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ABSTRACT

This paper presents a study on the vibration and stability of a read/write head disk system subjected to external axial and pitching oscillations and a parametric study on the optimum design of the coupled system. The solution is obtained by applying the Galerkin method with the multi-modal expansion approximation. In the analysis, the read/write head is modeled as a mass-spring system and the stiffness of the air film in the disk cover is approximated by the stiffness of the springs uniformly distributed under the disk. Both axial and pitching excitations are considered as external disturbances. Numerical results are given for the 5.25" floppy disk drive system.

INTRODUCTION

With the development of technology it has become commonly observed that the computers are installed in transport vehicles such as automobiles, ships and aircrafts. The vehicle vibration causes the excitation of the floppy disk drive unit. In some cases it leads to the signal loss in the computer and, as a result, decreases the reliability of the disk read/write process. It is therefore of technological importance to investigate the dynamic behavior of the disk system in such circumstances, in order to take suitable measures.

Many papers have been published during the past years on the vibration of rotating disks. Barasch and Chen(1) studied the transverse linear and nonlinear vibrations of circular and annular disks rotating freely about their axes with a constant angular velocity. Benson and Bogy(2) addressed the problem of steady deflection of flexible spinning disks subjected to transverse forces that are fixed in space. Hutton, Chonan and Lehmamn(3) analyzed the dynamic response characteristic of rotating disks when subjected to the effect of forces produced by stationary spring guides. As a related problem in this field, Iwan and stahl(4) and Mote(5) studied the vibration and stability of a stationary disk excited by mass-spring-damper systems moving around the disk in a circular path, with a constant angular velocity. Further Iwan and Moeller(6) and Good and Lowery(7) investigated the response of spinning disks with a stationary mass-spring-dashpot system or multiple degree-of-freedom structure, the read/write head floppy disk system.

This paper is concerned with the vibration and stability of a read/write head disk system subjected to external axial and pitching disturbances. The parametric study on the optimum design of the coupled system is also presented. The solution is obtained by applying the Galerkin method with the multi-modal expansion approximation. Numerical results are given for the 5.25" floppy disk drive system.

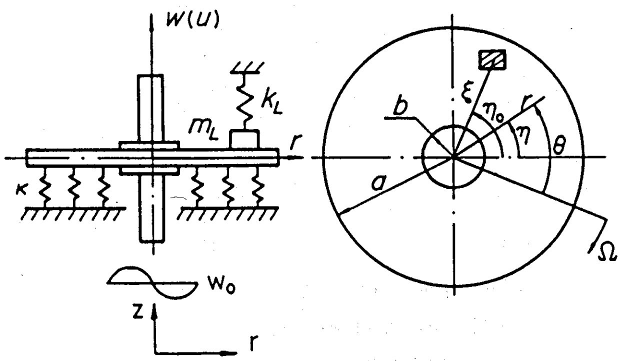

Fig. 1 Geometry of problem and co-ordinates

FORMULATION OF THE PROBLEM

Figure 1 shows a floppy disk of outer radius a and inner radius b, rotating with a constant angular speed Ω in the clock wise direction. The disk has a stationary R/W head, which is represented by a mass-spring system. The stiffness of the air film in the disk cover is approximated by the stiffness κ of the springs lying uniformly under the disk. Three coordinate frames are introduced in the following analysis, considering that the disk is spinning about the axis and at the same time it is excited by the external displacement input w_0. One denotes the frame rotating with the disk but stationary in the axial direction by (z , r , θ); the frame fixed on the rotating disk by (w , r , θ); and the frame fixed on the excited disk unit by (u , r , η). The radial coordinate r is common in three cases, while z and w are related through

$$z = w + w_0 \tag{1}$$

In the following one denotes the lateral displacement of the plate in the (z, r, θ) frame by z and the corresponding displacement in (w, r, θ) by w.

In this case the equation of motion of the plate is, with respect to the (z, r, θ) frame

$$\begin{aligned}
&\rho h(\partial^2 z/\partial t^2) r d\theta dr + k(z - w_0) r dr d\theta \\
&= (\partial/\partial r)(Q_r r d\theta) dr + (\partial/r\partial\theta)(Q_\theta dr) r d\theta \\
&+ (\partial/\partial r)(\sigma_r h r d\theta \partial w/\partial r) dr \\
&+ (\partial/r\partial\theta)(\sigma_\theta h dr \partial w/r\partial\theta) r d\theta \\
&+ (\partial/r\partial\theta)(\sigma_{r\theta} h dr \partial w/\partial r) r d\theta \\
&+ (\partial/\partial r)(\sigma_{r\theta} h r d\theta \partial w/r\partial\theta) dr \\
&+ q r dr d\theta
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
Q_r &= -D(\partial/\partial r)[\partial^2 w/\partial r^2 + (1/r)\partial w/\partial r \\
&\quad + (1/r^2)\partial^2 w/\partial\theta^2] \\
Q_\theta &= -D(\partial/r\partial\theta)[\partial^2 w/\partial r^2 + (1/r)\partial w/\partial r \\
&\quad + (1/r^2)\partial^2 w/\partial\theta^2]
\end{aligned} \tag{3}$$

Here, ρ is the mass density, h the thickness and κ the stiffness of uniform springs; q is the lateral force per unit area applied to the disk by the mass-spring system; σ_r and σ_θ are the centrifugal stresses generated by the disk rotation. By using the delta function, q is expressed as

$$\begin{aligned}
q(r,\theta,t) &= -[m_L \partial^2 z/\partial t^2 + k_L(z - w_0)] \\
&\times [\delta(\theta - \theta_0 - \Omega t + 2\hat{n}\pi)/r]\delta(r - \xi)
\end{aligned} \tag{4}$$

where m_L and k_L are the equivalent mass and stiffness of R/W head, (ξ, θ_0) the head location on the disk; $\hat{n}$ is chosen so that $2\hat{n}\pi \leq \Omega t \leq 2(\hat{n}+1)\pi$ Combining eqs.(2)-(4) and further introducing eq.(1) to transform the coordinate frame from (z, r, θ) to (w, r, θ), one has

$$\begin{aligned}
&D\nabla^4 w + \rho h \partial^2 w/\partial t^2 + \rho h \partial^2 w_0/\partial t^2 + \kappa w \\
&- h[(1/r)(\partial/\partial r)(r\sigma_r \partial w/\partial r) \\
&+ (1/r)^2(\partial/\partial\theta)(\sigma_\theta \partial w/\partial\theta)] \\
&= -(m_L \partial^2 w/\partial t^2 + m_L \partial^2 w_0/\partial t^2 + k_L w) \\
&\times [\delta(\theta - \theta_0 - \Omega t + 2\hat{n}\pi)/r]\delta(r - \xi)
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
D &= Eh^3/12(1 - \nu^2) \\
\nabla^4 &= [\partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r)^2\partial^2/\partial\theta^2]^2
\end{aligned} \tag{6}$$

Here, E is the Young's modulus and ν the Poisson's ratio.

To describe the system response with respect to the frame fixed on the disk unit, one further transforms the coordinate frame from (w, r, θ) to (u, r, η). With the use of

$$\begin{aligned}
&\theta = \eta + \Omega t - 2\hat{n}\pi \\
&w(r,\theta,t) = u(r,\eta,t) \\
&\sigma_r(r,\theta) = \sigma_r(r,\eta), \quad \sigma_\theta(r,\theta) = \sigma_\eta(r,\eta) \\
&\partial^m w/\partial r^m = \partial^m u/\partial r^m, \quad \partial^m w/\partial\theta^m = \partial^m u/\partial\eta^m \\
&\partial^m w/\partial t^m = (\partial/\partial t - \Omega\partial/\partial\eta)^m u \\
&\partial^m w_0/\partial t^m = (\partial/\partial t)^m w_0(r,\eta,t)
\end{aligned} \tag{7}$$

the equation of motion transformed is obtained as

$$\begin{aligned}
&D\nabla^4 u + \rho h(\partial^2/\partial t^2 - 2\Omega\partial^2/\partial\eta\partial t + \Omega^2\partial^2/\partial\eta^2)u \\
&+ \kappa u - h[(1/r)(\partial/\partial r)(r\sigma_r \partial u/\partial r) \\
&+ (1/r)^2(\partial/\partial\eta)(\sigma_\eta \partial u/\partial\eta)] \\
&+ [m_L(\partial^2/\partial t^2 - 2\Omega\partial^2/\partial\eta\partial t + \Omega^2\partial^2/\partial\eta^2)u \\
&+ k_L u]\delta(\eta - \eta_0)\delta(r - \xi)/r \\
&= -\rho h \partial^2 w_0/\partial t^2 - m_L \partial^2 w_0/ \\
&\partial t^2 \delta(\eta - \eta_0)\delta(r - \xi)/r
\end{aligned} \tag{8}$$

The solution of equation (8) may be written in the form

$$\begin{aligned}
u(r,\eta,t) &= \sum_{m=0}^{M}\sum_{n=0}^{N}[C_{mn}(t)\cos(n\eta) \\
&+ S_{mn}(t)\sin(n\eta)]R_{mn}(r/a)
\end{aligned} \tag{9}$$

where m and n are the number of nodal circles and diameters; M and N are the number of series terms to be considered; $C_{mn}(t)$ and $S_{mn}(t)$ are unknown functions to be determined in the following analysis; $R_{mn}(r/a)$ are space functions satisfying the boundary conditions. In the present analysis they are given by the mode functions of a non-rotating disk, i.e.,

$$\begin{aligned}
R_{mn}(r/a) &= J_n(k_{mn}r/a) + F_{mn}Y_n(k_{mn}r/a) \\
&+ G_{mn}I_n(k_{mn}r/a) + H_{mn}K_n(k_{mn}r/a)
\end{aligned} \tag{10}$$

where J_n and Y_n are Bessel functions and I_n and K_n are modified Bessel functions of the order n, F_{mn} through k_{mn} are constants determined from clamped-free boundary conditions.

Substituting eq.(9) into eq.(8), multiplying the resulted equation by $rR_{ql}(r)\cos l\eta d\eta dr$ and integrating from $r=b$ to a and from $\eta = 0$ to 2π, one has a simultaneous ordinary differential equations of C_{mn} and S_{mn} as

$$\begin{aligned}
&\sum_{m=0}^{M}\sum_{n=0}^{N}[\alpha_{mn}\partial^2 C_{mn}/\partial T^2 + \varepsilon_{mn}\partial^2 S_{mn}/\partial T^2 \\
&+ \lambda_{mn}\partial C_{mn}/\partial T + \mu_{mn}\partial S_{mn}/\partial T \\
&+ \phi_{mn}C_{mn}\tau_{mn}S_{mn}]_{ql} = f_{ql}\cos\omega_0 T \\
&q = 0, 1, 2, \cdots, M;\ l = 0, 1, 2, \cdots, N
\end{aligned} \tag{11}$$

In the same way, substituting eq.(9) into eq.(8), multiplying by $rR_{ql}(r)\sin l\eta d\eta dr$ and integrating one has

$$\begin{aligned}
&\sum_{m=0}^{M}\sum_{n=0}^{N}[\bar{\varepsilon}_{mn}\partial^2 C_{mn}/\partial T^2 + \bar{\alpha}_{mn}\partial^2 S_{mn}/\partial T^2 \\
&+ \bar{\mu}_{mn}\partial C_{mn}/\partial T + \bar{\lambda}_{mn}\partial S_{mn}/\partial T \\
&+ \bar{\tau}_{mn}C_{mn} + \bar{\phi}_{mn}S_{mn}]_{ql} = \bar{f}_{ql}\cos\omega_0 T \\
&q = 0, 1, 2, \cdots, M;\ l = 1, 2, \cdots, N
\end{aligned} \tag{12}$$

With the use of the matrix expression, equation (11) and (12) are rewritten as

$$\begin{aligned}
&\sum_{m=0}^{M}\sum_{n=0}^{N}\begin{bmatrix}\alpha_{mn} & \varepsilon_{mn}\\ \bar{\varepsilon}_{mn} & \bar{\alpha}_{mn}\end{bmatrix}\begin{Bmatrix}\ddot{C}_{mn}\\ \ddot{S}_{mn}\end{Bmatrix} + \begin{bmatrix}\lambda_{mn} & \mu_{mn}\\ \bar{\mu}_{mn} & \bar{\lambda}_{mn}\end{bmatrix}\begin{Bmatrix}\dot{C}_{mn}\\ \dot{S}_{mn}\end{Bmatrix} \\
&+ \begin{bmatrix}\phi_{mn} & \tau_{mn}\\ \bar{\tau}_{mn} & \bar{\phi}_{mn}\end{bmatrix}\begin{Bmatrix}C_{mn}\\ S_{mn}\end{Bmatrix}\Bigg|_{ql} = \begin{Bmatrix}f_{ql}\\ \bar{f}_{ql}\end{Bmatrix}\cos\omega_0 T
\end{aligned} \tag{13}$$

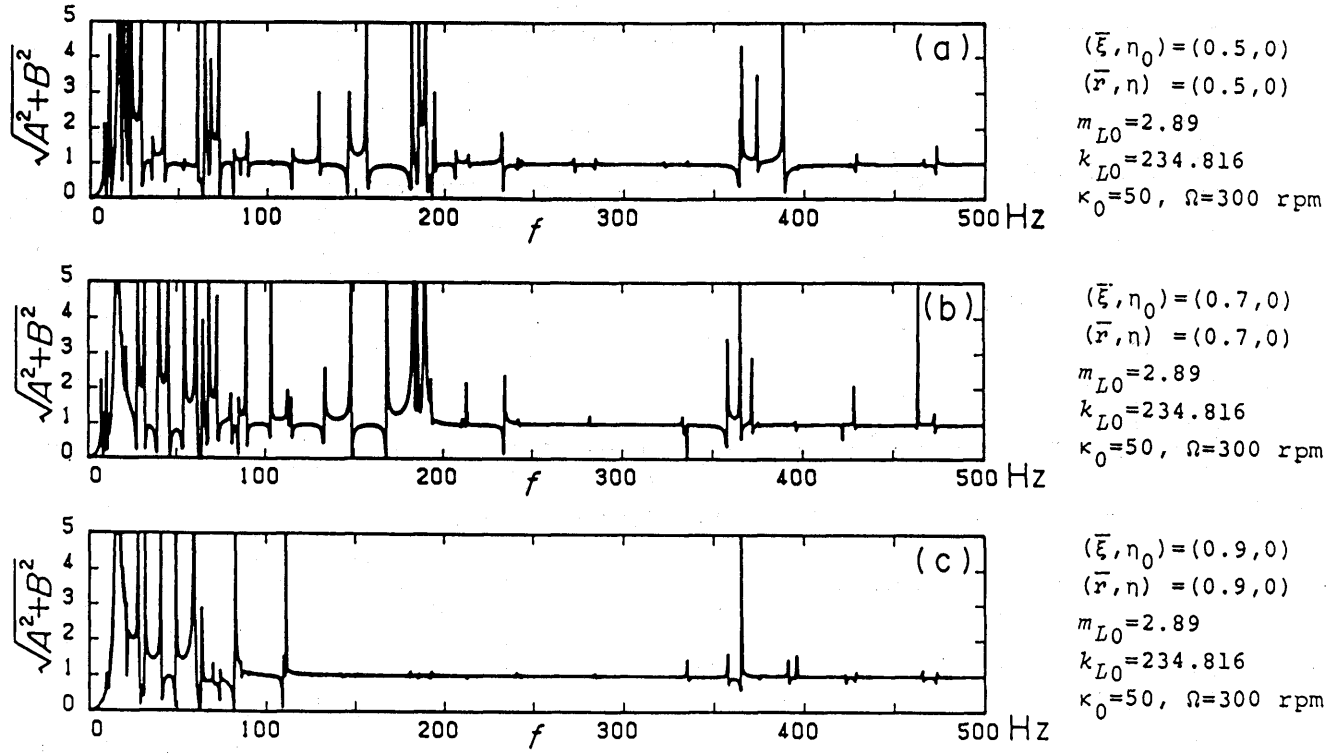

Fig. 2 Resonance curve of disk under axial excitation

RESULTS AND DISCUSSION

Numerical results that follow are for the 5.25" floppy disk of inner clamped radius b = 16.39 mm and outer free radius a = 65.09 mm. Physical parameters used are E= 5.3×10^9 N/m^2, ρ = 1.4×10^3 kg/m^3, ν =0.3 and h=0.077 mm. The R/W head, which is modeled as a mass-spring system, has equivalent parameters of m_L = 0.00415 kg and k_L = 38.58 N/m (m_{L0} = 2.89 and k_{L0} = 234.82 in nondimensional values). The results are given for the following two cases.

Disk response to external disturbances

When the disk unit is subjected to axial and pitching displacement excitation, one has w_0 as

$$w_0=A_a\cos\omega t+A_p(r/a)\cos\eta\cos\omega t \tag{14}$$

where A_a is the axial input amplitude, A_p the pitching amplitude and ω the angular frequency. In this case the right hand terms of eq.(13) become

$$f_{qi}=A_a\omega_0^2\langle rR_{qi}\rangle\delta_{i0}+A_p\omega_0^2\langle r^2R_{qi}\rangle\delta_{i1}+m_{L0}\omega_0^2(A_a+A_p\bar{\xi}\cos\eta_0)\langle CR_{qi}\rangle\beta_i$$
$$\bar{f}_{qi}=m_{L0}\omega_0^2(A_a+A_p\bar{\xi}\cos\eta_0)\langle SR_{qi}\rangle \tag{15}$$

The solution of eq.(13) is written in the form

$$[C_{mn},S_{mn}]^T=[C_{mn}^1,S_{mn}^1]^T\cos\omega_0 T+[C_{mn}^2,S_{mn}^2]^T\sin\omega_0 T \tag{16}$$

The constants C_{mn}^1, S_{mn}^1, C_{mn}^2, S_{mn}^2 are determined by substituting eq.(16) into eq.(13). Thus, the lateral displacement of disk is finally obtained in the form

$$u=A\cos\omega_0 T+B\sin\omega_0 T \tag{17}$$

where

$$A=\sum_{m=0}^{M}\sum_{n=0}^{N}(C_{mn}^1\cos n\eta+S_{mn}^1\sin n\eta)$$
$$B=\sum_{m=0}^{M}\sum_{n=0}^{N}(C_{mn}^2\cos n\eta+S_{mn}^2\sin n\eta) \tag{18}$$

Figure 2 shows the displacement amplitude $\sqrt{A^2+B^2}$ of the disk under axial excitation. Figures 2(a),(b) and (c) show the results when the R/W head are located at $(\bar{\xi},\eta_0)$ = (0.5,0), (0.7,0), (0.9,0) respectively. The excitation frequency is varied from 0 Hz to 500 Hz. It is found that the resonance frequencies of the disk are not much affected by the head location. However, as the head approaches the periphery the peaks of the resonance curve become concentrated in the low frequency range. Thus, it is said that when the head is located at $\bar{\xi}$ = 0.5,the displacement inputs with the excitation frequency 0-200 Hz should be isolated, while it should be 0-100 Hz for the case of $\bar{\xi}$ = 0.9 to make the disk unit stable.

Figure 3 shows the displacement of the disk when it is subjected to a pitching oscillation about the line $\eta = \pi/2$. Figure 3(a) is the case when the head is located at $(\bar{\xi},\eta_0)$ = (0.7,0), while Figure 3(b) the case when it is located on the nodal line of the pitching oscillation $(\bar{\xi},\eta_0)$ = (0.7, $\pi/2$). It is again observed that the resonance frequencies are not much affected by the head location. It is found however by comparing the figures that the response amplitude in Figure 3(b), on the whole, is lower than that in Figure 3(a), particularly in the high frequency range. This comes from the fact that in the latter case the head is located just on the pitching node. Thus, it is said when the system is under a

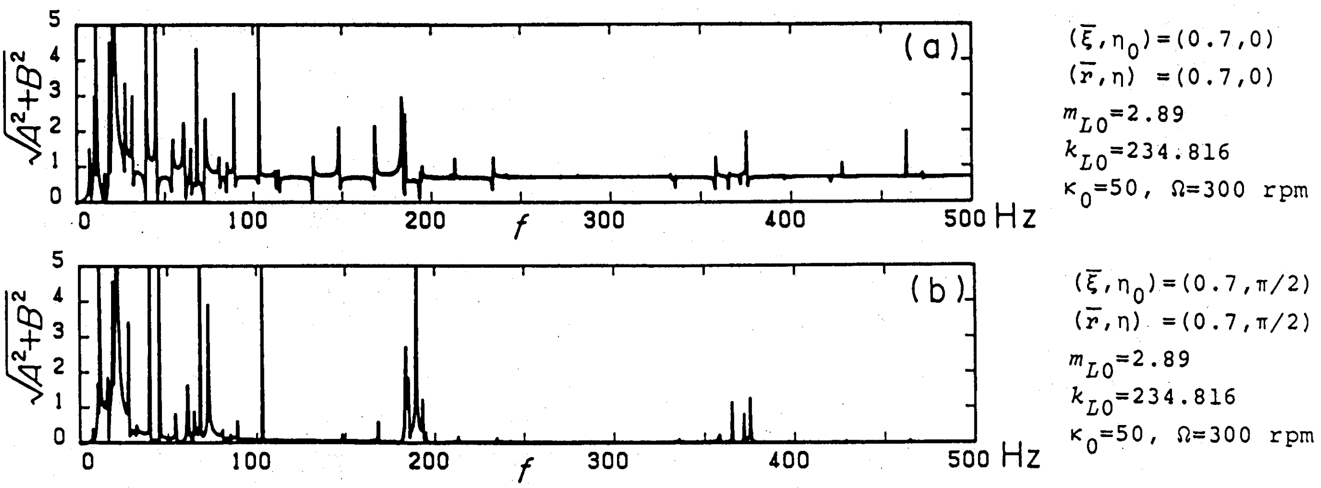

Fig. 3 Resonance curve of disk under pitching excitation

Thus, it is said when the system is under a pitching excitation the head should be located on the pitching nodal line.

Stability analysis and selection of optimum parameters

One puts the input amplitude to zero or $f_{ql} = \bar{f}_{ql} = 0$ in eq.(13). In this case the complex frequency of the system is written as

$$\alpha=\sigma+ip=f(\Omega_0,m_{L0},k_{L0},\kappa_0,\bar{\xi},\eta_0) \tag{19}$$

The disk system with no magnetic head is realized by putting $m_{L0} = 0$, $k_{L0} = 0$ and $\kappa_0=0$, the natural complex frequency of which is shown in Figure 4 by dashed lines. Figure 5 shows the frequency curves for the case of $m_{L0} = 2.89$, $k_{L0} = 234.816$, $\kappa_0= 0$, $\bar{\xi} = 0.7$ and $\eta_0= 0$. The solid lines in Figure 5(b) are just to make clear the AB and CD regions. For the case of the disk with no magnetic head (dashed lines in Figure 4) the frequency α have only imaginary parts, i.e.$\sigma = 0$, $p \neq 0$. In this case the frequency decreases monotonially with an increase of the rotation speed, and finally comes to zero, at which speed a static resonance(run-out) appears in the system when the disk is subjected to a static lateral load. The rotation speed at $p = 0$ is generally referred to as the critical speed Ω_{cr}. It is the maximum speed that can reach to operate the disk in stable. For the physical parameters under consideration, one has Ω_{cr}=5.5. When a head is attached to the disk, the frequency can have both real and imaginary parts (Figure 5). For example in the region AB one has $\sigma \neq 0$ and $p \neq 0$. In this case the disk vibrates sinusoidally with an amplitude increasing with time. The AB region is hereinafter called the flutter instability region and the speed at point A is called the flutter critical speed Ω_{fcr}. For the rotation speed within the region of CD, one has $\alpha = \pm\sigma$ with p=0. In this case the lateral displacement of the disk increases exponentially with time because of $\alpha=\sigma>0$. Thus, the CD region is called the divergence instability region and the speed at point C the divergence critical speed Ω_{dcr}. In Figure 5, the flutter critical speed Ω_{fcr} = 0.55 (56rpm) is much lower than the divergence critical

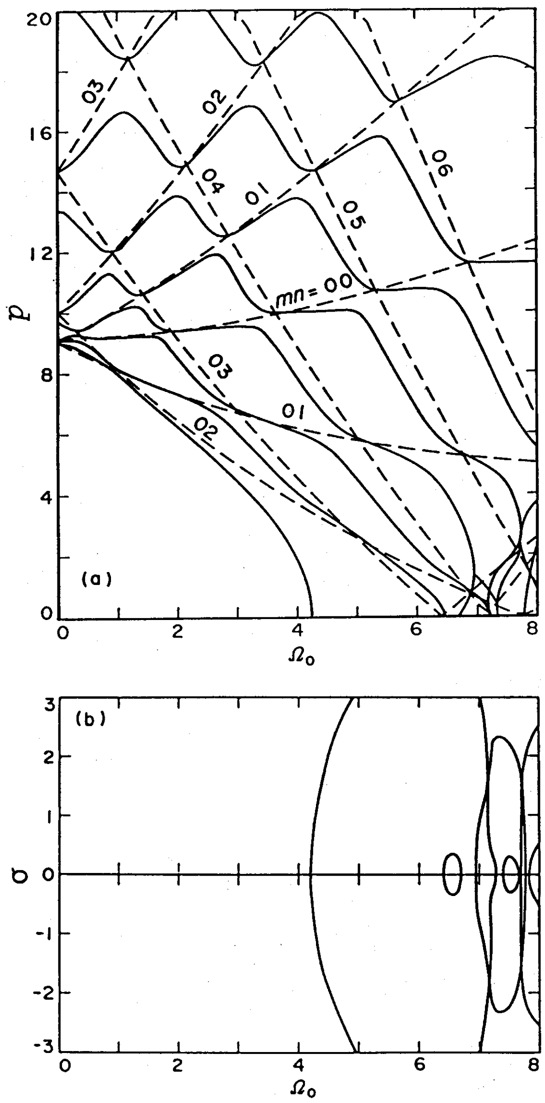

Fig. 4 Natural complex frequencies as functions of the disk rotation speed Ω_0. – – – –,Frequency of disk without R/W head; ———,frequency of disk with R/W head; M=3, N=6, m_{L0}=2.89, κ_0=50.0, k_{L0}=234.816, $\bar{\xi}$=0.7, η_0=0.

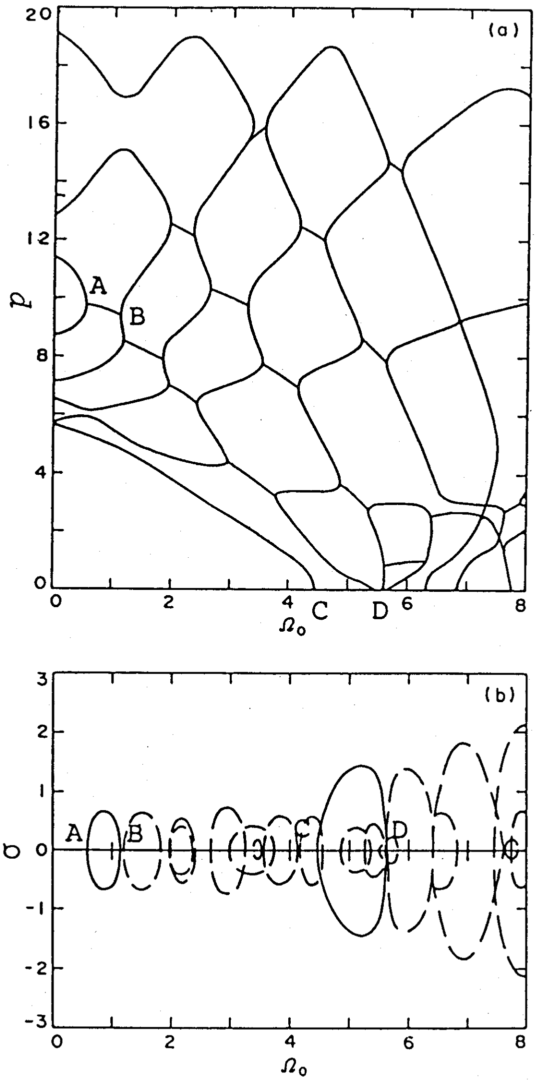

Fig. 5 Natural complex frequency as a function of the rotation speed of disk Ω_0. M=3, N=6, m_{L0}=2.89, k_{L0}=234.816, κ_0=0, $\bar{\xi}$=0.7, η_0=0.

speed Ω_{dcr} = 4.3 (430rpm), hence the maximum operation speed of disk is Ω_{fcr}=0.55. It is noted that in Figure 5 the stiffness of air film in the disk cover is assumed at κ_0=0. For the case κ_0=50 the corresponding frequency curves are shown in Figure 4 with the solid lines. In this case the maximum stable operation speed Ω_{cr} is Ω_{dcr}= 4.3. This means the stiffness of the air film κ_0 plays an important role in stabilizing the head disk system.

Figure 6 shows the maximum operation speed Ω_{cr} as a function of the mass m_{L0} and the stiffness k_{L0} of head for the parameters κ_0 = 0, $\bar{\xi}$ = 0.7 and η_0= 0. When k_{L0} = 0 (Figure 6a), the maximum operation speed Ω_{cr} is the divergence critical speed Ω_{dcr} regardless of the value of m_{L0}. In this case the speed Ω_{cr} have maximum value at m_{L0} = 0. Here, m_{L0} = 0 is the optimum mass for the case of k_{L0} = 0. When k_{L0} = 10.0 (Figure 6b), the maximum stable operation speed Ω_{cr} is the flutter critical speed Ω_{fcr} in the region $m_{L0} \leq 0.4$ while the divergence critical speed Ω_{dcr} for $m_{L0} \geq 0.4$. In this case the speed Ω_{cr} has its maximum value at m_{L0} = 0.4, hence the optimum mass corresponding to k_{L0} = 10.0 is m_{L0} = 0.4. Figure 6c is the case for k_{L0} = 100.0, which shows that the optimum mass is m_{L0} = 3.2. From those figures it is said that there is an optimum combination between the mass and the stiffness of the head to keep the disk-head system stable in operating at a high rotation speeds.

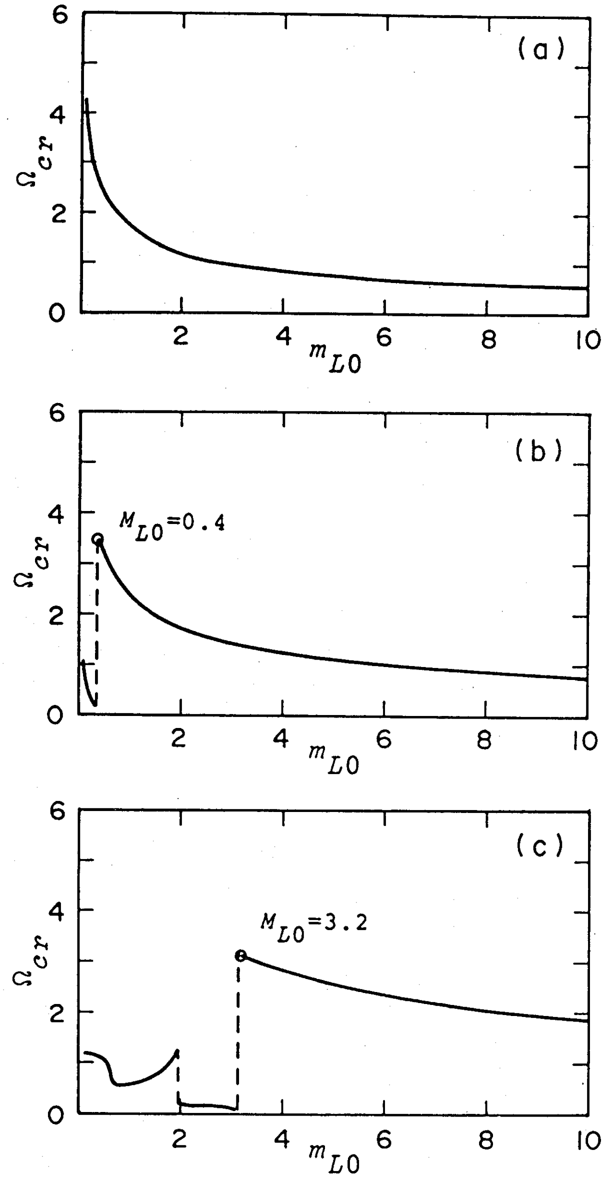

Fig. 6 Maximum operation speed of disk as a function of the mass m_{L0} and the stiffness k_{L0}. (a)k_{L0}=0, (b)k_{L0}=10, (c)k_{L0}=100; κ_0=0, η_0=0 , $\bar{\xi}$=0.7.

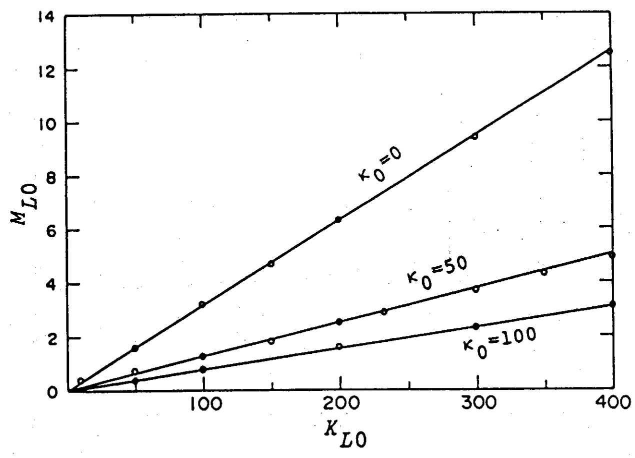

Fig.7 Optimum mass M_{L0} versus optimum stiffness K_{L0}. b/a=0.2518, ν=0.3, $\bar{\xi}$=0.7, η_0=0.

Figure 7 is a plot of the mass M_{L0} versus the stiffness K_{L0} , which brings the operation speed maximum. The slope of the curve is dependent on the stiffness κ_0 of the air film in disk cover, i.e.

$$M_{L0}=\lambda K_{L0} \tag{20}$$

Here, one has

$$\lambda=1.034/(32.802+\kappa_0) + 0.0001 \tag{21}$$

for a 5.25" floppy disk system. With the use of the relations

$$M_{L0}=M_L/\rho h a^2 \pi$$

$$K_{L0}=K_L a^2/D\pi$$

$$\kappa_0=\kappa a^4/D \; , \; D=Eh^3/12(1-\nu^2) \tag{22}$$

one has the corresponding dimensional equation as

$$M_L=\lambda \rho h(a^4/D)K_L \tag{23}$$

where E(N/m^2), ρ(kg/m^3), h(m), a(m), M_L(kg), K_L (N/m), κ(N/m^3) and ν = 0.3, b is given by b/a = 0.2578.

Equation (23) means that when the head stiffness K_L and the air film stiffness κ are given one should select the head mass M_L equal to or larger than $(\lambda\rho h a^4/D)K_L$ in order to make the disk rotate in stable at a high operation speed.

CONCLUSIONS

A theory has been developed for the response of a rotating R/W head floppy disk system subjected to the external axial and pitching oscillations. A parametric study has also been presented for the system to operate it in the stable condition. Results obtained can be summarized as follows:

1) The location of the R/W head on the rotating disk does not have much influence on the resonance frequencies of the system. However, for the disk excited in the axial direction, the high response amplitude becomes to be concentrated to the range of low frequencies as the head approaches the disk periphery. A similar tendency is observed for the disk under pitching excitation when the head comes near the nodal line of oscillation.

2) The air stiffness within the disk cover plays an important role on the stability of the system. There are optimum parameters for the R/W head to make the system response stable over a wide range of rotation speed. The equation to evaluate the optimum parameters is

$$M_L=\left(\frac{1.034}{32.802+\kappa a^4/D}+0.0001\right)(\rho h a^4/D)K_L \tag{24}$$

where E (N/m^2), ρ(kg/m^3), h(m), a(m), M_L(kg), K_L (N/m), κ (N/m^3) and ν = 0.3, b is given by b/a = 0.2578.

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APPENDIX

1. Non-dimensional parameters:

$$T=(D/\rho h a^4)^{1/2}t, \quad \Omega_0=(\rho h a^4/D)^{1/2}\Omega$$

$$\sigma_{r0}=(ha^2/D)\sigma_r, \quad \sigma_{\gamma 0}=(ha^2/D)\sigma_\gamma$$

$$x_0=(a^4/D)x, \quad \omega_0=(\rho h a^4/D)^{1/2}\omega$$

$$m_{L0}=m_L/\rho h a^2\pi, \quad k_{L0}=(a^2/\pi D)k_L$$

$$\bar{r}=r/a, \quad \bar{\xi}=\xi/a$$