

Remarks on concavity of the auxiliary function appearing in quantum reliability function in classical-quantum channels

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Abstract—This is an extension of results represented in ISIT2003. Concavity of the auxiliary function which appears in the random coding exponent as the lower bound of the quantum reliability function for general quantum states is proven for $0 \leq s \leq 1$.

Keywords: Quantum reliability function, random coding exponent and quantum information theory.

I. INTRODUCTION

In quantum information theory, it is important to study the properties of the auxiliary function $E_q(\pi, s)$, which will be defined in the below, appearing in the lower bound with respect to the random coding in the reliability function for general quantum states. In classical information theory [7], the random coding exponent $E_r^c(R)$, the lower bound of the reliability function, is defined by

$$E_r^c(R) = \max_{p,s} [E_c(p, s) - sR].$$

As for the classical auxiliary function $E_c(p, s)$, it is well-known the following properties [7].

- (a) $E_c(p, 0) = 0$.
- (b) $\frac{\partial E_c(p, s)}{\partial s} \Big|_{s=0} = I(X; Y)$, where $I(X; Y)$ presents the classical mutual information.
- (c) $E_c(p, s) > 0$ ($0 < s \leq 1$).
- $E_c(p, s) < 0$ ($-1 < s \leq 0$).
- (d) $\frac{\partial E_c(p, s)}{\partial s} > 0$, ($-1 < s \leq 1$).
- (e) $\frac{\partial^2 E_c(p, s)}{\partial s^2} \leq 0$, ($-1 < s \leq 1$).

In quantum case, the corresponding properties to (a),(b),(c) and (d) have been shown in [11], [10]. Also the concavity of the auxiliary function $E_q(\pi, s)$ is shown in the case when the signal states are pure [3], and when the expurgation method is adopted [10]. However, for general signal states, the concavity of the function $E_q(\pi, s)$ which corresponds to (e) in the above has remained as an open question [11] and still unsolved conjecture [10].

II. QUANTUM RELIABILITY FUNCTION

The reliability function of classical-quantum channel is defined by

$$E(R) \equiv - \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_e(2^{nR}, n), \quad (1)$$

$$0 < R < C,$$

where C is a classical-quantum capacity, R is a transmission rate $R = \frac{\log_2 M}{n}$ (n and M represent the length and the number of the code words, respectively), $P_e(M, n)$ can be taken any minimal error probabilities of $\min_{\mathcal{W}, \mathcal{X}} \bar{P}(\mathcal{W}, \mathcal{X})$ or $\min_{\mathcal{W}, \mathcal{X}} P_{\max}(\mathcal{W}, \mathcal{X})$. These error probabilities are defined by

$$\bar{P}(\mathcal{W}, \mathcal{X}) = \frac{1}{M} \sum_{j=1}^M P_j(\mathcal{W}, \mathcal{X}),$$

$$P_{\max}(\mathcal{W}, \mathcal{X}) = \max_{1 \leq j \leq M} P_j(\mathcal{W}, \mathcal{X}),$$

where

$$P_j(\mathcal{W}, \mathcal{X}) = 1 - \text{Tr } S_{w^j} X_j$$

is the usual error probability associated with the positive operator valued measurement $\mathcal{X} = \{X_j\}$ satisfying $\sum_{j=1}^M X_j \leq I$. Here we note S_{w^j} represents the density operator corresponding to the code word w^j chosen from the code(blook) $\mathcal{W} = \{w^1, w^2, \dots, w^M\}$. For details, see [9], [11], [10]. We assume that the words in the codebook \mathcal{W} are chosen at random, independently, and with the probability distribution

$$\mathcal{P}\{w = (i_1, \dots, i_n)\} = \pi_{i_1} \cdots \pi_{i_n}$$

for each word. We shall denote expectations with respect to this probability distribution by the symbol \mathcal{E} . In [3], it was conjectured that the random coding bound is given in the following;

$$\mathcal{E} \min_X \bar{P}(\mathcal{W}, \mathcal{X})$$

$$\leq c \inf_{0 < s \leq 1} (M-1)^s \left[\text{Tr} \left(\sum_{i=1}^a \pi_i S_i^{1+s} \right)^{1+s} \right]^n. \quad (2)$$

The bound (2) holds for pure states S_i in which case $S_i^{\frac{1}{1+s}} = S_i$ and $c = 2$. For commuting S_i it reduces to the classical bound of Theorem 5.6.2 in [7] with $c = 1$. By putting $M = 2^{nR}$, it implies the lower bound for the reliability function defined in Eq.(1), when we use random coding, is given by

$$E(R) \geq E_r^q(R) \equiv \max_{\pi} \sup_{0 < s \leq 1} [E_q(\pi, s) - sR],$$

where $\pi = \{\pi_1, \pi_2, \dots, \pi_a\}$ is *a priori* probability distribution satisfying $\sum_{i=1}^a \pi_i = 1$ and

$$\begin{aligned} E_q(\pi, s) &= -\log \text{Tr} \left[\left(\sum_{i=1}^a \pi_i S_i^{\frac{1}{1+s}} \right)^{1+s} \right], \end{aligned} \quad (3)$$

where each S_i is a non-degenerate density operator which corresponds to the output state of the classical-quantum channel $i \rightarrow S_i$ from the set of the input alphabet $A = \{1, 2, \dots, a\}$ to the set of the output quantum states in the Hilbert space \mathcal{H} . For the problem stated in previous section, a sufficient condition on concavity of the auxiliary function was given in the following.

Proposition 2.1 ([6]): If the trace inequality

$$\begin{aligned} &\text{Tr} \left[A(s)^s \left\{ \sum_{j=1}^a \pi_j S_j^{\frac{1}{1+s}} \left(\log S_j^{\frac{1}{1+s}} \right)^2 \right\} \right. \\ &\quad \left. - A(s)^{-1+s} \left\{ \sum_{j=1}^a \pi_j H \left(S_j^{\frac{1}{1+s}} \right) \right\}^2 \right] \\ &\geq 0. \end{aligned} \quad (4)$$

holds for any real number s ($-1 < s \leq 1$), any density matrices S_i ($i = 1, \dots, a$) and any probability distributions $\pi = \{\pi_i\}_{i=1}^a$, under the assumption that $A(s) \equiv \sum_{i=1}^a \pi_i S_i^{\frac{1}{1+s}}$ is invertible, then the auxiliary function $E_q(\pi, s)$ defined by Eq.(2) is concave for all s ($-1 < s \leq 1$). Where $H(x) = -x \log x$ is the matrix entropy.

We note that our assumption “ $A(s)$ is invertible” is not so special condition, because $A(s)$ becomes invertible if we have one invertible S_i at least. Moreover, we have the possibility such that $A(s)$ becomes invertible even if all S_i is not invertible for all $\pi_i \neq 0$.

In [13], Yanagi, Furuichi and Kuriyama proved the concavity of $E_q(\pi, s)$ in the special case $a = 2$ with $\pi_1 = \pi_2 = \frac{1}{2}$ under the assumption that the dimension of \mathcal{H} is two by proving the trace inequality (4). And recently in [5], Fujii proved (4) in the case $a = 2$ with $\pi_1 = \pi_2 = \frac{1}{2}$ under any dimension of \mathcal{H} . In this paper we prove (4) for any a under any dimension of \mathcal{H} . Then it is shown that $E_q(\pi, \cdot)$ is concave on $[0, 1]$.

III. MAIN RESULTS

We need several results in order to state the main theorem.

Definition 3.1 ([1],[2]): Let f, g be real valued continuous functions. Then (f, g) is called a monotone (resp. antimonotone) pair of functions on the domain $D \subset \mathbb{R}$ if

$$(f(a) - f(b))(g(a) - g(b)) \geq 0 \quad (\text{resp. } \leq)$$

for any $a, b \in D$.

Proposition 3.2 ([1],[2],[5]): If (f, g) is a monotone (resp. antimonotone) pair, then

$$\text{Tr}[f(A)Xg(A)X] \leq \text{Tr}[f(A)g(A)X^2] \quad (\text{resp. } \geq)$$

for selfadjoint matrices A and X whose spectra are included in D .

Now we state the main theorem.

Theorem 3.3: Let $S_i^{\frac{1}{1+s}} = A_i$ ($i = 1, \dots, a$). Then

$$\begin{aligned} &\text{Tr} \left[\left(\sum_{k=1}^a \pi_k A_k \right)^s \sum_{i=1}^a \pi_i A_i (\log A_i)^2 \right. \\ &\quad \left. - \left(\sum_{k=1}^a \pi_k A_k \right)^{s-1} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right)^2 \right] \\ &\geq 0, \end{aligned}$$

for $s \geq 0$.

We have to need the following lemma to prove the theorem.

Lemma 3.4 ([8]): For the continuous function $f : [0, \alpha) \rightarrow \mathbb{R}$, ($0 \leq \alpha \leq \infty$), the following statements are equivalent.

- (i) f is operator convex and $f(0) \leq 0$.
- (ii) For the bounded linear operators K_i , ($i = 1, 2, \dots, n$) satisfying $\sigma(K_i) \subset [0, \alpha)$, where $\sigma(Z)$ represents the set of all spectrums of the bounded linear operators Z , and the bounded linear operators C_i , ($i = 1, 2, \dots, n$) satisfying $\sum_{i=1}^n C_i^* C_i \leq I$, we have

$$f\left(\sum_{i=1}^n C_i^* K_i C_i\right) \leq \sum_{i=1}^n C_i^* f(K_i) C_i.$$

Proof. We apply Lemma 3.4. If $\sum_{i=1}^a C_i^* C_i = I$, then

$$\sum_{i=1}^a C_i^* X_i^2 C_i \geq \left(\sum_{i=1}^a C_i^* X_i C_i \right)^2$$

holds for any Hermitian operators X_i , since $f(x) = x^2$ is operator convex on any interval. We put

$$X_i = \log A_i, \quad C_i = (\pi_i A_i)^{1/2} \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2}$$

for $i = 1, 2, \dots, a$. Since $\sum_{i=1}^a C_i^* C_i = I$, we have

$$\begin{aligned} & \sum_{i=1}^a \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} (\pi_i A_i)^{1/2} (\log A_i)^2 \\ & (\pi_i A_i)^{1/2} \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \\ \geq & \left(\sum_{i=1}^a \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} (\pi_i A_i)^{1/2} \log A_i \right. \\ & \left. (\pi_i A_i)^{1/2} \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \right)^2. \end{aligned}$$

And so we have

$$\begin{aligned} & \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \sum_{i=1}^a (\pi_i A_i)^{1/2} (\log A_i)^2 \\ & (\pi_i A_i)^{1/2} \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \\ \geq & \left(\left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \right. \\ & \left. \left(\sum_{k=1}^a \pi_k A_k \right)^{-1/2} \right)^2. \end{aligned}$$

Hence it follows that

$$\begin{aligned} & \sum_{i=1}^a (\pi_i A_i)^{1/2} (\log A_i)^2 (\pi_i A_i)^{1/2} \\ \geq & \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \left(\sum_{k=1}^a \pi_k A_k \right)^{-1} \\ & \left(\sum_{i=1}^a \pi_i A_i \log A_i \right). \end{aligned}$$

Then we have

$$\begin{aligned} & \left(\sum_{k=1}^a \pi_k A_k \right)^{s/2} \sum_{i=1}^a \pi_i A_i (\log A_i)^2 \\ & \left(\sum_{k=1}^a \pi_k A_k \right)^{s/2} \\ \geq & \left(\sum_{k=1}^a \pi_k A_k \right)^{s/2} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \\ & \left(\sum_{k=1}^a \pi_k A_k \right)^{-1} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \\ & \left(\sum_{k=1}^a \pi_k A_k \right)^{s/2}. \end{aligned}$$

Thus

$$\begin{aligned} & \text{Tr} \left[\left(\sum_{k=1}^a \pi_k A_k \right)^s \sum_{i=1}^a \pi_i A_i (\log A_i)^2 \right] \\ \geq & \text{Tr} \left[\left(\sum_{k=1}^a \pi_k A_k \right)^s \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \right. \\ & \left. \left(\sum_{k=1}^a \pi_k A_k \right)^{-1} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right) \right]. \end{aligned}$$

Since $f(x) = x^s$ ($s \geq 0$) and $g(x) = x^{-1}$, it is clear that (f, g) is an antimotone pair. By Proposition 3.2,

$$\begin{aligned} & \text{Tr} \left[\left(\sum_{k=1}^a \pi_k A_k \right)^s \sum_{i=1}^a \pi_i A_i (\log A_i)^2 \right. \\ & \left. - \left(\sum_{k=1}^a \pi_k A_k \right)^{s-1} \left(\sum_{i=1}^a \pi_i A_i \log A_i \right)^2 \right] \\ \geq & 0. \end{aligned}$$

■

We conclude that in this paper we finally solved the open problem given by [10] [11] that $E_q(\pi, \cdot)$ is concave on $[0, 1]$.

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