Conceptual Extension of Stress Intensity to an Angled Defect I
— An Edge Notch with Arbitrary Included Angle —

Makio IINO and Ken KAMINISHI

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Abstract

Complex variable methods are applied to the plane elastic problems of semi-infinite sheet with a sharp edge notch of an arbitrary included angle 2β. The concept of the stress intensity in a crack problem is extended to the externally cut V-shaped notch. The difficulty of the problem would lie in the unavoidable introduction of a mapping function with singularities of branch-point type and related complex potentials, which is shown to be resolved by a power series development with expansion coefficients, which depend on the boundary-describing parameter, being smoothly continued from the traction-free boundary region to the local zone characterized by a stress singularity. General solutions for the stresses and the stresses local to the notch tip are given. In the light of the foregoing arguments the implications of the Westergaard solution for a crack are discussed.

Keywords : Stress singularity factor, Strength of singularity, Edge notch, Conformal mapping, Schwartz-Christoffel transformation

1. INTRODUCTION

Notwithstanding an engineering importance of the elastic analysis of stress singularities and distributions at and around a sharp edge notch, very limited significant contributions in the analysis approach have been made to that effect up to date. William [1] first solved the problem of elastic stresses induced around the apex of an infinite wedge and V-shaped sharp notch with an arbitrary angle, which might be the basis for the intended discussions. The concept of stress intensity in the realm of fracture mechanics might be extended to an internal and an external angled defect or notch. No attempts have been made to do this, and it seems what have been attempted even for an infinite V-shaped sharp notch are inadequate. You may find to date some numerical approaches toward this problem, but it may be that the understanding of the singular behaviors of the stress fields at notch tip does require the analytical approach as the basis.

In this work complex variable methods are applied to the analyses of general distributions and singularities of the stresses at and around the tip of an externally cut V-shaped notch with an arbitrary included angle. The difficulty of this problem would lie in the unavoidable introduction of a mapping function with singularities of branch-

Department of Mechanical Engineering, Yamaguchi University, 2557 Tokiwadai, Ube, 755 Japan

point type and related complex potentials, which is shown to be resolved by a power series expansion with expansion coefficients, which depend on a boundary- describing parameter, being smoothly continued from the traction-free boundary region to the local zone characterized by a stress singularity.

2. INITIAL FORMULATION

The semi-infinite sheet under tension with a V-shaped edge notch of an arbitrary included angle 2β and depth c, Figure 1, will be considered, as lying in the top, $Im(z+ic)\geqq 0$, of the complex z-plane, $z=x+iy$, with the tip of the notch described by $z=0$, where $i=[-1]^{1/2}$.

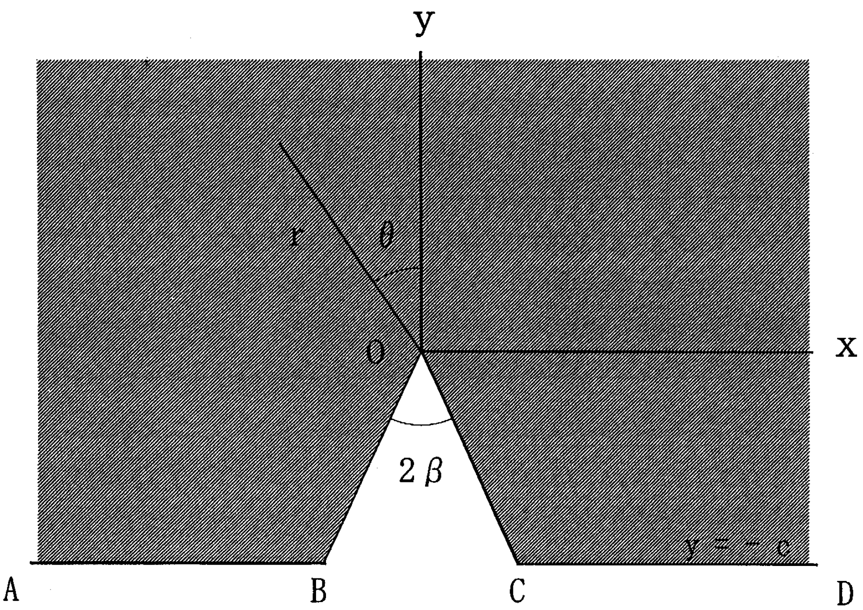

Figure 1 Semi-infinite sheet with a V-notch under tension in the x-direction

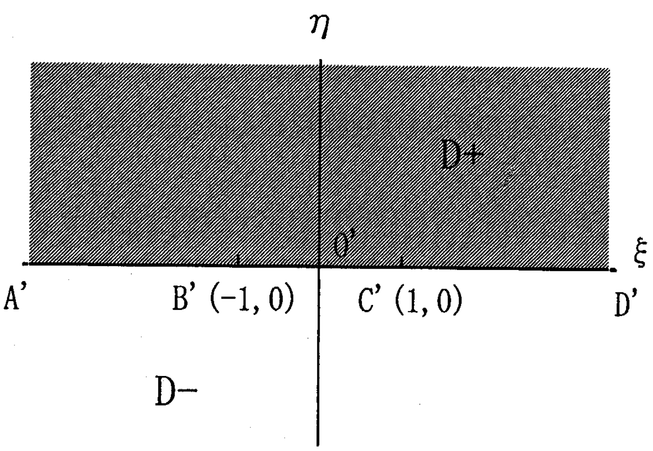

Figure 2 Auxiliary complex plane, $\zeta=\xi+i\eta$

For analyses complex potentials $\phi(z)$ and $x(z)$, known as Gursat's functions of the complex variable z, are used; both are arbitrarily chosen analytic functions but satisfy the required boundary conditions, and compose a bi-harmonic function well known as Airy's stress function, $F(z)=Re\ [\bar{z}\phi(z)+\int^{z}dz\chi(z)]$.

For convenience of the boundary condition consideration, an auxiliary complex plane, the ζ-plane, $\zeta=\xi+i\eta$, illustrated in Figure 2, is introduced, and a function relationship $z=\omega(\zeta)$ is searched such that the real axis A'B'O'C'D' and the upper-half plane D^{+} conformally map into the boundary ABOCD and the physical region occupied by the sheet, shown in Figure 1. By application of the Schwartz-Christoffel transformation, it was found that the mapping function, $\omega(\zeta)$, can be given as a principal branch of

$$z=\omega(\zeta)=C_0+C\int_{\zeta_0}^{\zeta}d\zeta\zeta^{2n}[\zeta^2-1]^{-n}, \tag{1}$$

where the exponent, n, is related to the included angle, 2β, as †

$$n=1/2\text{-}2\beta/2\pi. \tag{2}$$

The constants C, C_0 and ζ_0 are determined as

$$C=cB(1/2,\ n)/\pi\ ‡, \tag{3}$$

$$C_0=\omega(0)=0, \tag{3a}$$

and $\zeta_0=0$, by defining

$$\omega(-1)=-c\tan\beta-ic,\ \omega(1)=c\tan\beta-ic, \tag{4}$$

and

$$\omega(0)=0, \tag{4a}$$

where B(p,q) is the betha function. C can also be written as $C=c\Gamma(n)/\Gamma(1/2)\Gamma(1/2+n)$ by use of the gamma function $\Gamma(s)$ and remembering that $\Gamma(1/2)=\pi^{1/2}$.

Thus obtained mapping function, $\omega(\zeta)$, is analytic in the upper-half plane, Im $\zeta>0$, but contains singularities which describe branch points, A',B',C' and D', on the boundary, *Im* $\zeta=0$, itself. Other than these corner-describing singularities, the nocth tip is described by the root of $\omega'(\zeta)=0$, which occurs at $\zeta=0$. The prime is used to denote differentiation by the variable shown in the parentheses, thus $f'(z)=f'(\zeta)/\omega'(\zeta)$. To economize notations here we designate $f(z)=f[\omega(\zeta)]$ as $f(\zeta)$. In this way the stresses, σ_ξ, σ_η and $\tau_{\xi\eta}$, and displacements, u_ξ and u_η, in curvilinear coordinates, ξ and η, can be written as

$$\sigma_\xi+\sigma_\eta=2\phi'(\zeta)/\omega'(\zeta)+\text{complex conjugate}, \tag{5}$$

$$\sigma_\eta\text{-}\sigma_\xi+2i\tau_{\xi\eta}=\{2/\overline{\omega'(\zeta)}\}[\overline{\omega(\zeta)}d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta+\chi'(\zeta)], \tag{6}$$

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† Although the Schwartz-Christoffel transformation directly leads to this relation, the following reasoning would be more understandable. From equations(1) we find along the line segments BO and OC,

$$dz/d\zeta=C\xi^{2n}[\xi^2-1]^{-n}=C\xi^{2n}[1-\xi^2]^{-n}e^{\mp i\pi n} \tag{a}$$

On the other hand the slopes of the segments BO and OC are

$$\mp\ dy/dx=\tan(\pi/2\text{-}\beta) \tag{b}$$

Hence for the segments

$$dz=dx+idy=dxe^{\mp i(\pi/2-\beta)}/\cos(\pi/2\text{-}\beta) \tag{c}$$

Comparison of equation(c) with (a) leads to the relation(2)

‡ See APPENDIX I

$$2\mu(u_\xi - iu_\eta) = \{\omega'(\zeta)/|\omega'(\zeta)|\}[\kappa\phi(\zeta) - \overline{\omega(\zeta)}\overline{\phi'(\zeta)/\omega'(\zeta)} - \overline{\chi(\zeta)}], \quad (7)$$

where μ and κ are elastic constants of the material, and bars denote complex conjugates[2]. The bar notation $\bar{f}(\zeta)$, to appear below, is defined by $\overline{f(\bar{\zeta})}$. In terms of the functions $\phi(\xi)$ and $\chi(\xi)$ the traction-free boundary condition on ABOCD, Figure 1, can be written as

$$\bar{\phi}(\xi) + \bar{\omega}(\xi)\phi'(\xi)/\omega'(\xi) + \chi(\xi) = \text{constant}, \quad (8)$$

since the x- and y-components, $p_x ds$ and $p_y ds$, of the force acting on the arc ds of an arbitrary curve drawn on the sheet is expressed in terms of $\phi(\zeta)$ and $\chi(\zeta)$ as

$$(p_x - ip_y)\,ds = i\,d[\overline{\phi(\zeta)} + \overline{\omega(\zeta)}\phi'(\zeta)/\omega'(\zeta) + \chi(\zeta)], \quad (9)$$

and this equation is applicable to an arbitrary arc on the boundary under consideration. The solution of the problem requires the determination of the functions $\phi(\zeta)$ and $\chi(\zeta)$ which are analytic in $Im\zeta > 0$ and satisfy the boundary conditions(8) and loading conditions to appear in equation(13) below.

In a domain of interest around the notch-tip, $|\zeta| < 1$, the mapping function $\omega(\zeta)$, equation(1) with constants determined as (3), can be expressed in a power series as

$$\omega(\zeta) = (C/\nu)e^{\mp i\pi n}\zeta^{\nu}\sum_{k=0}^{\infty} a_k\zeta^{2k}, \quad \nu = 1 + 2n \quad (10)$$

$$a_0 = 1, \quad (10a)$$

$$a_k = [\nu/(\nu+2k)](n+k-1)\cdots(n+1)n/k! \quad (k=1,2,3\cdots), \quad (10b)$$

by term-by-term integration after expansion of the integrand in equation(1) for $|\zeta| < 1$, in a power series. And for large $|\zeta|$, $|\zeta| > 1$, as

$$\omega(\zeta) = C\zeta\sum_{k=0}^{\infty} b_k\zeta^{-2k} - ic, \quad (11)$$

$$b_0 = 1, \quad (11a)$$

$$b_k = -(n+k-1)\cdots(n+1)n/(2k-1)k! \quad (k=1,2,3\cdots), \quad (11b)$$

by term-by-term integration after expansion of the integrand in equation(1) for $|\zeta| > 1$, in a power series.

3. STRESSES AROUND TIP OF A NOTCH

In terms of the above-developed formulations the essential character of the stresses, namely the notch-tip singularity and the azimuth dependences of the stresses as functions of 2β, will be examined, which should be influenced by the presence of traction-free boundaries. From the boundary condition consideration with respect to equation(8), it follows that the function $\chi(\zeta)$ can be expressed as

$$\chi(\zeta) = -\bar{\phi}(\zeta) - \bar{\omega}(\zeta)\phi'(\zeta)/\omega'(\zeta), \quad (12)$$

which is analytic in the upper-half plane, Im $\zeta > 0$, letting the constant, which does not influence the stresses, zero. Thus, the problem reduces to the determination of $\phi(\zeta)$ which satisfies the loading conditions at infinity,

$$\sigma_x = \sigma, \ \sigma_y = \tau_{xy} = 0 \ (\xi \to \infty). \quad (13)$$

Examination of $\omega(\zeta)$, equation(10), will suggest that $2\phi(\zeta)$ can be assumed to be developed in a power series as

$$2\phi(\zeta) = iC\sum_{k=1}^{\infty} B_k[\nu\omega(\zeta)/iC]^{\lambda_k}. \quad (14)$$

From equations(12) $\chi(\zeta)$ is expressed as

$$2\chi(\zeta)=iC\sum_{k=0}^{\infty}E_k(\zeta)[\nu\omega(\zeta)/iC]^{\lambda_k}, \tag{15}$$

$$E_k(\zeta)=\bar{B}_k\bar{\varepsilon}^{\lambda_k}(\zeta)+B_k\lambda_k\bar{\varepsilon}(\zeta), \tag{15a}$$

where $\bar{\varepsilon}(\zeta)$ is defined by

$$\bar{\varepsilon}(\zeta)=-\bar{\omega}(\zeta)/\omega(\zeta), \tag{15b}$$

and each B_k is an arbitrary constant but real from symmetry consideration in the present loading. λ_k is assumed to be real and positive, $\lambda_k>0$, for the displacements to be bounded at the notch tip. Each term with the exponent λ_k is introduced for $2\phi(\zeta)$ and $2\chi(\zeta)$ to be capable of describing a right notch-tip singularity as well as satisfying the required traction-free boundary, which requires that λ_k is the real part of the k-th(k=1, 2, •••) solution of

$$\sin\lambda_k 2\alpha+\lambda_k\sin 2\alpha=0, \tag{16}$$

which accords with William's condition. λ_k is thus a function of 2α, where $2\alpha=2\pi-2\beta$. $E_k(\zeta)$ in equation(15) is related to B_k as

$$E_k(\xi)=B_k(\cos\lambda_k 2\alpha+\lambda_k\cos 2\alpha) \qquad (|\xi|\leqq 1), \tag{17}$$

and

$$E_k(\xi)=B_k[e^{-i\lambda_k 2\theta}+\lambda_k e^{-i2\theta}], \qquad (|\xi|>1), \tag{18}$$

Thus, $E_k(\zeta)$ is found to vary as a function of a single variable, θ, for $|\xi|>1$ as in equation(18), and smoothly continued onto $E_k(\xi)$ for $|\xi|\leqq 1$ in equation(17). See APPENDIX II for the implications of equations(16) to (18).

General distributions of the stresses in a domain of interest may be rendered by substituting equations(14) and (15), with $\omega(\zeta)$ given in equation(10) or (11), into (5) and (6) as

$$\sigma_\xi=\sum_{k=1}^{\infty}(B_k\lambda_k\nu/2C)Re[\{2-c_k(\zeta)\delta_1(\zeta)-(1-\lambda_k)\delta_1(\zeta)\overline{\delta(\zeta)}\}(\nu\omega(\zeta)/iC)^{\lambda_k-1}], \tag{19}$$

$$\sigma_\eta=\sum_{k=1}^{\infty}(B_k\lambda_k\nu/2C)Re[\{2+c_k(\zeta)\delta_1(\zeta)+(1-\lambda_k)\delta_1(\zeta)\overline{\delta(\zeta)}\}(\nu\omega(\zeta)/iC)^{\lambda_k-1}], \tag{20}$$

$$\sigma_{\xi\eta}=\sum_{k=1}^{\infty}(B_k\lambda_k\nu/2C)Im[\{c_k(\zeta)\delta_1(\zeta)+(1-\lambda_k)\delta_1(\zeta)\overline{\delta(\zeta)}\}(\nu\omega(\zeta)/iC)^{\lambda_k-1}], \tag{21}$$

where $\delta(\zeta)$, $\delta_1(\zeta)$ and $c_k(\zeta)$ signify $\delta(\zeta)=-\omega(\zeta)/\overline{\omega(\zeta)}$, $\delta_1(\zeta)=\omega'(\zeta)/\overline{\omega'(\zeta)}$ and $c_k(\zeta)=E_k(\zeta)/B_k$, respectively. A term of the maximum impotance in $2\phi(\zeta)$, equation(14), and in $2\chi(\zeta)$, equation(15), however, are $iCB_1[\nu\omega(\zeta)/iC]^{\lambda_1}$ and $iCE_1(\zeta)[\nu\omega(\zeta)/iC]^{\lambda_1}$, respectively, which will be discussed below.

4. NOTCH-TIP SINGULARITIES AND AZIMUTH DEPENDENCES OF LOCAL STRESSES

For the examination of notch tip singularities let attention be restricted to the domain $|\zeta|<<1$, where $\omega(\zeta)$ and $\omega'(\zeta)$ are closely approximated by

$$\omega(\zeta)=(C/\nu)e^{\mp i\pi n}\zeta^{\nu}, \tag{22}$$

$$\omega'(\zeta)=Ce^{\mp i\pi n}\zeta^{2n}, \tag{23}$$

If the z-plane is described by polar coordinates, r and θ, with pole at the notch tip and θ being the counter-clockwise angle with the y-axis, then

$$z=\omega(\zeta)=ire^{i\theta}, \tag{24}$$

The stresses in the immediate vicinity of the notch tip in polar coordinates can now be expressed, through the conversion formulae and writing λ_1 as λ, as follows:

$$\sigma_r=(B_1\lambda\nu/2)[C/\nu r]^{1-\lambda}[(3-\lambda)\cos(1-\lambda)\theta+(\cos2\lambda\alpha+\lambda\cos2\alpha)\cos(1+\lambda)\theta], \quad (25)$$

$$\sigma_\theta=(B_1\lambda\nu/2)[C/\nu r]^{1-\lambda}[(1+\lambda)\cos(1-\lambda)\theta-(\cos2\lambda\alpha+\lambda\cos2\alpha)\cos(1+\lambda)\theta], \quad (26)$$

$$\sigma_{r\theta}=(B_1\lambda\nu/2)[C/\nu r]^{1-\lambda}[(1-\lambda)\sin(1-\lambda)\theta-(\cos2\lambda\alpha+\lambda\cos2\alpha)\sin(1+\lambda)\theta]. \quad (27)$$

The amplitude of stress singularity at a crack tip, being termed as a stress intensity factor, is a wide-spread concept today, and a general definition of it will be given by

$$K_I=\lim_{\zeta\to\zeta_0} Re[e^{-i\delta}2\pi\{\omega(\zeta)-\omega(\zeta_0)\}]^{1/2}2\phi'(\zeta)/\omega'(\zeta), \quad (28)$$

where K_I is mode I stress-intensity factor, $\omega(\zeta_0)$ the location of the crack tip, and δ the angle which the normal of the crack plane makes against the y-axis. We will not refer to mode II stress intensity here. The extention of the concept to a general angled defect would define mode I stress singularity factor K_I for the defect as

$$K_I=\lim_{\zeta\to\zeta_0} Re[e^{-i\delta}\pi\nu\{\omega(\zeta)-\omega(\zeta_0)\}]^{1-\lambda}2\phi'(\zeta)/\omega'(\zeta), \quad (29)$$

where $1-\lambda$ denotes strength of the stress singularity. It is to be noted that a factor $\pi\nu$, which apperas in equation(29) in place of 2π for a crack, is related by definition with the notch-tip angle 2β as

$$\pi\nu=2\pi-2\beta. \quad (30)$$

By applying the general definition, equation(29), to $2\phi(\zeta)$ in equation(14), the quantity B_1 proves to be related with the stress singularity factor K_I as

$$B_1\lambda\nu[\pi C]^{1-\lambda}=K_I. \quad (31)$$

In a limiting case of a crack, $\lambda=1/2$, the above equations(25) to (27), and (31) reduces to the widely known conventional formulae.

5. IMPLICATIONS OF THE WESTERGAARD SOLUTION

In the light of the foregoing formulations and derivations the implications of the Westergaard solution in a crack problem[3] will be considered. Westergaard's method is characterized by an *a priori* representation of the solutions in the form ¶

$$\sigma_x=Re[\sigma c/\omega'(\zeta)]-xIm[\{\sigma c/\omega'(\zeta)\}d\{1/\omega'(\zeta)\}/d\zeta], \quad (32)$$

$$\sigma_y=Re[\sigma c/\omega'(\zeta)]+xIm[\{\sigma c/\omega'(\zeta)\}d\{1/\omega'(\zeta)\}/d\zeta], \quad (33)$$

$$\sigma_{xy}=-xRe[\{\sigma c/\omega'(\zeta)\}d\{1/\omega'(\zeta)\}/d\zeta], \quad (34)$$

where

$$\omega(\zeta)=c[\zeta^2-1]^{1/2}-ic. \quad (35)$$

In this way the Westergaard solution claims its engineering expediency by assigning itself the restrictive requirements,

$$\sigma_y-\sigma_x+i2\tau_{xy}=0 \text{ on the crack prolongation (at } x=0 \text{ and } |y|\geq c). \quad (36)$$

Note first that his method corresponds to having utilized $d\zeta/dz=1/\omega'(\zeta)$ itself as a stress function; the function $\omega(\zeta)$ conformally maps the ξ-axis in the ζ plane, Figure 2, into the boundary ABOCD, illustrated in Figure 3, although he probably does not intend to utilize the nature of $\omega(\zeta)$, equation(35). Secondly, he just imposes the restrictive requirements(36) other than the conditions of remotely applied stresses ‡,

$$\sigma_x=\sigma_y=\sigma \text{ and } \tau_{xy}=0 \text{ at infinity}, \quad (37)$$

to the solutions, without assigning any load-free boundary conditions.

¶ Note that the crack line is on the y-axis here in conformity with the foregoing

arguments, while Westergaard assumes the crack line on the x-axis.
‡ It must be added that a uniform compressive stress, σ, may be superposed in the y-direction without disturbing the remaining features of the solution, as he himself addresses[3].

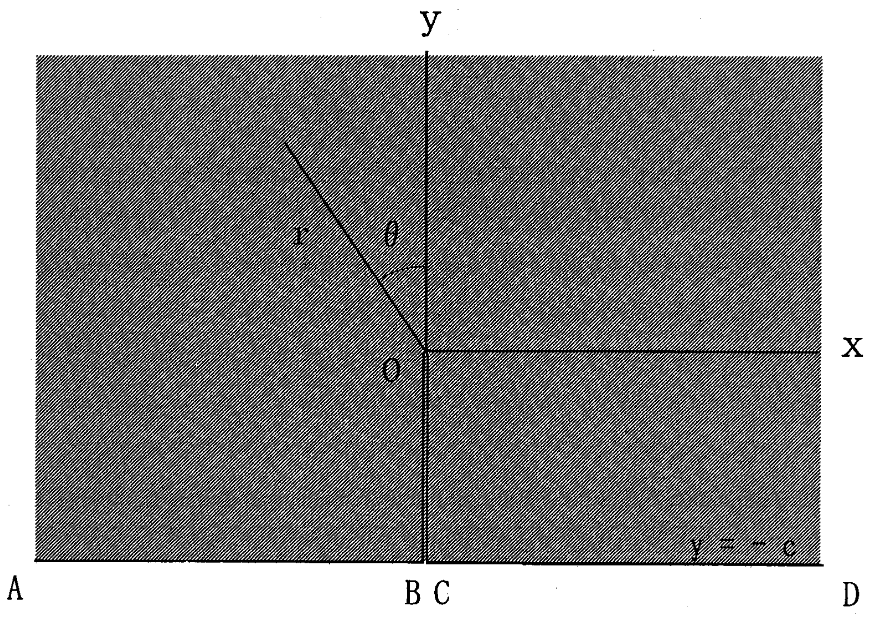

Figure 3 Semi-infinite sheet with an edge crack under tension in the x-direction

If the Westergaard's method is reformulated using complex potentials, then

$$2\phi(\xi) = \sigma c \xi, \tag{38}$$

$$2\chi(\xi) = -\sigma c/\xi \tag{39}$$

By substituting these functions into equations(5) and (6), you can confirm $2\phi(\xi)$, equation(38), and $2\chi(\xi)$, equation(39), produce the so-called Westergaard solutions which satisfy the restrictive requirements(36), and not the appropriate load-free boundary conditions to be alloted on the crack surface.

We are now not interested in the right treatment of an internal crack problem, and correcting the Westergaard approarch. But we will confine ourselves to the problem of an edge crack, which can make an exactly right use of the mapping function, equation(35). Getting back to equations(14) and (15), and letting $\lambda_k = k/2$, $2\phi(\xi)$ and $2\chi(\xi)$ for a crack reduces to

$$2\phi(\xi) = ic \sum_{k=1}^{\infty} B_k [2\omega(\xi)/ic]^{k/2}, \tag{14a}$$

$$2\chi(\xi) = ic \sum_{k=1}^{\infty} E_k(\xi) [2\omega(\xi)/ic]^{k/2}. \tag{15a}$$

Application of the general definition of stress singularity, equation(29), to this $2\phi(\xi)$, equation(14a), relates B_1 with the stress singularity factor K_I as

$$B_1 [\pi c]^{1/2} = K_I, \tag{40}$$

Thus, the significance of B_1 in this system is at once clear, but the above discussions exclusively could not determine B_1.

6. CONCLUSIONS

In this work the concept of the stress intensity in a crack problem is extended to an externally cut V-shaped notch with an arbitrary included angle, 2β. The difficulty in the present comlex analyses of the mapping functions with singularities of branch-point type and related complex potentials is shown to be resolved by a power series development with expansion coefficients, which depend on the boundary-describing parameter, being smoothly continued from the traction-free boundary region to the local zone characterized by a stress singularity. General solutions for the stresses and those in the vicinity of the notch tip are derived on the basis of the formulations developed in this work. In the light of the foregoing formulations and derivations the implications of the Westergaard soulution for a crack are discussed.

7. References

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[2] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticty,* pp. 171-190, P. Noordhoff Ltd., Groningen-the Netherlands, 1963.

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APPENDIX I Detrmination of C

Since equation(1) must satisfy $\omega(1)=c\tan\beta-ic=-ice^{i\beta}/\cos\beta$, equatiou (4),

$$-ic\, e^{i\beta}(1/\cos\beta)=C\int_0^1 d\xi\ \xi^{2n}(\xi^2-1)^{-n}=Ce^{\mp i\pi n}I \tag{A1}$$

holds, where the integral $I=\int_0^1 d\xi\ \xi^{2n}(1-\xi^2)^{-n}$ is known to be given by the betha function as

$$I=\int_0^1 d\xi\ \xi^{2n}(1-\xi^2)^{-n}=(1/2)B[(1+2n)/2,\ 1-n],$$

which can be deformed in the manner

$$\begin{aligned}&=(1/2)\Gamma(1/2+n)\Gamma(1-n)/\Gamma(3/2)\\&=\{\Gamma(1/2+n)/\Gamma(1/2)\Gamma(n)\}\cdot\Gamma(n)\Gamma(1-n)\\&=\{1/B(1/2,\ n)\}(\pi/\sin\pi n)\\&=\{\pi/B(1/2,\ n)\}(1/\cos\beta).\end{aligned} \tag{A2}$$

Thus, remembering $n=1/2-\beta/\pi$, C in equation(A1) is lead to

$$C=cB(1/2,\ n)/\pi, \tag{A3}$$

where it is assumed without loss of generality that C is real.

APPENDIX II Derivation of equations(16) to (18)

The traction-free boundary condition(8) can be expressed as

$$ds(p_x-ip_y)=id\sum_{k=1}^{\infty} f_k(\xi,\xi)=0, \tag{B1}$$

where

$$f_k(\bar{\xi},\xi)=\overline{\phi_k(\xi)}+\overline{\omega(\xi)}\phi_k'(\xi)/\omega'(\xi)+\chi_k(\xi)+\text{a constant}, \tag{B2}$$

with $\phi_k(\xi)$ and $\chi_k(\xi)$ being the k-th term of $\phi(\xi)$ and $\chi(\xi)$, respectively. By substituting $2\phi(\xi)$, equation(14), and $2\chi(\zeta)$, equation(15), into $f_k(\bar{\zeta},\zeta)$ in equation(B2), and letting $\eta=0$, $f_k(\xi,\xi)$ is found to be written as

$$f_k(\xi,\xi)=[-\bar{B}_k\bar{\varepsilon}^{\lambda_k}(\xi)-B_k\lambda_k\bar{\varepsilon}(\xi)+E_k(\xi)][\nu\omega(\xi)/iC]^{\lambda_k}+\text{a constant}, \quad \text{(B3)}$$

Where $\bar{\varepsilon}(\xi)=-\bar{\omega}(\xi)/\omega(\xi)$. In order for the condition(B1) to be true for an arbitrary value of $\omega(\xi)$, the coefficient of $[\nu\omega(\xi)/iC]^{\lambda_k}$ must vanish, i.e.,

$$E_k(\xi)=\bar{B}_k\bar{\varepsilon}^{\lambda_k}(\xi)+B_k\lambda_k\bar{\varepsilon}(\xi). \quad \text{(B4)}$$

On BO and OC in Figure 1, $\bar{\varepsilon}(\xi)=e^{\pm i2\alpha}$, where the condition(B4) requies

$$E_k(\xi)=\bar{B}_k e^{\pm i\lambda_k 2\alpha}+B_k\lambda_k e^{\pm i2\alpha} \quad (|\xi|\leqq 1). \quad \text{(B5)}$$

Thus $E_k(\xi)$ must be constant, on BO and OC. Equating both the constants in relation(B5), and remembering that B_k are real, it is at once clear that

$$\sin\lambda_k 2\alpha+\lambda_k\sin 2\alpha=0 \quad (|\xi|\leqq 1), \quad \text{(B6)}$$

which should and do agree with William's results[1], and consequently

$$E_k(\xi)=B_k(\cos\lambda_k 2\alpha+\lambda_k\cos 2\alpha) \quad (|\xi|\leqq 1). \quad \text{(B7)}$$

Thus, $E_k(\xi)$ must also be real. On AB and CD in Figure 1, on the other hand, it is found that $E_k(\xi)$ varies, because $\bar{\varepsilon}(\xi)$ there is

$$\bar{\varepsilon}(\xi)=-(x+ic)/(x-ic)=(c^2-x^2)/(c^2-x^2)-i2cx/(c^2+x^2)=\exp[i\tan^{-1}\{-2cx/(c^2-x^2)\}], \quad \text{(B8)}$$

which varies with x. If the counter-clockwise angle, in polar coordinates with pole at $z=0$, with the positive y-axis is denoted by θ, then $x=c\tan(\pi+\theta)$, and $\tan^{-1}\{-2cx/(c^2-x^2)\}=\tan^{-1}[-\tan(2\pi+2\theta)]=-(2\pi+2\theta)$. It follows that

$$\bar{\varepsilon}(\xi)=e^{-i(2\pi+2\theta)}=e^{-i2\theta}. \quad \text{(B9)}$$

By substituting this into equation(B4), $E_k(\xi)$ is lead to

$$E_k(\xi)=B_k[e^{-i\lambda_k 2\theta}+\lambda_k\, e^{-i2\theta}]. \quad (|\xi|>1). \quad \text{(B10)}$$

Thus, $E_k(\zeta)$ is found to be expressed as a function of a single variable, θ, on AB and CD. Further it is seen that the $E_k(\zeta)$ is smoothly continued at $\theta=\pm\alpha$ onto the value for $|\xi|\leqq 1$, equation(B7). In the complex ζ-plane, where the mapping function, $\omega(\zeta)$, is expressed as in equation(10) or (11) depending on the ζ area, it is understood that

$$E_k(\zeta)=B_k(\cos\lambda_k 2\alpha+\lambda_k\cos 2\alpha)=\text{a real constant} \quad (|\zeta|\leqq 1), \quad \text{(B11)}$$

and

$$E_k(\zeta)=B_k\bar{\varepsilon}^{\lambda_k}(\zeta)+B_k\lambda_k\bar{\varepsilon}(\zeta)=\text{a complex variable} \quad (|\zeta|>1). \quad \text{(B12)}$$