ and $L \leq_\oplus M$, then $K \ll L$.

Proof. (i) Assume $M = K + X$, $X \leq M$. Since $K \leq L$, $M = L + X$, and hence, by assumption, $M = X$. So $K \ll M$. (ii) Assume that $f(L) + N' = N$, $N' \leq N$. Put $N'' = \{m \in M \mid f(m) \in N'\} \leq M$. Then $N'' + L = M$. Since $L \ll M$, $N'' = M$. Since $L \leq M = N''$, $f(L) \leq N'$. Hence $N' = N$. Therefore $f(L) \ll N$. (iii) Suppose $L = K + Y$, $Y \leq L$. Since $L \leq_\oplus M$, $M = L \oplus {}^\exists L' = K + Y + L'$. Since $K \ll M$, $M = Y + L'$. By the modular law, $L = Y$. Hence $K \ll L$. ■

Proposition 1.3 ([2, Proposition 5.20]). *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then*

$K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.

Proof. ($\Longrightarrow$) Assume that $K_1 \oplus K_2 \ll M_1 \oplus M_2$. Let $p_i : M \to M_i$ be a projection, $i = 1, 2$. By Proposition 1.2(ii), $p_i(K_1 \oplus K_2) = K_i \ll M_i$. ($\Longleftarrow$) Suppose that $K_1 \ll M_1$ and $K_2 \ll M_2$. Consider an injection $f_i : M_i \to M$, $i = 1, 2$. By Proposition 1.2(ii), $f_i(K_i) = K_i \ll M$. Suppose $M = L + (K_1 \oplus K_2)$, $L \leq M$. Since $K_i \ll M_i$, $M = L$. Therefore $K_1 \oplus K_2 \ll M_1 \oplus M_2$. ■

A submodule N of M is said to be *closed* in M (or a *closed* submodule of M), if N has no proper essential extentions in M, that is, $N \leq_e N'$ in M implies $N = N'$.

Lemma 1.4 ([16, Theorem 2.6]). *Let M be a module and $K \le L$ be submodules of M. If K is closed in L and L is closed in M, then K is closed in M.*

For $N' \le N \le M$, N is called a *closure* of N' in M if N is closed in M and $N' \le_e N$ in M.

A module M is said to be *extending* (or *CS*) if, for any submodule A of M, there exists a direct summand A^* of M such that $A \le_e A^*$ in M.

Lemma 1.5 (cf., [39, Proposition 1.4]). *Any direct summand of an extending module M is extending.*

Proof. Let $M = M_1 \oplus M_2$ and let A_1 be a submodule of M_1. Since M is extending, there exists a direct summand A_1^* of M such that $A_1 \le_e A_1^*$ in M. Let $\pi_i : M = M_1 \oplus M_2 \to M_i$ be a projection, $i = 1, 2$. Then $A_1 \le \pi_1(A_1^*)$. Moreover, we can see from $A_1 \le_e A_1^*$ and $A_1^* \cap M_2 = 0$ that $A_1 \le_e \pi_1(A_1^*)$; whence $A_1 = \pi_1(A_1^*)$. This implies that $A_1^* = A_1 \oplus \pi_2(A_1^*)$ and hence $\pi_2(A_1^*) = 0$ and $A_1^* = \pi_1(A_1^*) = A_1$. Thus $A_1 \le_\oplus M_1$. ■

By Lemma 1.5, the following holds:

Lemma 1.6 ([32, Proposition 2.4]). *A module M is extending if and only if any closed submodule of M is a direct summand.*

Proof. Obvious. ■

A module M is said to have the *(finite) exchange property* if, for any (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for some submodules $B_i \le A_i$. A module M has the *(finite) internal exchange property* if, for any (finite) direct sum decomposition $M = \oplus_{i \in I} M_i$ and any direct summand X of M, there exist submodules $\overline{M_i} \le M_i$ such that $M = X \oplus (\oplus_{i \in I} \overline{M_i})$.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_{i \in I} M_i$. Then M is said to be an *extending module for* the decomposition $M = \oplus_{i \in I} M_i$ if, for any submodule X of M, there exist a direct summand X^* of M and direct summands M_i' of M_i $(i \in I)$ such that $M = X^* \oplus (\oplus_{i\in I} M_i')$ and $X \leq_e X^*$, that is, M is an extending module and satisfies the internal exchange property in the direct decomposition $M = \oplus_{i\in I} M_i$.

Lemma 1.7 ([42, Lemma 2.1]). *Let P be a module with a decomposition $P = \oplus_{i\in I} M_i$ such that each M_i is extending. We consider the index set I as a well ordered set: $I = \{1, 2, \cdots, \omega, \omega+1, \cdots\}$, and let X be a submodule of M. Then there exist submodules $T(i) \leq_e T(i)^* \leq_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\oplus_{i\in I} X(i) \leq_e X$ for which the following properties hold:*

(1) $X(1) = T(1) \leq_e T(1)^*$.

(2) $X(k) \leq T(k) \oplus (\oplus_{i<k} N_i)$ for all $k \in I$.

(3) $\sigma(X(k)) = T(k) \leq_e T(k)^*$, $X(k) \simeq \sigma(X(k))$ (by $\sigma\,|_{X(k)}$) for all $k \in I$, where σ is the projection: $P = \oplus_{i\in I} T(i)^* \oplus (\oplus_{i\in I} N_i) \to \oplus_{i\in I} T(i)^*$.

(4) $X \simeq \sigma(X)$ (by $\sigma\,|_X$).

A module E is *injective* if for every R-module A, any monomorphism $g : X \to A$ and any homomorphism $f : X \to E$, there exists a homomorphism $h : A \to E$ such that $hg = f$.

Let M and N be R-modules. M is called to be *N-injective* if, for any monomorphism $g : X \to N$ and homomorphism $f : X \to M$, there exists a homomorphism $h : N \to M$ such that $hg = f$. A module M is *quasi-injective* (or *self-injective*) if M is M-injective.

Proposition 1.8 ([2, pp.204-206]). *For a right R-module E, the following statements are equivalent:*

(i) E is injective;

(ii) Every homomorphism of a right ideal I of R to E can be extended to a homomorphism of R to E;

(iii) For any module M, every monomorphism $0 \to E \to M$ splits;

(iv) E has no proper essential extensions.

For a given right R-module M, there exists an injective module $E(M)$

containing M as an essential submodule. Here, $E(M)$ is called the *injective hull* of M. This existence theory is known as the Eckmann-Schopf theorem ([14]).

Proposition 1.9 ([32, Proposition 2.1]). *Any (quasi-)injective module M is extending with the following condition:*

(C_2) If a submodule X of M is isomorphic to a direct summand of M, then X is a direct summand of M.

Proposition 1.10 ([32, Proposition 2.2]). *If a module M has (C_2), then it satisfies the following condition:*

(C_3) If M_1 and M_2 are direct summand of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called *continuous* if it is extending with (C_2). M is called *quasi-continuous* if it is extending with (C_3). It is well-known from [32] that the following implications hold:

"injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ extending".

In general, the converse is not true.

Example 1.11. (1) A $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is quasi-injective, but not injective.

(2) Let F be a field with a proper subfield K. Put $Q = \Pi_{i=1}^{\infty} F_i$ ($F_i = F$) and $R = \{(f_i) \in Q \mid f_m \in K, m > n \text{ for some } n \in \mathbb{N}\}$. Then R_R is continuous, but not quasi-injective.

(3) $\mathbb{Z}_{\mathbb{Z}}$ is quasi-continuous but not continuous.

(4) A $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ is extending but not quasi-continuous.

Now, we introduce the generalized relative injectivity as follows.

Let A and B be R-modules. A is said to be *B-ojective* (or *generalized B-injective*) if, for any submodule $X \leq B$ and any homomorphism $f : X \to A$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism

$h_1 : B_1 \to A_1$ and an monomorphism $h_2 : A_2 \to B_2$, and for $x = b_1 + b_2$ and $f(x) = a_1 + a_2$ one has $a_1 = h_1(b_1)$ and $b_2 = h_2(a_2)$. (cf., [18]).

A non-zero module M is said to be *uniform* if every non-zero submodule is essential in M. We see that any uniform module is indecomposable extending (quasi-continuous).

Remark. Let A and B be indecomposable modules. Then A is B-ojective if and only if, for any homomorphism $f : X \to A$ and any monomorphism $g : X \to B$, (i) if Ker $f \neq 0$, then f can be extended to $B \to A$. (ii) if Ker $f = 0$, then either f is extended to $B \to A$ or there exists an monomorphism $h : A \to B$ such that $hf = g$. Note that in the case A is a uniform module, A is B-ojective if and only if A is almost B-injective (cf., [4]).

Proposition 1.12 ([18, Proposition 1.4], [34, Proposition 8]). *Let $A_1 \leq_\oplus A$ and $B_1 \leq_\oplus B$. Suppose B is A-ojective. Then B_1 is A_1-ojective.*

For an R-module M, the (*Jacobson*) *radical* of M is defined as the intersection of all maximal submodules of M, and denoted by Rad(M), i.e., Rad$(M) = \cap\{K \leq M \mid K$ is a maximal submodule of $M\}$. If M has no maximal submodule, we define Rad$(M) = M$.

For a ring R, we say that Rad(R_R)(= Rad$({}_RR)$) is the *Jacobson radical* of R and denote it by $J(R)$.

Proposition 1.13 ([2, Proposition 9.13]). *Let M be a right R-module. Then Rad$(M) = \Sigma\{L \leq M \mid L \ll M\}$.*

For an R-module M, the *socle* of M is defined as the sum of all simple submodules of M and is denoted by Soc(M), i.e., Soc$(M) = \Sigma\{K \leq M \mid K$ is a simple submodule of $M\}$.

A dual version of Proposition 1.13 is now given by the following characterization of the socle.

Proposition 1.14 ([2, Proposition 9.7]). *Let M be a right R-module.*

Then $Soc(M) = \cap\{L \leq M \mid L \leq_e M\}$.

Proposition 1.15 (cf., [2, Proposition 5.2]). *Let* $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ *be a short exact sequence of R-homomorphism. Then the following conditions are equivalent:*

(i) There is an R-homomorphism $k : B \to A_1$ *with* $kf = 1_{A_1}$;

(ii) There is an R-homomorphism $h : A_2 \to B$ *with* $gh = 1_{A_2}$;

(iii) Imf *is a direct summand of* B;

(iv) $Kerg$ *is a direct summand of* B.

Proof. (i) $\Longrightarrow$ (iii) Let $b \in B$. Then $b = (b - fk(b)) + fk(b)$. Since $k(b - fk(b)) = k(b) - kfk(b) = 0$, $b - fk(b) \in \text{Ker } k$. Thus $b = (b - fk(b)) + fk(b) \in \text{Ker } k + \text{Im } f$. Hence $B \subseteq \text{Ker } k + \text{Im } f$. Therefore $B = \text{Ker } k + \text{Im } f$. It is sufficient to show that $\text{Ker } k \cap \text{Im } f = 0$. Let $b = f(a_1) \in \text{Ker } k \cap \text{Im } f$, where $a_1 \in A_1$. Then $0 = k(b) = kf(a_1) = a_1$. Thus $b = f(a_1) = f(0) = 0$. Hence $\text{Im } f \oplus \text{Ker } k = B$. (iii) $\Longrightarrow$ (i) As $\text{Im } f \leq_{\oplus} B$, $B = \text{Im } f \oplus {}^{\exists}C$. Thus $b = f(a_1) + c$, where $a_1 \in A_1$, $c \in C$. Define a map $k : B \to A_1$ by $k(f(a_1) + c) = a_1$. Then k is an R-homomorphism. Moreover, $kf = 1_{A_1}$. (ii) $\Longrightarrow$ (iv) The proof of this part is similar to one of the part (i) $\Longrightarrow$ (iii). (iv) $\Longrightarrow$ (ii) Since $\text{Ker } g \leq_{\oplus} B$, there exists a direct summand $K \leq_{\oplus} B$ such that $B = \text{Ker } g \oplus K$. Since $g|_K : K \to A_2$ is an isomorphism, we put $(g|_K)^{-1} = h$. Then $gh = 1_{A_2}$. (iii) $\Longleftrightarrow$ (iv) This is trivial. ■

Proposition 1.16 ([2, Proposition 7.1]). *For the R-module* R_R*, there is a decomposition* $R_R = A_1 \oplus A_2$ *if and only if there exists an idempotent* $e \in R$ *with* $A_1 = eR$ *and* $A_2 = (1 - e)R$.

Proposition 1.17 ([2, Theorem 2.8]). *Let* M *be a finitely generated R-module and let* K *be a proper submodule of* M*. Then there exists a maximal submodule* L *of* M *such that* $K \subseteq L$.

This can be easily shown using Zorn's Lemma.

§1.3 Co-closed submodules and lifting modules

Let $N_1 \leq N_2 \leq M$. N_1 is a *co-essential* submodule of N_2 in M, abbreviated $N_1 \leq_c N_2$ in M, if the kernel of the canonical map $M/N_1 \to M/N_2 \to 0$ is small in M/N_1, or equivalently, if $M = N_2 + X$ with $N_1 \leq X$ implies $M = X$.

Proposition 1.18. *(i) Let A, B and C be submodules of M with $A \leq B \leq C$. Then $A \leq_c B$ in M and $B \leq_c C$ in M if and only if $A \leq_c C$ in M.*

(ii) Let $A \leq B \leq M$. Then $A \leq_c B$ in M if and only if $M = A + K$ for any submodule K of M with $M = B + K$.

(iii) Let $A \leq C \leq M$ and let $M = A + B$. If $C \cap B \ll M$, then $A \leq_c C$ in M. The converse is true if $A \cap B = 0$.

Proof. (i) follows from [38, Proposition 1.1] and (ii) is clear.

(iii) ($\Longrightarrow$) Let $M = C + K$. Since $C = A + (C \cap B)$ and $C \cap B \ll M$, $M = A + (C \cap B) + K = A + K$. Hence $A \leq_c C$ in M by (ii). ($\Longleftarrow$) Assume $A \cap B = 0$ and $A \leq_c C$. Then $C = A \oplus (B \cap C)$. Put $B = (B \cap C) + K$. Then $M = A \oplus B = A + (B \cap K) + K = C + K = A + K$. Thus $K = B$. Therefore $(B \cap C) \ll M$. ■

A submodule N of M is said to be *co-closed* in M (or a *co-closed* submodule of M), if N has no proper co-essential submodule in M. i.e., $N' \leq_c N$ in M implies $N = N'$. It is easy to see that any direct summand of a module M is co-closed in M.

Lemma 1.19. *Let M be a module and $K \leq L$ be submodules of M. Then the following hold:*

(i) If K is co-closed in L and L is co-closed in M, then K is co-closed in M.

(ii) If $K \ll M$ and L is co-closed in M, $K \ll L$.

Proof. (i) follows from [12, Section 5] and [16, Lemma 2.6]. (ii) follows from [16, Lemma 2.5]. ■

For $N' \leq N \leq M$, N' is called a *co-closure* of N in M if N' is a co-closed submodule of M with $N' \leq_c N$ in M. Any submodule of a module has a closure, however, co-closure does not exist in general, for example, $2\mathbb{Z}$ does not have co-closure in $\mathbb{Z}_{\mathbb{Z}}$.

A module M is said to be *lifting* if, for any submodule A, there exists a direct summand A^* of M such that $A^* \leq_c A$ in M.

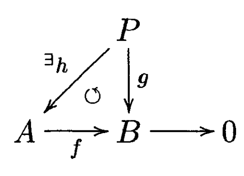
Lemma 1.20 (cf., [32] and [38]). *Any direct summand of a lifting module M is lifting.*

Proof. Let $N \leq_{\oplus} M$. Assume $X \leq N$. Then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $X \cap M_2 \ll M_2$. From $M_1 \subseteq N$, $N = M_1 \oplus (M_2 \cap N)$. Moreover, $X \cap (M_2 \cap N) = X \cap M_2 \ll M_2$. Since $M_2 \cap N \leq_{\oplus} M$, by Proposition 1.2(iii), $X \cap (M_2 \cap N) \ll M_2 \cap N$. Hence N is a lifting module. ■

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_{i \in I} M_i$. Then M is said to be a *lifting module for* the decomposition $M = \oplus_{i \in I} M_i$ if, for any submodule X of M, there exist a direct summand X^* of M and direct summands M_i' of M_i $(i \in I)$ such that $M = X^* \oplus (\oplus_{i \in I} M_i')$ and $X^* \leq_c X$, that is, M is a lifting module and satisfies the internal exchange property in the direct decomposition $M = \oplus_{i \in I} M_i$.

A module F is *free* if F has a free basis $\{b_i\}_{i \in I}$, namely, each $b_i R_R \simeq R_R$ canonically and every element $f \in F$ can be expressed uniquely in the form $f = \Sigma_{i \in I} b_i r_i$ where $r_i \in R$ and all but a finite number of the r_i are 0.

A module P is *projective* if given any epimorphism $f : A \to B$ and any homomorphism $g : P \to B$, there exists a homomorphism $h : P \to A$ such that the diagram



commutes.

Let M and N be R-modules. M is called to be N-*projective* if, for any epimorphism $f : N \to X$ and homomorphism $g : M \to X$, there exists a homomorphism $h : M \to N$ such that $fh = g$. A module M is *quasi-projective* (or *self-projective*) if M is M-projective.

Lemma 1.21 ([2, Corollary 16.11]). *Let $\{P_\alpha\}_{\alpha \in A}$ be a set of R-modules. Then $\oplus_{\alpha \in A} P_\alpha$ is projective if and only if each P_α is projective.*

We frequently use the following fact.

Proposition 1.22 (cf., [32, pp.68-69]). *(i) Let A and B be modules. If A is B-projective, then A is C-projective for any submodule C of B.*

(ii) Let A be a module and let $\{B_i \mid i = 1, \cdots, n\}$ be a family of modules. Then A is $\oplus_{i=1}^n B_i$-projective if and only if A is B_i-projective, $(i = 1, \cdots, n)$.

(iii) Let I be any set and let $\{A_i \mid i \in I\}$ be a family of modules. Then $\oplus_{i \in I} A_i$ is B-projective if and only if A_i is B-projective for all $i \in I$.

Proposition 1.23 ([2, Proposition 17.2]). *For an R-module P, the following statements are equivalent:*

(i) P is projective;

(ii) Every epimorphism $M \to P \to 0$ splits;

(iii) P is isomorphic to a direct summand of a free R-module.

Proof. (i) $\Longrightarrow$ (ii) Suppose that $f : M \to P$ is an epimorphism. If P is projective, then there is a homomorphism g such that $fg = 1_P$, so the epimorphism f splits.

(ii) $\Longrightarrow$ (iii) This follows from the fact that every module is an epimorphic image of a free module.

(iii) $\Longrightarrow$ (i) Every free module is projective. ■

The following two lemmas are due to Oshiro [38].

Lemma 1.24 (cf., [46, 41.14]). *Any projective module satisfies the following condition:*

(D) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 \ll M$ and $M = M_1 + M_2$, then $M = M_1 \oplus M_2$.

Lemma 1.25 ([38, Theorem 3.5]). *If M is a lifting module with the condition (D), then M can be expressed as a direct sum of hollow modules.*

Let M and P be R-modules. An epimorphism $g : P \to M$ is called *superfluous* if Ker $g \ll M$. A pair (P, g) is called a *projective cover* of the module M if P is projective and there exists a superfluous epimorphism $g : P \to M$. In the case when we simply say that $g : P \to M$ is a projective cover. This notion is dual to that of an injective hull. Projective covers do not exist in general. For example, $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ does not have a projective cover.

Proposition 1.26 ([32, Proposition 4.38]). *Any quasi-projective module satisfies the following condition:*

(D_2) *If $X \leq M$ such that M/X is isomorphic to a direct summand of M, then X is a direct summand of M.*

Proposition 1.27 ([32, Lemma 4.6]). *If a module M has (D_2), then it satisfies the following condition:*

(D_3) *If M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.*

A module M is called *discrete* (or *semiperfect*) if it is lifting with (D_2). A module M is called *quasi-discrete* (or *quasi-semiperfect*) if it is lifting with (D_3). It is well-known from [32] that the following implications hold:

"projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ discrete $\Rightarrow$ quasi-discrete $\Rightarrow$ lifting".

The converse implications are not true in general.

Example 1.28. (1) A $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is quasi-projective, but not projective.

(2) $\mathbb{Z}_{\mathbb{Z}}$ is quasi-projective but not discrete.

(3) Let R be a discrete variation ring with a prime ideal P. Then an injective hull $E(R/P)$ of R/P is quasi-discrete but not discrete.

(4) Put $R = \mathbb{Z}/4\mathbb{Z}$ and $Q_R = R \oplus R$. Then a submodule $M_R = (1,2)R \oplus (1,0)R$ of Q_R is lifting but not quasi-discrete.

Let M be a right R-module and N a submodule of M. We say N is a *fully invariant* submodule of M if N is a right R-, left $\mathrm{End}(M_R)$-bimodule of M.

Theorem 1.29 ([5, Theorem 1.1.24]). *For an R-module M, the following hold:*

(1) If M is a quasi-injective module, then M is a fully invariant submodule of E(M).

(2) If M is a quasi-injective module, then any direct decomposition $E(M) = E_1 \oplus \cdots \oplus E_n$ *induces* $M = (M \cap E_1) \oplus \cdots \oplus (M \cap E_n)$.

(3) If M is a quasi-projective module with a projective cover $\varphi : P \to M$, *Ker* φ *is a fully invariant submodule of P; whence any endomorphism of P induces an endomorphism of M.*

(4) If M is a quasi-projective module with a projective cover $\varphi : P \to M$, *then any direct decomposition* $P = P_1 \oplus \cdots \oplus P_n$ *induces* $M = \varphi(P_1) \oplus \cdots \oplus \varphi(P_n)$.

We note that (1), (3) can be easily verified, and (2), (4) can be proved by (1), (3), respectively.

Now we introduce the generalized relative projectivity as follows.

Let A and B be modules. A is said to be *dual B-ojective* (or *generalized B-projective*) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $gh_1 = f\,|_{A_1}$ and $fh_2 = g\,|_{B_2}$ (cf., [33]).

A non-zero module M is said to be *hollow* if every proper submodule is small in M. We see that any hollow module is indecomposable lifting (quasi-discrete).

Remark. Let A and B be indecomposable modules. Then A is dual B-ojective if and only if, for any homomorphism $f\colon A \to X$ and any epimorphism $g : B \to X$, (i) if Im $f \neq X$, then f is liftable to $A \to B$. (ii) if Im $f = X$, then either f is liftable to $A \to B$ or there exists an epimorphism $h : B \to A$ such that $fh = g$. Note that in the case A is a hollow module, A is dual B-ojective if and only if A is almost B-projective (cf., [4]).

Proposition 1.30. *Suppose A is B-projective. Then A is dual B-ojective.*

Proof. Obvious. ■

Proposition 1.31 (cf., [33]). *Let C be a direct summand of B. Suppose A is dual B-ojective. Then A is dual C-ojective.*

Proposition 1.32 ([28, Proposition 2.2]). *Let A be a module with the finite internal exchange property and let A^* be a direct summand of A.*

Suppose A is dual B-ojective. A^* is dual B-ojective.

A right R-module M is said to be *semisimple* if M can be expressed as a direct sum of simple submodules. In particular, a ring R is said to be *semisimple* if R_R is semisimple, or equivalently, ${}_RR$ is semisimple.

Proposition 1.33 ([2, Theorem 9.6]). *For an R-module M the following statements are equivalent:*

(i) M is semisimple;

(ii) Every submodule of M is a direct summand;

(iii) Every submodule of M is semisimple;

(iv) Every homomorphic image of M is semisimple.

Proposition 1.34. *Let R be a ring such that every maximal right ideal of R is a direct summand of R_R. Then R is semisimple.*

Proof. Assume that $\mathrm{Soc}(R_R) \lneq R_R$. By Proposition 1.17, there is a maximal submodule I_R such that $\mathrm{Soc}(R_R) \subseteq I_R$. By hypothesis, there exists a decomposition $R_R = I \oplus X$. Then, since X is a simple submodule of R_R, we see $X \subseteq \mathrm{Soc}(R_R) \subseteq I$, which is a contradiction. Hence $R = \mathrm{Soc}(R_R)$. ■

Proposition 1.35 ([2, Proposition 17.10]). *Let R be a ring with $J = J(R)$ and P a non-zero projective right R-module. Then Rad(P) $= PJ$.*

Proof. Proposition 1.23 allows us to assume that P is a direct summand of a free module $P \oplus P' = F = R^{(A)}$. Then $\mathrm{Rad}(P) \oplus \mathrm{Rad}(P') = \mathrm{Rad}(R^{(A)}) = (\mathrm{Rad}(R))^{(A)} = J^{(A)} = R^{(A)}J = PJ \oplus P'J$. So, since $PJ \leq \mathrm{Rad}(P)$ and $P'J \leq \mathrm{Rad}(P')$, we must have $\mathrm{Rad}(P) = PJ$. ■

Proposition 1.36 ([24, Lemma 3]). *For a projective R-module $P(\neq 0)$ over a ring R, we have $PJ \neq P$, where $J = J(R)$. That is, every non-zero projective module contains a maximal submodule.*

Proof. Let F be a free module such that $F = P \oplus Q$ and let $x \in P$. Select a basis $\{u_i\}$ of F such that the expression of x in terms of that basis has the smallest possible number of non-zero entries. Assume that $PJ = P$, i.e., $P \subseteq FJ$, $x = \Sigma_{i=1}^{n} u_i r_i, r_i \neq 0, r_i \in R$; $u_i = p_i + q_i$, $p_i \in P$, $q_i \in Q$; $p_i = \Sigma_{j=1}^{m} u_j s_{ij}, s_{ij} \in R$, $i = 1, 2, \cdots, n$. Then we have $x = \Sigma_i u_i r_i = \Sigma_i p_i r_i = \Sigma_{i,j} u_j s_{ij} r_i$. Thus we have $r_1 = \Sigma_{i=1}^{n} s_{i1} r_i$, i.e., $(1 - s_{11})r_1 = \Sigma_{i=2}^{n} s_{i1} r_i$. By assumption, $s_{11} \in J$, whence $1 - s_{11}$ is invertible in R. Put $s = 1/(1 - s_{11})$, and we have $r_1 = \Sigma_{i=2}^{n} s s_{i1} r_i$. Therefore $x = \Sigma_{i=2}^{n}(u_i + u_1 s_{i1} s) r_i$. This is a shorter expression for x, a contradiction if $x \neq 0$, since $\{u_1, u_2 + u_1 s_{21} s, \cdots, u_n + u_1 s_{n1} s, \cdots\}$ is a free basis. ■

§1.4 Characterizations of semiperfect and perfect rings

A ring R is called *semiperfect* (resp. *right perfect*) if every finitely generated right R-module (resp. every right R-module) has a projective cover.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be an R-homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called the *graph* with respect to φ. Note that $M = M_1 \oplus M_2 =$

$\langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

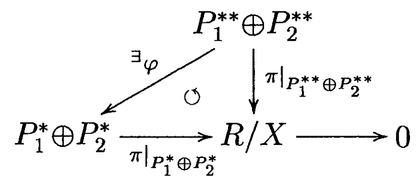
Proposition 1.37. *$P_1, \cdots, P_n$ are projective lifting R-modules if and only if $P = P_1 \oplus \cdots \oplus P_n$ is projective lifting.*

Proof. ($\Longleftarrow$) This part is a direct consequence of Lemma 1.20 and 1.21. ($\Longrightarrow$) It is enough to show that $P = P_1 \oplus P_2$ is lifting. Assume that P_1 and P_2 are projective lifting R-modules. Let $X \leq P$. For $(X + P_1) \cap P_2 \leq P_2$, since P_2 is lifting, there exists a decomposition $P_2 = P_2^* \oplus P_2^{**}$ such that $P_2^{**} \leq (X + P_1) \cap P_2$ and $[(X + P_1) \cap P_2] \cap P_2^* \ll P_2^*$. This implies that $P_2 = [(X + P_1) \cap P_2] + P_2^*$ such that $(X + P_1) \cap P_2^* \ll P_2^*$. Hence $P = X + P_1 + P_2 = X + P_1 + [(X + P_1) \cap P_2] + P_2^* = X + P_1 + P_2^*$. Similarly, for $(X + P_2^*) \cap P_1 \leq P_1$, there is a decomposition $P_1 = P_1^{**} \oplus P_1^*$ such that $P_1^{**} \leq (X + P_2^*) \cap P_1$ and $[(X + P_2^*) \cap P_1] \cap P_1^* \ll P_1^*$. This implies that $P_1 = [(X + P_2^*) \cap P_1] + P_1^*$ such that $(X + P_2^*) \cap P_1^* \ll P_1^*$. Therefore $P = X + P_1 + P_2^* = X + P_1^* + [(X + P_2) \cap P_1] + P_2^* = X + (P_1^* \oplus P_2^*)$.

Furthermore, $(P_1^* \oplus P_2^*) \cap X \leq [P_1^* \cap (X + P_2^*)] \oplus [P_2^* \cap (X + P_1^*)] \leq [P_1^* \cap (X + P_2^*)] \oplus [P_2^* \cap (X + P_1^*)]$. Since $(X + P_1) \cap P_2^* \ll P_2^*$ and $(X + P_2^*) \cap P_1^* \ll P_1^*$, by Proposition 1.2(i) and 1.3, $(P_1^* \oplus P_2^*) \cap X \ll P_1^* \oplus P_2^*$.

On the other hand, $P = X + (P_1^* \oplus P_2^*) = P_1^{**} \oplus P_2^{**} \oplus P_1^* \oplus P_2^*$.

Consider the canonical epimorphism $\pi : P \to P/X \to 0$. Then $\pi |_{P_1^* \oplus P_2^*} : P_1^* \oplus P_2^* \to P/X \to 0$ is an epimorphism and $\pi |_{P_1^{**} \oplus P_2^{**}} : P_1^{**} \oplus P_2^{**} \to P/X$ is a homomorphism. By Lemma 1.21, $P_1^{**} \oplus P_2^{**}$ is projective, hence there exists a homomorphism $\varphi : P_1^{**} \oplus P_2^{**} \to P_1^* \oplus P_2^*$ such that the diagram



commutes.

For any $y - \varphi(y) \in \langle P_1^{**} \oplus P_2^{**} \xrightarrow{\varphi} P_1^* \oplus P_2^* \rangle$, $\pi(y - \varphi(y)) = (\pi |_{P_1^{**} \oplus P_2^{**}})(y) - (\pi |_{P_1^* \oplus P_2^*})(\varphi(y)) = (\pi |_{P_1^{**} \oplus P_2^{**}})(y) - (\pi |_{P_1^{**} \oplus P_2^{**}})(y) = 0$. Hence $\langle P_1^{**} \oplus P_2^{**} \xrightarrow{\varphi} P_1^* \oplus P_2^* \rangle \in \text{Ker } \pi = X$. Thus $P = \langle P_1^{**} \oplus P_2^{**} \xrightarrow{\varphi} P_1^* \oplus P_2^* \rangle \oplus P_1^* \oplus P_2^*$.

■

Lemma 1.38. *Let e be an idempotent of a ring R. For any $s \in eRe$, $s \in J(eRe) = eJe$ if and only if $sR \ll eR$.*

Proof. ($\Longrightarrow$) Let $s \in J(eRe) = eJe$. Then $sR \subseteq eJ(R)$, so $sR \ll R$. Therefore $sR \ll eR$. ($\Longleftarrow$) Assume that $sR \ll eR$. Then $sR \ll R$ and hence $sR \subseteq J(R)$. Hence $esRe \subseteq eJ(R)e$. Therefore $s = ese \in eJ(R)e$. ■

A ring R is said to be *local* in case R has a unique maximal left (or right) ideal.

For a subset S of a ring R and $a \in R$, the left multiplication map : $S \to aS$ defined by $s \mapsto as$ is denoted by $(a)_L$. (Similarly, the right multiplication map : $S \to Sa$ defined by $s \mapsto sa$ is denoted by $(a)_R$.)

Proposition 1.39 ([2, Corollary 17.20]). *For an idempotent e of a ring R, the following conditions are equivalent:*

(i) eRe is a local ring;

(ii) $J(R)e$ is the unique maximal submodule of ${}_RRe$;

(iii) $eJ(R)$ is the unique maximal submodule of eR_R.

Proof. We may show only (i) $\Longleftrightarrow$ (iii).

(i) $\Longrightarrow$ (iii) Let K be a proper submodule of eR_R and let $eR = K + L$. Then $eR/K \simeq L/(L \cap K)$. Consider the canonical epimorphism $f : L \to L/(L \cap K)$. Since eR_R is projective, there exists a homomorphism $\rho : eR \to L$ such that Im f + Ker $f = L$. Since $\rho \in \text{End}(eR_R)$, ρ is realized by a left multiplication $(s)_L$ for some $s \in eRe$. Since Ker $f \neq L$, Im ρ is not small in L. By Lemma 1.38, $s \notin J(eRe) = eJ(R)e$. Since eRe is a local ring, s is unit. So Im $\rho = L = eR$, and hence $K \ll eR$, and $K \subseteq eJ(R)$.

(iii) $\Longrightarrow$ (i) Since $eRe \simeq \text{Hom}_R(eR, eR)$, it suffices to show that if $f, g \in \text{Hom}_R(eR, eR)$ are a non-unit, then $f + g$ is a non-unit. Let $f : eR \to eR$ be an epimorphism. Then f is an isomorphism. Thus f is not an epimorphism if and only if f is a non-unit. Moreover, f is a non-unit if and only if $f(eR) \subseteq eJ(R)$. Then $(f+g)(eR) = f(eR)+g(eR) \subseteq eJ(R)+eJ(R) \subseteq eJ(R)$. Hence

$f + g$ is a non-unit. ■

An idempotent e of R is called *primitive* if eR_R is an indecomposable module, or equivalently, if ${}_RRe$ is an indecomposable module. If $\{e_1, \cdots, e_n\}$ is a set of orthogonal primitive idempotents of R with $1 = e_1 + \cdots + e_n$, then the set is said to be a *complete set of primitive idempotents* of R.

Let I be an ideal in a ring R and let $g + I$ be an idempotent of R/I. We say that this idempotent can be *lifted (to e) modulo I* in case there is an idempotent $e \in R$ such that $g + I = e + I$. We say that *idempotents lift modulo I* in case every idempotent in R/I can be lifted to an idempotent in R.

Lemma 1.40. *Let R be a ring such that R_R is a lifting module. Then the following statements hold:*

(i) $\overline{R} = R/J(R)$ is semisimple.

(ii) If e is a primitive idempotent of R, then $eJ(R)$ is the unique maximal submodule of eR_R, i.e., eRe is a local ring.

(iii) Every complete set of orthogonal (primitive) idempotents of $\overline{R} = R/J(R)$ lifts to a complete set of orthogonal (primitive) idempotents of R.

Proof. (i) Let A be a submodule of R_R with $A \supseteq J(R)$. We put $\overline{A} = A/J(R)$ and $\overline{R} = R/J(R)$. We may show $\overline{A} \leq_{\oplus} \overline{R}$. Since R_R is lifting, there exists a decomposition $R_R = A^* \oplus A^{**}$ such that $A^* \leq A$ and $A \cap A^{**} \ll R$. Consider the canonical map $\varphi = \varphi \mid_{J(R)} : R \to R/J(R) \to 0$. Then $\overline{R} = \varphi(A) \oplus \varphi(A^{**})$. In fact, $\varphi(A) = \overline{A}$. Hence $\overline{A} \leq_{\oplus} \overline{R}$, i.e., Therefore $\overline{R}$ is semisimple.

(ii) Consider $K_R \lneq eR$. Since eR_R is indecomposable lifting, $K \ll eR$. Thus $K \subseteq eJ(R)$. Therefore $eJ(R)$ is the unique maximal submodule of eR_R.

(iii) Let $\overline{R} = \overline{g_1R} \oplus \cdots \oplus \overline{g_nR}$, where $\{\overline{g_1}, \cdots, \overline{g_n}\}$ is a complete set of orthogonal idempotents in $\overline{R}$. We consider the canonical epimorphism $R \xrightarrow{\varphi} \overline{R} \to 0$. Since R_R is lifting, there exists $R_R = A_i \oplus A_i^*$ such that $A \leq_c \varphi^{-1}(\overline{g_iR})$ $(i = 1, 2, \cdots, n)$. Then $R_R = A_1 + \cdots + A_n + \text{Ker } \varphi$. Since

Ker $\varphi \ll R_R$, $R_R = A_1 + \cdots + A_n$. Moreover, $A_j \cap \Sigma_{i \neq j} A_i \ll R_R$. By Lemma 1.25, $R_R = A_1 \oplus \cdots \oplus A_n$. Thus there exists a (necessarily) complete set $\{e_1, \cdots, e_n\}$ of pairwise orthogonal idempotents in R with $A_i = e_i R$ $(i = 1, 2, \cdots, n)$. Then $\overline{1} = \overline{e_1} + \cdots + \overline{e_n}$, $\overline{e_i} \in \overline{g_i R}$ $(i = 1, 2, \cdots, n)$. On the other hand, $\overline{1} = \overline{g_1} + \cdots + \overline{g_n}$. By the uniqueness, $\overline{e_i} = \overline{g_i}$ $(i = 1, 2, \cdots, n)$. ■

Proposition 1.41. *Let P be a projective lifting module and let $P_1, \cdots, P_n$ be indecomposable direct summands of P such that $P = P_1 + \cdots + P_n$ and $\overline{P} = \overline{P_1} \oplus \cdots \oplus \overline{P_n}$. Then $P = P_1 \oplus \cdots \oplus P_n$.*

Proof. First we show $P_1 \oplus P_2 \leq_{\oplus} P$. Since $P_1 \leq_{\oplus} P$, there exists a decomposition $P = P_1 \oplus P_1^*$. Let $\pi_{P_1} : P \to P_1$ and $\pi_{P_1^*} : P \to P_1^*$ be projections, respectively. We consider $\pi_{P_1^*}|_{P_2} : P_2 \to P_1^*$. Then $\pi_{P_1^*}(P_2)$ is not small in P_1^*. As P_1^* is lifting, there is a decomposition $P_1^* = \overline{P_1^*} \oplus \overline{\overline{P_1^*}}$ such that $\pi_{P_1^*}(P_2) \leq_c \overline{P_1^*}$. Then $\pi_{P_1^*}(P_2) = \overline{P_1^*} \oplus (\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}})$. Since $\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}} \ll P_1^* \leq P$, $\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}} \ll P$. Hence $\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}} \subseteq$ Rad(P). On the other hand, $P = P_1 + P_2 = P_1 \oplus P_1^* = P_1 \oplus \overline{P_1^*} \oplus \overline{\overline{P_1^*}}$. Let $\pi_{\overline{P_1^*}} : P \to \overline{P_1^*}$ and $\pi_{\overline{\overline{P_1^*}}} : P \to \overline{\overline{P_1^*}}$ be projections, respectively. Then $\pi_{P_1^*}(P_2) = \pi_{\overline{P_1^*}}(P_2) \oplus \pi_{\overline{\overline{P_1^*}}}(P_2)$ and $\pi_{\overline{P_1^*}}(P_2) = \overline{P_1^*}$. Since $\overline{P_1^*}$ is projective, the sequence $P_2 \xrightarrow{\pi_{\overline{P_1^*}}} \pi_{\overline{P_1^*}}(P_2) \to 0$ splits. Thus Ker $(\pi_{\overline{P_1^*}}) \leq_{\oplus} P_2$. Since P_2 is indecomposable, Ker $(\pi_{\overline{P_1^*}}) = 0$. Hence $P_2 \overset{\pi_{\overline{P_1^*}}}{\simeq} \pi_{\overline{P_1^*}}(P_2)$. Now, we define a map $\varphi : \pi_{\overline{P_1^*}}(P_2) \to P_1 \oplus \overline{\overline{P_1^*}}$ by $\pi_{\overline{P_1^*}}(p_2) \to \pi_{P_1}(p_2) + \pi_{\overline{\overline{P_1^*}}}(p_2)$. Then φ is well-defined. Since $P_2 \subseteq \langle \overline{P_1^*} \xrightarrow{\varphi} P_1 \oplus \overline{\overline{P_1^*}} \rangle$, $\langle \overline{P_1^*} \xrightarrow{\varphi} P_1 \oplus \overline{\overline{P_1^*}} \rangle = P_2 \oplus X$ for some X. Hence we get $P_1 + P_2 = P_1 \oplus P_2 \leq_{\oplus} P$. We put $P_1 \oplus P_2 = Q$.

Using the case $n - 1$, we obtain $P = P_1 + \cdots + P_n = Q \oplus P_3 \oplus \cdots \oplus P_n$. Thus, the induction works. ■

Lemma 1.42 (cf., [2, Lemma 17.17]). *Suppose that M has a projective cover. If P is projective with an epimorphism $\varphi : P \to M$, then P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq$ Ker φ and $\varphi|_{P_2} : P_2 \to M$ is a projective cover of M.*

Proof. Let $Q \xrightarrow{f} M \to 0$ be a projective cover. Then we have a homomorphism $h : P \to Q$ satisfying $fh = \varphi$. Since Ker $f \ll Q$, we see that h is an epimorphism. Since Q is projective, h splits, i.e., there exists an R-homomorphism $g : Q \to P$ such that $hg = 1_Q$, and hence $P =$ Ker $h \oplus$ Im g. Put $P_2 =$ Im g and $P_1 =$ Ker h. Then $P_1 \leq$ Ker φ since Ker $h \subseteq$ Ker $fh =$ Ker φ. Since $P_2 \simeq Q$ by $h \mid_{P_2}$ and $fg \mid_{P_2} = \varphi \mid_{P_2}$, we see that $\varphi \mid_{P_2} : P_2 \to M$ is a projective cover. ■

Proposition 1.43. *Let R be a ring such that A a right ideal of R. If $R/(A + J(R))$ has a projective cover, then so does R/A.*

Proof. Consider the canonical epimorphisms : $R \stackrel{\pi_A}{\to} R/A \stackrel{\pi_{(A+J(R))}}{\to} R/(A+J(R))$. Then, by Lemma 1.42, we can take an idempotent $e \in R$ for which $\pi_{(A+J(R))}\pi_A \mid_{eR} : eR \to R/(A + J(R))$ is a projective cover, hence Ker $(\pi_{(A+J(R))}\pi_A \mid_{eR}) \ll eR$. Since $R = eR + A + J(R)$, we obtain $R = eR + A$. Hence $\pi_A \mid_{eR} : eR \to R/A$ is an epimorphism. Since Ker $(\pi_A \mid_{eR}) \subseteq$ Ker $(\pi_{(A+J(R))}\pi_A \mid_{eR}) \ll eR$, $\pi_A \mid_{eR} : eR \to R/A$ is a projective cover. ■

Proposition 1.44. *The following statements are equivalent:*

(i) Every cyclic right R-module has a projective cover;

(ii) R_R is a lifting module.

Proof. (i) $\Longrightarrow$ (ii) Let A be a submodule of R_R and let $\varphi : R \to R/A$ be the canonical epimorphism. Since R/A has a projective cover, by Lemma 1.42, there exists a decomposition $R_R = eR \oplus (1-e)R$ such that $(\varphi \mid_{eR})$: $eR \to R/A \to 0$ a projective cover and $(1-e)R \leq A$. This implies Ker $(\varphi \mid_{eR}) = A \cap eR \ll eR$. i.e., $R = eR \oplus (1-e)R$ such that $A \cap eR \ll eR$. Thus R_R is lifting.

(ii) $\Longrightarrow$ (i) Suppose that R_R is lifting. We claim that R/A has a projective cover. Since R_R is lifting, for any $A \leq R$, there exists $A^* \leq_c A$ such that $R = A^* \oplus A^{**}$. Then $\pi \mid_{A^{**}} : A^{**} \to R/A \to 0$ is a projective cover of R/A, where $\pi : R \to R/A \to 0$ is the canonical epimorphism. ■

As corollaries of Proposition 1.44, we obtain the following two results.

Corollary 1.45. *Let P be a projective module. Then the following* *statements are equivalent:*

(i) Every factor module of P has a projective cover;

(ii) P is lifting.

Corollary 1.46. *The following statements are equivalent:*

(i) Every simple right R-module has a projective cover;

(ii) R_R satisfies the lifting property for simple factor modules.

Proposition 1.47. *Let R be a ring such that $R/J(R)$ is semisimple and every idempotent lift modulo $J(R)$. Then R_R satisfies the lifting property for simple factor modules.*

Proof. Let M be a maximal right ideal of R. By the assumption, we can take an idempotent e of R such that $\overline{eR} = \overline{M}$. Then $\overline{(1-e)R}$ is simple and $eR + J(R) = M + J(R) = M$. Hence $M = eR \oplus (M \cap (1-e)R) \subseteq eR \oplus (1-e)J(R)$. Because $\overline{(1-e)R}$ is simple, $(1-e)J(R)$ is the unique maximal submodule of $(1-e)R_R$. Hence $M \cap (1-e)R \ll (1-e)R$ as desired. ■

Proposition 1.48. *Let R be a ring such that R_R satisfies the lifting property for simple factor modules. Then R_R is a lifting module.*

In other words, if every simple right R-module has a projective cover, then every cyclic right R-module has a projective cover.

Proof. Let $A_R \leq R_R$. We show that R/A has a projective cover. By Proposition 1.43, we may assume that $J(R) \subseteq A$. By Proposition 1.17 and 1.34, $R/J(R)$ is semisimple. By Proposition 1.33, $(R/J(R))/(A/J(R)) \simeq R/A$, we see that R/A can be expressed as a direct sum of simple submodules. Since any simple right R-module has a projective cover, R/A has a projective cover. ■

An idempotent e of R is called to be *local* if eRe is a local ring.

Theorem 1.49 (cf., [2, Theorem 27.6] or [7, Theorem 2.1]). *Let R be a ring. Then the following conditions are equivalent:*

(1) R is semiperfect;

(2) $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$;

(3) $R/J(R)$ is semisimple and every complete set of orthogonal (primitive) idempotents of $R/J(R)$ lifts to a complete set of orthogonal (primitive) idempotents of R;

(4) R can be expressed as $R_R = e_1R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents of R and each e_i is a local idempotent;

(5) R can be expressed as $R_R = e_1R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a complete set of orthogonal primitive idempotents of R and each $e_iJ(R)$ is the unique maximal submodule of e_iR_R;

(6) Every cyclic right R-module has a projective cover;

(7) Every simple right R-module has a projective cover;

(8) Every finitely generated projective right R-module is a lifting module;

(9) R_R is a lifting module.

(10) R_R satisfies the lifting property for simple factor modules.

Proof. Let $J = J(R)$ be the radical of R.

(1) $\Longrightarrow$ (6) $\Longrightarrow$ (7) are obvious. (1) $\Longleftrightarrow$ (8) By Corollary 1.45, this part is clear. (6) $\Longleftrightarrow$ (9) This part is a direct consequence of Proposition 1.44. (7) $\Longleftrightarrow$ (10) This part is a direct consequence of Corollary 1.46. (8) $\Longrightarrow$ (9) $\Longrightarrow$ (10) are trivial. (10) $\Longrightarrow$ (9) By Proposition 1.48, this part is clear.

(9) $\Longrightarrow$ (8) Let M be a finitely generated projective right R-module. Now we can consider $\oplus_{i=1}^n R_i \to M \to 0$, where $R_i = R_R$. Since M is projective, this epimorphism splits, i.e., there is a direct summand $K \leq_\oplus \oplus_{i=1}^n R_i$ such that $K \simeq M$. By Proposition 1.37, $\oplus_{i=1}^n R_i$ is projective lifting. Hence M is lifting. (9) $\Longrightarrow$ (3), (4) are clear by Lemma 1.40. (2) $\Longrightarrow$ (10) This part is a direct consequence of Proposition 1.47. (3) $\Longrightarrow$ (2) is obvious. (4) $\Longleftrightarrow$ (5) This part is a direct consequence of Proposition 1.39. (5) $\Longrightarrow$ (9) By (4), $R_R = e_1R \oplus \cdots \oplus e_nR$. Since e_iJ is the unique maximal submodule of e_iR, for any proper submodule K of e_iR, $K \subseteq e_iJ \ll e_iR$. Thus $K \ll e_iR$. Hence e_iR is a lifting module. By Proposition 1.37, $R_R = e_1R \oplus \cdots \oplus e_nR$ is lifting.

■

A subset I of a ring R is *right T-nilpotent* if for every sequence $a_1, a_2, \cdots$ in I, there is a positive integer n such that $a_n a_{n-1} \cdots a_1 = 0$. (Similarly, I is *left T-nilpotent* if for any sequence $a_1, a_2, \cdots$ in I, we have $a_1 a_2 \cdots a_n = 0$ for some n.) We note that if I is left or right T-nilpotent, then it is nil because $a, a, \cdots$ is a sequence in I whenever $a \in I$.

Lemma 1.50 (cf., [2, Lemma 28.1]). *Let $F = \oplus_{n=1}^{\infty} x_n R$ be a free right R-module and put $y_n = x_n - x_{n+1}a_n$ and $G = \sum_{n=1}^{\infty} y_n R \leq F$. Then*

(i) G is free with free basis $y_1, y_2, \cdots$;

(ii) $G = F$ iff for each $k \in \mathbb{N}$, there is an $n \geq k$ such that $a_n \cdots a_k = 0$.

Proof. Let $n \geq k$ and let $c_k, \cdots, c_n \in R$. Then $y_k c_k + \cdots + y_n c_n = x_k c_k + x_{k+1}(c_{k+1} - a_k c_k) + \cdots + x_n(c_n - a_{n-1}c_{n-1}) - x_{n+1}a_n c_n$. Thus if $y_k c_k + \cdots + y_n c_n = 0$, then from the independence of the x's we have $c_k = \cdots = c_n = 0$, which implies (i). Next, suppose that $x_k \in G$, say $x_k = y_1 c_1 + \cdots + y_n c_n$. Then clearly $c_1 = \cdots = c_{k-1} = 0$. Comparing the coefficients of $x_k, \cdots, x_n$ in this equation we see that $c_k = 1$, $c_{k+1} = a_k c_k$, $c_{k+2} = a_{k+1}c_{k+1}, \cdots, c_n = a_{n-1}c_{n-1}$, $a_n c_n = 0$. So $a_n a_{n-1} \cdots a_k = 0$. This gives the necessity in (ii). For the converse, let $k \leq n$. Since for each $i \geq 1$, $x_i = y_i + x_{i+1}a_i$, we have $x_k = y_k + y_{k+1}a_k + \cdots + y_n(a_{n-1} \cdots a_k) + x_{n+1}(a_n \cdots a_k)$. So if $a_n \cdots a_k = 0$, then $x_k \in G$. ■

Lemma 1.51 (cf., [2, Lemma 28.2]). *With the hypotheses of Lemma 1.48 if G is a direct summand of F, then the chain $Ra_1 \geq Ra_1a_2 \geq \cdots$ of principal left ideals terminates.*

Proof. By Lemma 1.48, there is an isomorphism $F \to G$ via $x_n \mapsto y_n$. Suppose the inclusion map $G \to F$ is split. Then there is an endomorphism $s : F \to F$ such that $s(y_n) = x_n$ ($n \in \mathbb{N}$). For each $m \in \mathbb{N}$, we write $s(x_m) = \Sigma_k x_k c_{mk}$ as a linear combination of $x_1, x_2, \cdots$. Then $x_n = s(y_n) = s(x_n - x_{n+1}a_n) = \Sigma_k x_k(c_{nk} - c_{n+1k}a_n)$. Hence $c_{nk} - c_{n+1k}a_n = \delta_{nk}$, so $-c_{n+1n}a_n = 1 - c_{nn}$ and $c_{in}a_{i-1} = c_{i-1n}$ for each $i \leq n$ and in particular, $c_{nn}a_{n-1} \cdots a_1 = c_{1n}$.

Now for some k, $c_{1n} = 0$ for all $n \geq k$. So for each $n \geq k$, $-c_{n+1n}a_n \cdots a_1 = (1 - c_{nn})a_{n-1} \cdots a_1 = a_{n-1} \cdots a_1 - c_{nn}a_{n-1} \cdots a_1 = a_{n-1} \cdots a_1$. That is, for each $n \geq k$, $a_{n-1} \cdots a_1 \in Ra_n \cdots a_1$. ■

Lemma 1.52 ([2, Lemma 28.3]). *Let I be a right ideal of R. Then the following statements are equivalent:*

(i) I is right T-nilpotent;

(ii) $MI \neq M$ for every non-zero right R-module M;

(iii) $MI \ll M$ for every non-zero right R-module M;

(iv) $FI \ll F$ for the countably generated free module $F = R^{(\mathbb{N})}$.

Let M be a module, and let N and L be submodules of M. N is called a *supplement* of L if it is minimal with respect to the property $M = N + L$, equivalently, $M = N + L$ and $N \cap L \ll N$. Note that any supplement submodule (hence any direct summand) of a module M is co-closed in M. Following [46], A module M is *supplemented* if every submodule of M has a supplement. A module M is said to be *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$.

Propositon 1.53 ([32, Proposition A.2]). *(i) Any lifting module is amply supplemented.*

(ii) Any amply supplemented module is supplemented.

Propositon 1.54 ([33, Corollaries 1.9 and 1.14]. *(i) Every factor module of a (amply) supplemented module is (amply) supplemented.*

(ii) If A and B are supplemented modules, then $M = A + B$ is supplemented.

Now we consider the following condition:

($*$) Every submodule of M has a co-closure in M.

The following is due to Oshiro [38, Proposition 1.3].

Propositon 1.55. *Any module M over a right perfect ring satisfies* *condition* $(*)$.

Propositon 1.56 ([26, Lemma 1.7]). *A module M is amply supplemented if and only if M is supplemented with* $(*)$.

Lemma 1.57. *Let P be a projective lifting module and A be a submodule of P. Then there exists a maximal submodule A^* of P such that $A \subseteq A^*$.*

Proof. It is suffcient to show that P/A has a maximal submodule. By Corollary 1.45, P/A has a projective cover. Say $Q \xrightarrow{f} P/A \to 0$. Thus $Q/\text{Ker } f \simeq P/A$. Since Q is projective, Q has a maximal submodule L. Hence $\text{Ker } f \subseteq \text{Rad}(Q) \subseteq L$. This implies that $L/\text{Ker } f$ is a maximal submodule of $Q/\text{Ker } f$. Therefore P/A has a maximal submodule. ■

We state the following Bass's theorem which is one of fundamental facts in ring theory.

Theorem 1.58 ([2, Theorem 28.4]). *Let R be a ring. Then the following conditions are equivalent:*

(1) R is right perfect;

(2) $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent;

(3) $R/J(R)$ is semisimple and every non-zero right R-module contains a maximal submodule;

(4) Every flat right R-module is projective;

(5) R satisfies DCC on principal left ideals;

(6) R contains no infinite orthogonal set of idempotents and every non-zero left R-module contains a minimal submodule.

Moreover, we give characterizations for right perfect rings.

Theorem 1.59. *Let R be a ring. The following conditions are equivalent:*

(1) R is right perfect;

(2) Every projective right R-module is lifting;

(3) Every quasi-projective right R-module is lifting;

(4) Every countably generated free right R-module is lifting.

Proof. (1) $\Longleftrightarrow$ (2) This follows from Corollary 1.45.

(2) $\Longrightarrow$ (3) Let Q_R be a quasi-projective module and let A be a submodule of Q. Consider the canonical epimorphism $f : Q \to Q/A$. We can take a projective module P_R such that Q is a homomorphic image of P, i.e., we have an epimorphism $g : P \to Q$. Since P is a lifting module, by Lemma 1.42, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq g^{-1}(A)$, $fg\,|_{P_2} : P_2 \to Q/A$ is a projective cover. Because Q is a quasi-projective module, the decomposition $P = P_1 \oplus P_2$ induces a direct decomposition $Q = g(P_1) \oplus g(P_2)$ by Theorem 1.29. Then $g(P_1) \leq A$ and $g(P_2) \cap A \ll g(P_2)$ hold.

(3) $\Longrightarrow$ (2) Obvious.

(1) $\Longrightarrow$ (4) This follows from Theorem 1.58.

(4) $\Longrightarrow$ (1) By (4), R is semiperfect and $R/J(R)$ is semisimple. Since $R^{(\mathbb{N})}$ is lifting, there exists a decomposition $R^{(\mathbb{N})} = X \oplus Y$ such that $X \leq \mathrm{Rad}(R^{(\mathbb{N})})$ and $\mathrm{Rad}(R^{(\mathbb{N})}) \cap Y \ll Y$. Because $\mathrm{Rad}(R^{(\mathbb{N})}) = \mathrm{Rad}(X) \oplus \mathrm{Rad}(Y)$ and $X \leq \mathrm{Rad}(R^{(\mathbb{N})})$, we see $\mathrm{Rad}(X) = X$, which implies $X = 0$ and $R^{(\mathbb{N})}J(R) = \mathrm{Rad}(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}$. Hence, by Lemma 1.52, $J(R)$ is right T-nilpotent. Thus R is right perfect. ■

Chapter 2

Lifting Modules over Right Perfect Rings

Okado [36] has studied the decomposition of extending modules over right noetherian rings and, by using Oshiro's lemma (Lemma 2), he obtained the following: A ring R is right noetherian if and only if every extending R-module can be expressed as a direct sum of indecomposable (uniform) modules. As a dual problem, we consider the following: Which ring R has the property that every lifting R-module has an indecomposable decomposition? Our purpose of this paper is to study this problem. Our main results can be summarized as follows:

(1) Every (finitely generated) lifting module over a right perfect (semiperfect) ring can be expressed as a direct sum of indecomposable modules.

(2) Let R be a right perfect ring and let M be a lifting module. If every hollow summand of M has a local endomorphism ring, then M has the exchange property.

§ 2.1 Local summands

Definition. $\Sigma \oplus_{\lambda \in \Lambda} X_\lambda \leq X$ is called a *local summand* of X, if $\Sigma \oplus_{\lambda \in F} X_\lambda \leq_\oplus X$ for every finite subset $F \subseteq \Lambda$.

Lemma 2.1.1 (cf., [40]). *If every local summand of M is a direct summand, then M has an indecomposable decomposition.*

By Lemma 1.24 and [38, Proposition 3.2], the following holds:

Lemma 2.1.2. *Every local summand of projective lifting modules is a direct summand.*

A family of modules $\{M_i \mid i \in I\}$ is said to be *locally semi-T-nilpotent* if, for any subfamily M_{i_k} ($k \in \mathbb{N}$) with distinct i_k and any family of non-isomorphisms $f_k : M_{i_k} \to M_{i_{k+1}}$, and for every $x \in M_{i_1}$, there exists $n \in \mathbb{N}$ (depending on x) such that $f_n \cdots f_2 f_1(x) = 0$.

The following is essentially due to Harada [19].

Theorem 2.1.3. *Let $M = \oplus_{\alpha \in I} M_\alpha$, where each M_α has a local endomorphism ring. Then the following conditions are equivalent:*

(i) *M has the internal exchange property (in the direct sum $M = \oplus_{\alpha \in I} M_\alpha$);*

(ii) *M has the (finite) exchange property;*

(iii) *Every local summand of M is a direct summand;*

(iv) *$\{M_\alpha\}_{\alpha \in I}$ is locally semi-T-nilpotent.*

§ 2.2 Main results

In 1984, Okado [36] showed the following: A ring R is right noetherian if and only if every extending R-module can be expressed as a direct sum of indecomposable (uniform) modules. In this section, as a dual problem, we consider the following: Which ring R has the property that every lifting R-module has an indecomposable decomposition?

To consider this problem, we need some lemmas.

Lemma 2.2.1 (cf., [10, 3.2]). *Let $f : M \to N$ be an epimorphism. Suppose $K \leq_c K'$ in M. Then $f(K) \leq_c f(K')$ in N.*

Proof. Assume that $N = f(K') + L$ such that L is a submodule of N. Since f is an epimorphism, there exists a submodule T of M with $f(T) = L$. Then $M = K' + T + \text{Ker } f$. Since $K \leq_c K'$ in M, $M = K + T + \text{Ker } f$. Hence $N = f(M) = f(K) + f(T) = f(K) + L$. By Proposition 1.18(ii), $f(K) \leq_c f(K')$ in N. ■

Lemma 2.2.2. *Let M be an amply supplemented module and let $f : M \to N$ be an epimorphism with Ker $f \ll M$. If K is co-closed in M, then $f(K)$ is co-closed in N.*

Proof. Since M is amply supplemented, there exists a supplement submodule L of K in M. As $N = f(M)$ is amply supplemented, there is a co-closure T of $f(K)$ in N. Then there exists a submodule K' of K such that $f(K') = T$. This implies $N = f(M) = f(L) + f(K) = f(L) + T = f(L) + f(K')$. Thus $M = L + K' + \text{Ker } f$. By Proposition 1.18(iii), $K' \leq_c K$ in M. Since K is co-closed in M, $K = K'$. Therefore $f(K) = f(K') = T$ is co-closed in N. ■

We show the following result.

Theorem 2.2.3. *If R is a right perfect (semiperfect) ring, then every local summand of (finitely generated) lifting modules is a direct summand.*

Proof. First assume that R is a right perfect ring. Let M be a lifting module and let $\Sigma \oplus_{i \in I} X_i$ be a local summand of M. Since R is a right perfect ring, M has a projective cover, say $\text{Ker } f \ll P \xrightarrow{f} M \to 0$. By Theorem 1.58, P is projective lifting. So there exists a decomposition $P = P_i \oplus P_i^*$ $(i \in I)$ such that $P_i \leq_c f^{-1}(X_i)$ in P. By Lemma 2.2.1, $f(P_i) \leq_c f(f^{-1}(X_i)) = X_i$ in M. As X_i is co-closed in M, $f(P_i) = X_i$. First we prove that $\Sigma_{i \in I} P_i$ is direct. Let F be a finite subset of $I - \{i\}$. Since $\Sigma \oplus_{i \in I} X_i$ is a local summand of M, we see

$$f(P_i + \Sigma_{j \in F} P_j) = X_i \oplus (\Sigma \oplus_{j \in F} X_j) \leq_\oplus M.$$

So there exists a direct summand Y of M such that $M = X_i \oplus (\Sigma \oplus_{j \in F} X_j) \oplus Y$. As P is lifting, there exists a decomposition $P = Q \oplus Q^*$ such that $Q \leq_c f^{-1}(Y)$ in P. Then $f(Q) = Y$. Thus we see

$$P = P_i + \Sigma_{j \in F} P_j + Q + \text{Ker } f = P_i + \Sigma_{j \in F} P_j + Q.$$

Then $P_i \cap (\Sigma_{j \in F} P_j + Q) \subseteq \text{Ker } f \ll P$. Similarly, we see $Q \cap (P_i + \Sigma_{j \in F} P_j) \ll P$ and $P_j \cap (P_i + \Sigma_{l \in F - \{j\}} P_l + Q) \ll P$. By Lemma 1.25, we obtain $P = P_i \oplus (\Sigma_{j \in F} P_j) \oplus Q$. Hence $\Sigma_{i \in I} P_i$ is direct. By the same argument, we see $\Sigma \oplus_{i \in I} P_i$ is a local summand of P. By Lemma 2.1.2, $\Sigma \oplus_{i \in I} P_i \leq_\oplus P$. So

$f(\Sigma \oplus_{i\in I} P_i)$ is co-closed in M by Lemma 2.2.2. Since M is lifting, we see

$$\Sigma \oplus_{i\in I} X_i = f(\Sigma \oplus_{i\in I} P_i) \leq_\oplus M.$$

Thus any local summand of M is a direct summand.

Next, we assume that R is a semiperfect ring. Let M be a finitely generated lifting module and let $\text{Ker } f \ll P \xrightarrow{f} M \to 0$ be a projective cover of M. Since M is finitely generated, there exist a finitely generated projective module Q and an epimorphism $g : Q \to M$. As Q is projective, there exists a homomorphism $h : Q \to P$ such that $fh = g$. By $\text{Ker } f \ll P$, we see $P = h(Q) + \text{Ker } f = h(Q)$. Hence h is an epimorphism and so h is split. Thus P is finitely generated. By the same argument as the case of right perfect rings, we see that any local summand of P is a direct summand. ■

By Lemma 2.1.1. and Theorem 2.2.3, we obtain the first main theorem.

Theorem 2.2.4. *Every (finitely generated) lifting module over right perfect (semiperfect) rings has an indecomposable decomposition.*

By Theorem 2.2.4, "R is a right perfect ring $\Longrightarrow$ Every lifting R-module has an indecomposable decomposition." But the converse is not true, as the following shows:

Proposition 2.2.5. *Every lifting module over a commutative semi-artinian von Neumann regular ring R is semisimple.*

Proof. Let M be a lifting R-module. For $\text{Soc}(M)$, M has a decomposition $M = X \oplus Y$ with $X \leq_c \text{Soc}(M)$. Then $\text{Soc}(M) = X \oplus (Y \cap \text{Soc}(M))$ and $Y \cap \text{Soc}(M) \ll Y$. Assume $0 \neq Y \cap \text{Soc}(M)$, then we can take a non-zero simple module T in $Y \cap \text{Soc}(M)$. But R is a commutative von Neumann regular ring, as is well known, T is injective; hence $0 \neq T \leq_\oplus Y$, which contradicts to $T \ll M$. Therefore $\text{Soc}(M) = X$. As $\text{Soc}(M) \leq_e M$, we see $\text{Soc}(M) = M$. ■

Example 2.2.6 (cf., [6, Proposition 4.7] or [13, Lemma 17]). Let F be a field and $R := \sum_{i=1}^{\infty} \oplus F + F \cdot 1 = \{(f_1, \cdots, f_n, f, f, \cdots)\} \subseteq \prod_{i=1}^{\infty} F$. Then

R is a commutative von Neumann regular ring. As $\mathrm{Soc}(R) = \sum_{i=1}^{\infty} \oplus F$, it is easy to see that $\mathrm{Soc}(R) \leq_e R$. We show that R is semi-artinian, that is, every non-zero right R-module has an essential socle. To show this, it suffices to show that every cyclic right R-module has an essential socle. Consider the cyclic right R-module xR. Let $Z(xR)$ be a singular submodule of xR. First assume that xR is singular, that is, $Z(xR) = xR$. Then $xR \cdot \mathrm{Soc}(R) = 0$, hence xR is $R/\mathrm{Soc}(R)$-module. Therefore $\mathrm{Soc}(xR) = xR$. Next assume that xR is not singular. If $Z(xR) \leq_e xR$, then $\mathrm{Soc}(xR) \leq_e xR$. Now we assume $Z(xR)$ is not essetial in xR. Take a submodule T of xR such that $Z(xR) \oplus T \leq_e xR$. Then T is non-singular. Let $0 \neq t \in T$. Then tR is non-singular, as is well known, tR is embedded in the maximal quotient ring Q of R. Since $\mathrm{Soc}(R) \leq_e R$, we see that any submodule of Q_R has an essential socle; whence tR has an essential socle for any $0 \neq t \in T$. Hence we see that $\mathrm{Soc}(T) \leq_e T$, hence it follows $\mathrm{Soc}(xR) = Z(xR) \oplus \mathrm{Soc}(T) \leq_e xR$.

For this ring R, by Proposition 2.2.5, every lifting R-module has an indecomposable decomposition. But R is not right perfect.

We recall that a module H is called hollow if H is indecomposable lifting. By Theorem 2.1.3, 2.2.3 and 2.2.4, we obtain the following.

Theorem 2.2.7. *Let R be a right perfect ring and let M be a lifting module. If every hollow summand of M has a local endomorphism ring, then M has the exchange property.*

By the proof of [45, Proposition 1], we have the following.

Lemma 2.2.8. *Let H be a hollow module. If $H \oplus H$ has the internal exchange property, then H has a local endomorphism ring.*

Lemma 2.2.9 (cf., [28, Theorem 3.7]). *Let H be a hollow module. Then $H \oplus H$ is lifting with the internal exchange property if and only if H is dual H-ojective.*

By Lemma 2.2.8, 2.2.9 and Theorem 2.2.7, we have the following.

Theorem 2.2.10. *Let R be a right perfect ring and let M be a lifting module satisfying one of the following:*

(a) M has the finite exchange property,

(b) $M \oplus M$ has the finite internal exchange property,

(c) If M is dual M-ojective.

Then M has the exchange property.

The following follows from Theorem 2.2.10.

Corollary 2.2.11 (cf., [20], [47]). *Any projective module over right perfect rings has the exchange property.*

Chapter 3

Direct Sums of Relative (Quasi-) Continuous Modules

In [30], relative (quasi-)continuous modules are introduced, and several fundamental results are given. In this chapter, we shall give necessary and sufficient conditions for direct sums of relative (quasi-)continuous modules to be relative (quasi-)continuous modules.

Let N and M be R-modules. By $\mathcal{A}(N, M)$, we denote the family of all submodule A of M such that $f(X) \leq_e A$ for some $X \leq N$ and some f in $\mathrm{Hom}_R(X, M)$. It is easy to see that this family $\mathcal{A}(N, M)$ is closed under submodules, essential extensions and isomorphic images.

Definition. For $\mathcal{A}(N, M)$, we consider the following conditions :

(C_1) For any $A \in \mathcal{A}(N, M)$, there exists a direct summand $A^* \leq_\oplus M$ such that $A \leq_e A^*$

(C_2) For any $A \in \mathcal{A}(N, M)$ and $X \leq_\oplus M$, $A \simeq X$ implies $A \leq_\oplus M$

(C_3) For any $A \in \mathcal{A}(N, M)$ and $X \leq_\oplus M$, if $A \leq_\oplus M$ and $A \cap X = 0$ then $A \oplus X \leq_\oplus M$

M is said to be N-*continuous* if (C_1) and (C_2) hold, and it is said to be N-*quasi-continuous* if (C_1) and (C_3) hold. Furthermore, M is said to be N-*extending* if (C_1) holds.

We note that these modules are closed under direct summands (cf., [39]).

One easily obtains the following implications:

"M **is** N**-continuous** $\Rightarrow$ M **is** N**-quasi-continuous** $\Rightarrow$ M **is** N**-extending**".

Clearly, the notion of relative continuity generalizes the concept of continuity. On the other hand, relative injectivity does not imply relative continuity. For example, all modules are S-injective whenever S is a simple module while, on the other hand, if S is a simple module which is not injective then $M = S \oplus E(S)$ is not S-continuous since $0 \oplus S$ is isomorphic to the direct summand $S \oplus S$ of M, an element of $\mathcal{A}(S, M)$.

For R-modules $M = \sum \oplus_{i \in I} M_i$ and X, we use the following conditions :

(A) For every choice of distinct $k_i \in I$ (i$\in \mathbb{N}$) and $m_i \in M_{k_i}$, if the sequence $(0 : m_i)$ is ascending, then it becomes stationary.

(B) For every choice of $m_i \in M_{k_i}$ ($i \in \mathbb{N}$) for distinct $k_i \in I$ such that $(0 : x) \subseteq \cap_{i=1}^{\infty}(0 : m_i)$ for some $x \in X$, the ascending sequence $\cap_{i \geq n}(0 : m_i)$ (n$\in \mathbb{N}$) becomes stationary.

(C) For any $x \in X$ and for every choice of distinct $k_i \in I$ (i$\in \mathbb{N}$) and $m_i \in M_{k_i}$, with $(0 : x) \subseteq (0 : m_i)$, if the sequence $(0 : m_i)$ is ascending, then it becomes stationary.

For these conditions, the reader can refer [32]. We note that (B) implies (C).

Lemma 3.1.1 (cf., [3]). *For R-modules X and $\{M_i\}_{i \in I}$, the following are equivalent:*

(1) $\sum \oplus_{i \in I} M_i$ is X-injective;

(2) (a) each M_i is X-injective

(b) the condition (B) holds for X and $\{M_i\}_{i \in I}$.

So, in this case (C) holds.

Lemma 3.1.2 (cf., [30]). *For an N-(quasi-)continuous module M, the following hold:*

(1) Any direct summand of M is N-(quasi-)continuous.

(2) For any $X \leq_{\oplus} M$ and $A \in \mathcal{A}(N, M)$ with $X \cap A = 0$, X is A-injective.

(3) For any $A, B \in \mathcal{A}(N, M)$ with $A \cap B = 0$ if $A \leq_{\oplus} M$ and $A \simeq B$ then $B \leq_{\oplus} M$.

Lemma 3.1.3. *Consider two modules $P = (\sum \oplus_{i \in I} T_i) \oplus N$ and $Q = (\sum \oplus_{i \in I} W_i) \oplus N$ such that $Q \leq_e P$. If $\sum \oplus_{i \in I} T_i$ satisfies (A) and, for any finite subset $F \subseteq I$, if $P = (\sum \oplus_{i \in F} W_i) \oplus (\sum \oplus_{j \in I-F} T_j) \oplus N$ then $P = Q$.*

Proof. Assume that $P \neq Q$. Since $\sum \oplus_{i \in I} T_i$ satisfies (A), we can take a finite subset F of I and an element $t \in \sum \oplus_{i \in F} T_i$ such that $t \notin Q$ and, for any $j \in I - F$ and $s \in T_j$, if $(0 : t) \subsetneq (0 : s)$ then $s \in Q$.

Since $Q \leq_e P$, we can take $r \in R$ such that $0 \neq tr \in Q$. So there exists a finite subset $G \subseteq I$ such that $tr \in \sum \oplus_{i \in G} W_i \oplus N$.

We take G as $G \supseteq F$. We express t in $P = (\sum \oplus_{i \in G} W_i) \oplus (\sum \oplus_{j \in I-G} T_j) \oplus N$ as $t = w + s + n$, where $w \in \sum \oplus_{i \in G} W_i$, $s \in \sum \oplus_{j \in I-G} T_j$ and $n \in N$.

Since $\sum \oplus_{j \in I-G} T_j \ni sr = tr - wr - nr \in (\sum \oplus_{i \in G} W_i) \oplus N$, we see $sr = 0$; so $(0 : t) \subsetneq (0 : s)$. This implies $s \in Q$ and hence $t = w + s + n \in Q$, which is a contradiction. Hence $P = Q$. ■

By a slight modification, we can prove the following result by using Lemma 1.7.

Lemma 3.1.4. *Let $\{M_i\}_{i \in I}$ be a family of N-extending modules and let $A \in \mathcal{A}(N, P = \sum \oplus_{i \in I} M_i)$. Then there exist submodules $T(i) \leq_e T(i)^* \leq_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum \oplus_{i \in I} A(i) \leq_e A$ for which the following properties hold:*

(1) $A(i) \leq T(i) \oplus (\sum \oplus_{j<i} N_j)$

(2) $\sigma(A(i)) = T(i)$ and $A(i) \overset{\sigma|_{A(i)}}{\simeq} T(i)$ (by $\sigma |_{A(i)}$) for each $i \in I$, where σ is the projection : $P = (\sum \oplus_{i \in I} T(i)^*) \oplus (\sum \oplus_{i \in I} N_i) \to \sum \oplus_{i \in I} T(i)^*$.

So, $T(i), T(i)^* \in \mathcal{A}(N, M_i)$ and $A \overset{\sigma|_A}{\simeq} \sigma(A) \leq_e \sum \oplus_{i \in I} T(i)^*$.

We first show the following theorem which is a generalization of [32, Theorem 2.13].

Theorem 3.1.5. *Let $\{M_i\}_{i \in I}$ be a family of R-modules. Then the following are equivalent:*

(1) $P = \sum \oplus_{i \in I} M_i$ is N-quasi-continuous;

(2) (a) Each M_i is N-quasi-continuous.

(b) $\sum \oplus_{j \in I - \{i\}} M_j$ is A_i-injective, for any $i \in I$ and any $A_i \in \mathcal{A}(N, M_i)$;

(3) (a) Each M_i is N-quasi-continuous.

(b) For any distinct $i, j \in I$ and $A_i \in \mathcal{A}(N, M)$, M_j is A_i-injective.

(c) For any $i \in I$ and $A_i \in \mathcal{A}(N, M_i)$, the condition (B) holds for $(A_i, \sum \oplus_{j \in I - \{i\}} M_j)$.

Proof. (1)$\Longrightarrow$(2) follows from Lemma 3.1.2.

(2)$\Longleftrightarrow$(3) follows from Lemma 3.1.1.

(2)$\Longrightarrow$(1). First we show that $P = \sum \oplus_{i\in I} M_i$ is N-extending. Let $X \in \mathcal{A}(N, P)$. By Lemma 3.1.4, we have submodules $T(i) \leq_e T(i)^* \leq_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum \oplus_{i\in I} X(i) \leq_e X$ such that, for each $i \in I$,

(i) $\sigma(X(i)) = T(i)$

(ii) $X(i) \simeq T(i)$ (by $\sigma|_{X_i}$)

, where σ is the projection: $P = (\sum \oplus_{i\in I} T(i)^*) \oplus (\sum \oplus_{i\in I} N_i) \to \sum \oplus_{i\in I} T(i)^*$. So, we see

(iii) $X \overset{\sigma|X}{\simeq} \sigma(X) \leq_e \sum \oplus_{i\in I} T(i)^*$

Since $X \in \mathcal{A}(N, P)$, we see that $X(i) \in \mathcal{A}(N, M_i)$, whence $T(i)^* \in \mathcal{A}(N, M_i)$ for each $i \in I$. So, by (b), $\sum \oplus_{j\in I-\{i\}} N_j$ is $T(i)^*$-injective for each $i \in I$. On the other hand, by (a), N_i is $T(i)^*$-injective. Hence $\sum \oplus_{i\in I} N_i$ is $T(i)^*$-injective for each $i \in I$.

Now, by (iii), the mapping $\varphi : \sigma(X) \to \sum \oplus_{i\in I} N_i$ given by $\varphi(\sigma(x)) = \tau(x)$ is a homomorphism, where τ is the projection: $P = (\sum \oplus_{i\in I} T(i)^*) \oplus (\sum \oplus_{i\in I} N_i) \to \sum \oplus_{i\in I} N_i$.

Since $\sum \oplus_{i\in I} N_i$ is $\sum \oplus_{i\in I} T(i)^*$-injective, φ can be extended to a homomorphism $\varphi^* : \sum \oplus_{i\in I} T(i)^* \to \sum \oplus_{i\in I} N_i$. We put

$X^* = \{x + \varphi^*(x) \mid x \in \sum \oplus_{i\in I} T(i)^*\}$.

Then $P = X^* \oplus (\sum \oplus_{i\in I} N_i)$ and moreover, we see from $X \overset{\sigma|X}{\simeq} \sigma(X) \leq_e \sum \oplus_{i\in I} T(i)^*$ that $X \leq_e X^*$. Accordingly, P is N-extending. Here we note that if $X \leq_\oplus P$, then $X = X^*$, $X \simeq \sum \oplus_{i\in I} T(i)^*$, $P = X \oplus (\sum \oplus_{i\in I} N_i)$, and moreover $\sum \oplus_{i\in I} N_i$ is X-injective.

Next we will show that $P = \sum \oplus_{i\in I} M_i$ satisfies (C_3) for N.

Let $A \in \mathcal{A}(N, P)$ and $X \leq P$, and assume that both A and X are direct summands with $A \cap X = 0$; Put $P = X \oplus Q = Y \oplus A$ and let π_Q and π_X be the projections: $P = X \oplus Q \to Q$ and $P = X \oplus Q \to X$, respectively. Since $X \cap A = 0$, $A \simeq \pi_Q(A)$ by $\pi_Q|_A$. Since Q is N-extending and $\pi_Q(A) \in \mathcal{A}(N, P)$, there exists a direct summand $\pi_Q(A)^* \leq_\oplus Q$ such that $\pi_Q(A) \leq_e \pi_Q(A)^*$.

Since $\pi_Q(A)^* \leq_\oplus P$, as we noted above, $P = \pi_Q(A)^* \oplus (\sum \oplus_{i \in I} N_i)$ for some $N_i \leq_\oplus M_i$ and $\sum \oplus_{i \in I} N_i$ is $\pi_Q(A)^*$-injective.

Since $X \cap \pi_Q(A)^* = 0$, X is isomorphic to a submodule of $\sum \oplus_{i \in I} N_i$. Hence X is $\pi_Q(A)^*$-injective. Here consider the mapping $\varphi : \pi_Q(A) \to X$ given by $\varphi(\pi_Q(a)) = \pi_X(a)$. Then φ is a homomorphism. So φ can be extended to a homomorphism $\varphi^* : \pi_Q(A)^* \to X$.

Putting $A^* = \{q + \varphi^*(q) \mid q \in \pi_Q(A)^*\}$, we see that $X \oplus A^* \leq_\oplus P$ and $A \leq_e A^*$. Since $A \leq_\oplus P$, it follows $A = A^*$ and hence $X \oplus A \leq_\oplus P$ as required. ■

We generalize [32, Theorem 3.16] as follows:

Theorem 3.1.6. *Let $\{M_i\}_{i \in I}$ be a family of R-modules. Then the following are equivalent:*

(1) $P = \sum \oplus_{i \in I} M_i$ is N-continuous;

(2) (a) Each M_i is N-continuous.

(b) $\sum \oplus_{j \in I - \{i\}} M_j$ is A_i-injective, for any $i \in I$ and any $A_i \in \mathcal{A}(N, M_i)$;

(3) (a) Each M_i is N-continuous.

(b) For any distinct $i, j \in I$ and $A_i \in \mathcal{A}(N, M_i)$, M_j is A_i-injective.

(c) For any $i \in I$ and $A_i \in \mathcal{A}(N, M_i)$, the condition (B) holds for $(A_i, P = \sum \oplus_{j \in I - \{i\}} M_j)$.

Proof. As in the proof of Theorem 3.1.5, we may only show that $P = \sum \oplus_{i \in I} M_i$ satisfies (C_2) for $\mathcal{A} = \mathcal{A}(N, P)$. So, let $A, B \in \mathcal{A}$ such that $A \leq_\oplus P$ and $A \overset{\tau}{\simeq} B$.

By Lemma 3.1.4, there exist submodules $T(i) \leq_e T(i)^* \leq_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum \oplus_{i \in I} B(i) \leq_e B$ such that, for each $i \in I$, (i) $\sigma(B(i)) = T(i)$

(ii) $B(i) \simeq T(i)$ (*by* $\sigma\,|_{B(i)}$)

, where σ is the projection: $P = (\sum \oplus_{i \in I} T(i)^*) \oplus (\sum \oplus_{i \in I} N_i) \to \sum \oplus_{i \in I} T(i)^*$.

Put $A(i) = \tau^{-1}(B(i))$ for each $i \in I$. Since A is N-extending, for each $i \in I$, there exists direct summand $A(i)^* \leq_\oplus A$ such that $A(i) \leq_e A(i)^*$.

Fix $i_0 \in I$. By the proof of Theorem 3.1.5, there exist direct summands $K_j \leq_\oplus M_j$ such that $A(i_0)^*$ is isomorphic to $\sum \oplus_{j\in I} K_j$; say $A(i_0)^* \overset{\varphi}{\simeq} \sum \oplus_{j\in I} K_j$.

Now put $B(i_0)^* = \tau(A(i_0)^*)$. Then $B(i_0) \leq_e B(i_0)^*$ and $B(i_0)^* \overset{\sigma_{i_0}|_{B(i_0)^*}}{\simeq} \sigma_{i_0}(B(i_0)^*) \leq_e T(i_0)^*$, where σ_{i_0} is the projection: $P = (\sum \oplus_{i\in I} T(i)^*) \oplus (\sum \oplus_{i\in I} N_i) \to T(i_0)^*$.

Since $\sum \oplus_{j\in I-\{i_0\}} M_j \oplus \geq \sum \oplus_{j\in I-\{i_0\}} K_j \simeq \sigma_{i_0}\tau\varphi(\sum \oplus_{j\in I-\{i_0\}} K_j)$ (by $\sigma_{i_0}\tau\varphi \mid_{\sum \oplus_{j\in I-\{i_0\}} K_j}$) and $\sigma_{i_0}\tau\varphi(\sum \oplus_{j\in I-\{i_0\}} K_j) \subseteq T(i_0)^* \leq_\oplus M_{i_0}$, we see from (b) that $\sigma_{i_0}\tau\varphi(\sum \oplus_{j\in I-\{i_0\}} K_j) \leq_\oplus T(i_0)^*$.

On the other hand, $\sigma_{i_0}\tau\varphi(K_{i_0}) \leq_\oplus T(i_0)^*$ by (a). As a result we see that $T(i_0) \leq_e \sigma_{i_0}\tau\varphi(\sum \oplus_{i\in I} K_i) \leq_\oplus T(i_0)^*$; whence $\sigma_{i_0}\tau\varphi(\sum \oplus_{i\in I} K_i) = \sigma_{i_0}(B(i_0)^*) = T(i_0)^*$. Thus we have $P = B(i_0)^* \oplus (\sum \oplus_{i\in I-\{i_0\}} T(i)^*) \oplus (\sum \oplus_{i\in I} N_i)$.

Inductively, we see that, for any finite subset F of I, $P = (\sum \oplus_{i\in F} B(i)^*) \oplus (\sum \oplus_{i\in I-F} T(i)^*) \oplus (\sum \oplus_{i\in I} N_i)$.

Here using Lemma 3.1.3 we get $P = (\sum \oplus_{i\in I} B(i)^*) \oplus (\sum \oplus_{i\in I} N_i)$ and hence $B = \sum \oplus_{i\in I} B(i)^* \leq_\oplus P$. ■

Remark 1. In Theorem 3.1.5, $\sum \oplus_{i\in F} M_i$ is N-quasi-continuous for any finite subset F of I if and only if (a), (b) in (3) hold.

Remark 2. In Theorem 3.1.6, $\sum \oplus_{i\in F} M_i$ is N-continuous if and only if (a), (b) in (3) hold (See [30]).

Remark. Recently, D. Keskin and A. Harmanci [27] defined the family $\mathcal{B}(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in \mathrm{Hom}_R(M, X/Y), \mathrm{Ker}\ f/A \ll M/A\}$, for two R-modules M and X and they considered the following conditions:

$\mathcal{B}(M,X)$-(D_1): For any $A \in \mathcal{B}(M,X)$, there exists a direct summand $A^* \leq_\oplus M$ such that $A/A^* \ll M/A^*$

$\mathcal{B}(M,X)$-(D_2): For any $A \in \mathcal{B}(M,X)$, if $B \leq_\oplus M$, $M/A \simeq B$ implies $A \leq_\oplus M$

$\mathcal{B}(M,X)$-(D_3): For any $A \in \mathcal{B}(M,X)$ and $B \leq_\oplus M$, if $A \leq_\oplus M$ and $M = A + B$ then $A \cap B \leq_\oplus M$

They defined that M is said to be X-*discrete* if $\mathcal{B}(M,X)$-(D_1) and $\mathcal{B}(M,X)$-

(D_2) hold, and is said to be X-*quasi-discrete* if $\mathcal{B}(M,X)$-(D_1) and $\mathcal{B}(M,X)$-(D_3) hold. Furthermore, M is said to be X-*lifting* if $\mathcal{B}(M,X)$-(D_1) holds. The notions of $\mathcal{B}(M,X)$-(D_1), $\mathcal{B}(M,X)$-(D_2), and $\mathcal{B}(M,X)$-(D_3) are dual to those of (C_1), (C_2), and (C_3). Further, N. Orhan and D. Keskin [37] generalized dual ojective modules via the class $\mathcal{B}(M,X)$ and they obtained the following results: (1) Let $M = M_1 \oplus M_2$ be an X-amply supplemented module with the finite internal exchange property. Then for every direct decomposition of $M = M_i \oplus M_j$, M_i is dual $\mathcal{B}(M_j,X)$-ojective for $i \neq j$, M_1 and M_2 are X-lifting if and only if M is X-lifting.

(2) Let $M = M_1 \oplus M_2$ be an X-amply supplemented module such that M_1 and M_2 are indecomposable X-lifting modules, if M_2 is dual $\mathcal{B}(M_1,X)$-ojective and M_1 is small-dual $\mathcal{B}(M_2,X)$-ojective then M is X-lifting.

Open problem

Last in this thesis, we state the following problems with new problem;

Problem A. When is a direct sum of lifting modules lifting?

Also, Problem A relates to the following problem;

Problem B. When is an infinite direct sum of lifting modules lifting for this decomposition?

Problem C. Does any lifting module have the (finite) internal exchange property?

Problem D. Let R be a von Neumann regular ring and let M be a lifting right R-module. Is M semisimple? This question is open even for right semi-artinian von Neumann regular rings.

Problem E. When is a direct sum of relative lifting modules relative lifting?

Recently N. Orhan and D. Keskin [37] generalized dual ojective modules via the class $\mathcal{B}(M, X)$ and they gave the following results:

(1) Let $M = M_1 \oplus M_2$ be an X-amply supplemented module with the finite internal exchange property. Then for every direct decomposition of $M = M_i \oplus M_j$, M_i is dual $\mathcal{B}(M_j, X)$-ojective for $i \neq j$, M_1 and M_2 are X-lifting if and only if M is X-lifting.

(2) Let $M = M_1 \oplus M_2$ be an X-amply supplemented module such that M_1 and M_2 are indecomposable X-lifting modules, if M_2 is dual $\mathcal{B}(M_1, X)$-ojective and M_1 is small-dual $\mathcal{B}(M_2, X)$-ojective then M is X-lifting.

Problem F. Does any lifting module over a semiperfect ring have an indecomposable decomposition?

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