On some double coset decompositions of the Weyl group of the simple Lie algebra of type F_4

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Introduction. Let $\mathfrak{G}$ be a complex semisimple Lie algebra, R the root system of $\mathfrak{G}$, $\Pi = \{ \alpha_1, \cdots, \alpha_l \}$ the simple roots of $\mathfrak{G}$. The group $W = W(R)$ generated by the reflections r_{α_i} with respect to α_i ($1 \leqq i \leqq l$) is called the Weyl group of $\mathfrak{G}$. Put $r_i = r_{\alpha_i}$ and sometimes we write simply i for r_i. Let $S = \{ r_1, \cdots, r_l \}$ and the length $\ell(w)$ of $w \in W$ with respect to S is defined to be the smallest t for which $w = \tau_1 \cdots \tau_t$, where τ_i is a reflection with respect to some simple root. For a subset I of S, let W_I be the subgroup generated by I. For subsets I and J of S, let $(W_J \backslash W/W_I)_s$ be the representatives of $W_J \backslash W/ W_I$ consisting of the shortest element in each double coset. We write $(W/W_I)_s$ for $(W_\emptyset \backslash W/W_I)_s$.

In this note, we consider the case where $\mathfrak{G}$ is the simple Lie algebra of type F_4 : $\overset{\alpha_1}{\circ}\!-\!\!-\overset{\alpha_2}{\circ}\!\Longrightarrow\!\overset{\alpha_3}{\circ}\!-\!\!-\overset{\alpha_4}{\circ}$ and $S - I$ consists of one simple reflection r_i and we shall obtain the coset representatives in $(W/W_I)_s$ and $(W_I \backslash W/W_I)_s$.

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From [1] and [2], we have the following:

Lemma. *(1) For subsets I and J of S, we have* $(W_J \backslash W/W_I)_s = \{ w \in W$; (i) $\ell(xw) > \ell(w)$ *for* $\forall x \in J$, (ii) $\ell(wy) > \ell(w)$ *for* $\forall y \in I \}$.

(2) For $I \subset S$, put $\Pi_I = \{ \alpha \in \Pi \,; r_\alpha \in I \}$. Let W'_{Π_I} be the set of $w \in W$ such that $w\Pi_I > 0$. Then each $w \in W$ can be written uniquely $w = w'w''$ with $w' \in W'_{\Pi_I}$ and $w'' \in W_I$. Hence $(W/W_I)_s = W'_{\Pi_I}$.

By simple calculation, we obtain the following propositions.

Proposition 1.1. *In the case of $S - I = \{ 1 \}$, the coset representatives of $(W/W_I)_s$ consist of 24 elements that are given by the diagram as in Figure 1.*

Proposition 1.2. *In the case of $S-I = \{ 4 \}$, if we replace i by $5-i$ $(1 \leqq i \leqq 4)$ in Figure 1, we obtain the diagram which gives the coset representatives of $(W/W_I)_s$.*

Proposition 1.3. *In the case of $S - I = \{ 2 \}$, the coset representatives of $(W/W_I)_s$ consist of 96 elements that are given by the diagram as in Figure 2.*

Proposition 1.4. *In the case of $S-I = \{ 3 \}$, if we replace i by $5-i$ $(1 \leqq i \leqq 4)$ in Figure 2, we obtain the diagram which gives the coset representatives of $(W/W_I)_s$.*

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By propositions in §1, we obtain the following results.

Proposition 2.1. *In the case of $S - I = \{ 1 \}$, $(W_I \backslash W/W_I)_s$ consists the following five elements.*

(i) e (= *the identity element*) (ii) 1 (iii) 12321
(iv) 12342321 (v) 123243212342321

Proposition 2.2. *In the case of $S - I = \{ 4 \}$, $(W_I \backslash W/W_I)_s$ consists the following five elements.*

(i) e (ii) 4 (iii) 43234
(iv) 43213234 (v) 432312343213234

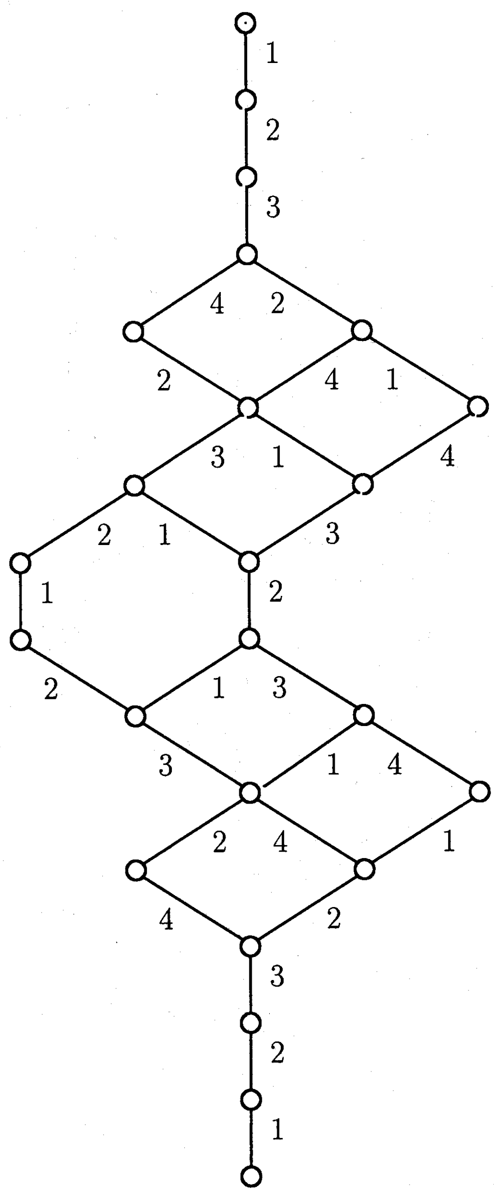

Figure 1: ●—○⇒○—○

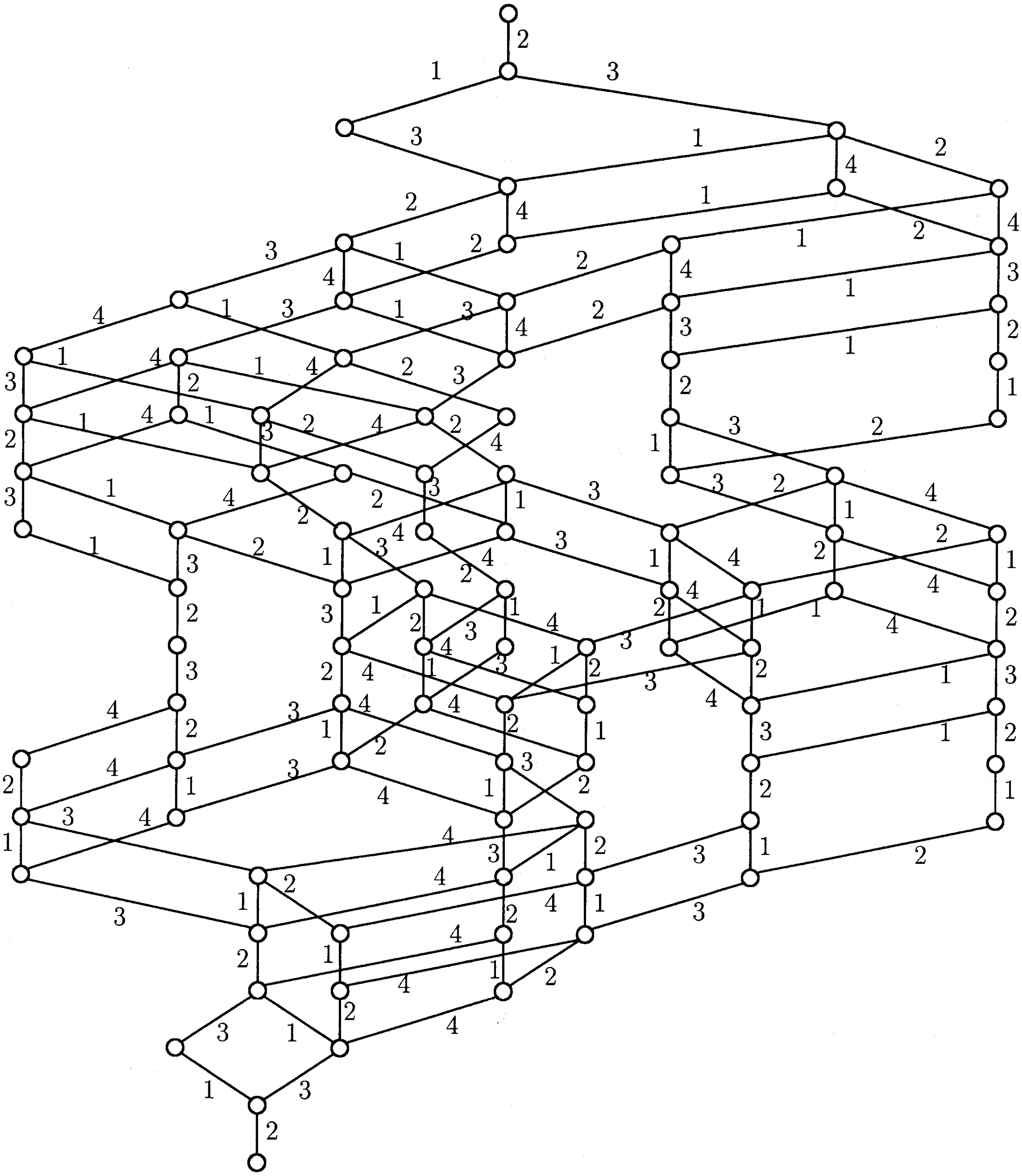

Figure 2: ○—●⟹○—○

Proposition 2.3. *In the case of* $S - I = \{ 2 \}$, $(W_I \backslash W / W_I)_s$ *consists the following 17 elements.*

(1) e	(2) 2	(3) 232
(4) 2132	(5) 234232	(6) 2321232
(7) 2324132	(8) 2314232	(9) 23214232
(10) 2321234232	(11) 2324321232	(12) 23412321432
(13) 231231432132	(14) 2324132314232	(15) 23241232314232
(16) 2324123243214232	(17) 232412342132432 14232	

Proposition 2.4. *In the case of* $S - I = \{ 3 \}$, $(W_I \backslash W / W_I)_s$ *consists the following 17 elements.*

(1) e	(2) 3	(3) 323
(4) 3423	(5) 321323	(6) 3234323
(7) 3231423	(8) 3241323	(9) 32341323
(10) 3234321323	(11) 3231234323	(12) 32143234123
(13) 324324123423	(14) 3231423241323	(15) 32314323241323
(16) 3231432312341323	(17) 323143213423 12341323	

References

[1] Bourbaki, N., Groupes et algèbres de Lie, Ch. 4, 5 et 6, 1968.

[2] Steinberg, R., Endomorphisms of linear algebraic groups, Memoirs of the AMS, 80, 1968.