Space Hierarchies of Three-Dimensional Turing Machines

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Abstract

We investigate space complexity hierarchies of three-dimensional Turing machines whose input tapes are restricted to cubic ones, and show that there exists an infinite hierarchy among the classes of sets accepted by space-bounded three-dimensional deterministic or nondeterministic Turing machines with cubic inputs.

1 Introduction

In general, computational complexity is a study of considering how the computational powers of various types of automata are characterized by space complexity, time complexity, or some other related measures. Especially, the concept of space complexity is very useful to characterize various types of antomata from a point of view of memory requirements. This study was motivated by Stearns, Hartmanis, and Lewis [19]. They introduced an $L(n)$ space-bounded one-dimensional Turing machine to formalize the notion of space complexity, and investigated its computing ability. Moreover, some results were refined by Hopcroft and Ullman [6-8].

After that, the problem of computational complexity was also arisen in the two-dimensional information processing. Blum and Hewitt first proposed two- dimensional antomata, and investigated their pattern recognition abilities [1]. Morita, Umeo, and Sugata proposed an $L(m,n)$ space-bounded two-dimensional Turing machine and its variants to formalize memory limited computations in the two-dimensional information processing [14,15].

Recently, due to the advances in computer vision, robotics and so forth, it has become increasingly apparent that the study of three-dimensional pattern processing should be very important. Thus, the research of three-dimensional automata as the computational model of three-dimensional pattern processing hat also been meaningful [2,7,16-18,21-24].

In this paper, we investigate the space complexity hierarchies of three- dimensional

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deterministic or nondeterministic Turing machines with cubic inputs. Section 2 gives several preliminaries necessary for this paper. Section 3 presents the hierarchy theorem for the space $\geq$ log n. Section 4 presents the hierarchy theorem for the space $<$ log n.

2 Preliminaries

Definition 2.1. Let Σ be a finite set of symbols. A *three-dimensional tape* over Σ is a three-dimensional rectangular array of elements of Σ. The set of all three-dimensional tapes over Σ is denoted by $\Sigma^{(3)}$.

Given a tape $x \in \Sigma^{(3)}$, for each $j\,(1 \leq j \leq 3)$, we let $l_j(x)$ be the length of x along the j-th axis. The set of all $x \in \Sigma^{(3)}$ with $l_1(x)=m_1$, $l_2(x)=m_2$ and $l_3(x)=m_3$ is denoted by $\Sigma^{(m_1,m_2,m_3)}$. When $1 \leq i_j \leq l_j(x)$ for each $j\,(1 \leq j \leq 3)$, let $x(i_1,i_2,i_3,)$ denote the symbol in x with coordinates (i_1,i_2,i_3). Furthermore, we define

$$x[(i_1,i_2,i_3),(i'_1,\ i'_2,i'_3)],$$

when $1 \leq i_j \leq i'_j \leq l_j(x)$ for each integer $j\,(1 \leq j \leq 3)$, as the three-dimensional tape y satisfying the following (i) and (ii):

(i) for each $j\,(1 \leq j \leq 3)$, $l_j(y)=i'_j-i_j+1$;

(ii) for each $r_1,r_2,r_3\,(1 \leq r_1 \leq l_1(y),\ 1 \leq r_2 \leq l_2(y), 1 \leq r_3 \leq l_3(y)), y(r_1,r_2,r_3)=x(r_1+i_1-1, r_2+i_2-1, r_3+i_3-1)$.

(We call $x[(i_1,i_2,i_3),\ (i'_1,i'_2,i'_3)]$ *the* $[(i_1,i_2,i_3),(i'_1,i'_2,i'_3)]$*-segment* of x.) When a three - dimensional tape x is given to any three-dimensional automaton, we assume that x is surrounded by the boundary symbol #.

We now introduce a three-dimensional Turing machine.

Definition 2.2. A *three-dimensional Turing machine* (3-*TM*) M has a read-only three -dimensional input tape with boundary symbols #'s and one semi-infinite *storage tape* initially blank. Of course, M has a *finite control*, an *input head*, and a *storage-tape head*. A *position* is assigned to each cell of the read-only input tape and to each cell of the storage tape, as shown in Fig.1. Formally, M is defined by the six-tuple

$$M=(Q,q_0,F,\Sigma,\Gamma,\delta),$$

where

(1) Q is a finite set of states,

(2) $q_0 \in Q$ is the initial state,

(3) $F \subseteq Q$ is the set of *accepting states,*

(4) Σ is a finite *input alphabet* ($\# \notin \Sigma$ is the *boundary symbol*),

(5) Γ is a finite *storage-tape alphabet* ($B \in \Gamma$ is the *blank symbol*), and

(6) $\delta \subseteq (Q \times (\Sigma \cup \{\#\}) \times \Gamma) \times (Q \times (\Gamma - \{B\}) \times$ {east, west, south, north, up, down, no move}$\times$ {right, left, no move}) is the *next-move relation.*

The action of M is similar to that of one- or two- dimensional Turing machine [4-6], except that the input head of M can move in six directions. That is, when an input tape $x \in \Sigma^{(3)}$ with boundary symbols #'s is presented to M, M starts in its initial state q_0, with the input head on $x(1,1,1)$, with all cells of the storage tape blank, and with the storage head on the leftmost cell of the storage tape. Then M determines the next state

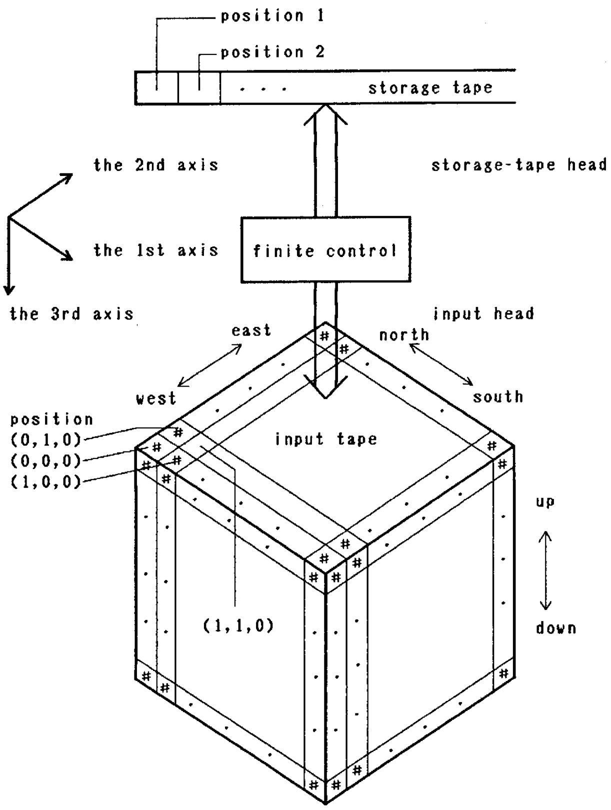

Fig.1. Three-dimensional Turing machine.

of the finite control, the move direction of the input head, the symbol written by the storage head, and the move direction of the storage head, depending on the present state of the finite control and the symbols read by the input and storage heads. We say that M *accepts* the tape x if it eventually enters an accepting state. Note that the machine cannot write the blank symbol. If the input head falls off the input tape or if the storage head falls off the storage tape (by moving left), then the machine M can make no further move. If M move deterministically (nondeterministically), we call M a *three-dimensional deterministic (nondeterministic) Turing machine,* denoted by 3-*DTM* (3-*NTM*).

Definition 2.3. For $X \in \{D,N\}$, a *configuration* of a 3-*XTM* $M=(Q,q_0, F,\Sigma,\Gamma,\delta)$ is an element of

$$\Sigma^{(3)} \times (\mathbf{N} \cup \{0\})^3 \times S_M,$$

where $S_M = Q \times (\Gamma-\{B\})^* \times \mathbf{N}$ and $\mathbf{N}$ denotes the set of all positive integers. The first component x of a configuration[1] $c = (x,(i_1,i_2,i_3),(q,\alpha,j))$ represents the input to M. The second component (i_1,i_2,i_3) of c represents the inpnt-head position. The third component (q,a,j) of c represents the state of the finite control, nonblank contents of the storage tape, and the storage-head position. An element of S_M is called a *storage state* of M.

Next, we consider a restrited type of 3-*TM*, called a space bounded 3-*TM*, which can be considered as a natural extension of the space-bounded one- or two-dimensional

Turing machine [7,8,12] to three dimensions.

Definition 2.4. Let $L(n)$: $\mathbf{N} \to \mathbf{R}$ be a function of one variable n, where $\mathbf{N}$ is the set of all positive integers and $\mathbf{R}$ is the set of all nonnegative real numbers. A three dimensional Turing machine M is said to be $L(n)$ *space-bounded* if for no three-dimensional input tape $x \in \Sigma^{(3)}$ with $l_1(x) = l_2(x) = l_3(x) = n$ does M scan more than $L(n)$ cells[2] on the storage tape. We denote an $L(n)$ space-bounded 3-*DTM* (3-*NTM*) by 3-*DTM*$(L(n))$ (3-*NTM*$(L(n))$).

A determlinistic (nondeterministic) three-dimensional finite automaton [21], denoted by 3-*DFA* (3-*NFA*), is a 3-*DTM*(0) (3-*NTM*(0)).

It has often been noticed that we can easily get several properties of three-dimensional automata by directly applying the results of one- or two- dimensional case, if the three-dimensional inpnt tapes are not restricted to cubic ones. So we let the three -dimensional input tapes, throughout this paper, be restricted to cubic ones in order to increase the theoretical interest.

For each $X \in \{D,N\}$, we denote a 3-*XTM* [3-*XTM*$(L(n))$, 3-*XFA*] whose three -dimensional input tapes are retstricted to cubic ones by 3-XTM^c [3-$XTM^c(L(n))$, 3-XFA^c].

Definition 2.5. For any three-dimensional automaton M with input alphabet Σ, define $T(M) = \{x \in \Sigma^{(3)} \mid M \textit{ accepts } x\}$. Furthermore, for each $X \in \{D,N\}$, define $\pounds[\text{3-}XTM^c] = \{T \mid T = T(M)$ for some 3-XTM^c $M\}$. $\pounds[\text{3-}XTM^c(L(n))]$ and $\pounds[\text{3-}XFA^c]$ also have analogous meanings.

Moreover, we need the following definitions.

Definition 2.6. A function $L : \mathbf{N} \to \mathbf{R}$ is *three-dimensionally space constructible* if there is an $L(n)$ space-bounded 3-DTM^c M such that for each $n \geq 1$, there exists some input tape x with $l_1(x) = l_2(x) = l_3(x) = n$ on which M halts after its storage tape head has marked off exactly $L(n)$ cells of tle storage tape. (In this case, we say that M constructs the function L.)

Definition 2.7. Let Σ_1, Σ_2 be finite sets of symbols. A *projection* is a mapping $\tilde{\tau}: \Sigma_1^{(3)} \to \Sigma_2^{(3)}$ which is obtained by extending a mapping $\tau: \Sigma_1 \to \Sigma_2$ as follows : $\tilde{\tau}(x) = x'$ if and only if (i) $l_i(x) = l_i(x')$ for each $i(1<i<3)$, and (ii) $\tau(x(i_1,i_2,i_3)) = x'(i_1,i_2,i_3)$ for each $(i_1,i_2,i_3)[1 \leq i_1 \leq l_1(x),\ 1 \leq i_2 \leq l_2(x),\ 1<i_3<l_3(x)]$. If $T \subseteq \Sigma_1^{(3)}$, we let $\tilde{\tau}(T) = \{\tilde{\tau}(x) \mid x \in T\}$.

The following theorem shows that the acceptability is not affected by adding a constant factor to the space function $L(n)$.

Theorem 2.1. For any $X \in \{D,N\}$, for any function $L : \mathbf{N} \to \mathbf{R}$, and for any constanlt $d>0$,

$$\pounds[\text{3-}XTM^c(L(n))] = \pounds[\text{3-}XTM^c(L(n)+d)].$$

The next theorem shows that the acceptability is not affected by multiplying a

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[1]We note that $0 \leq i_1 \leq l_1(x)+1$, $0 \leq i_2 \leq l_2(x)+1$, $0 \leq i_3 \leq l_3(x)+1$, *and* $1 \leq j \leq |\alpha|+1$, where for any string w, $|w|$ denotes the length of w (with $|\lambda|=0$, where λ is the null string).

[2]Rigorously, "$L(n)$ cells" should be replaced with "$\lceil L(n) \rceil$ cells" where $\lceil r \rceil$ means the smallest integer greater than or equal to r. Below we omit $\lceil\ \rceil$ if no confusion results.

constant factor to the space function $L(n)$.

Theorem 2.2. (Tape Reduction Theorem of 3-TM^c) For any $X \in \{D,N\}$, for any funetion $L : \mathbf{N} \rightarrow \mathbf{R}$, and for any constant $d>0$,

$$\pounds[3\text{-}XTM^c(L(n))] = \pounds[3\text{-}XTM^c(dL(n))].$$

These two theorems are easily proved in the same way as in the one- or two-dimensional case [14,19], and so the proofs are omitted here.

3 Hierarchy Theorem for $> \log n$

In this section, we investigate the space hierarchy among the classes of sets accepted by 3-TM^c's with spaces equal to or larger than log n, and show that there exists an infinite hierarchy among those classes.

Theorem 3.1. Let $L_1(n)$ and $L_2(n)$ be any three-dimensionally space constructible functions such that

(1) $\lim_{i\to\infty} L_1(n_i)/L_2(n_i) = 0$, and

(2) $L_2(n_i)/logn_i > k\,(i=1,2,\cdots)$

for some increasing sequence of natural numbers $\{n_i\}$ and for some constant $k >$ O. Then there exists a language T such that $T \in \pounds[3\text{-}DTM^c(L_2(n))]$ but $T \notin \pounds[3\text{-}DTM^c(L_1(n))]$.

Proof. This lemma can be proved by using the *diagonalization* [7,8]. We will construct a 3-$DTM^c(L_2(n))$ A which accepts a language not accepted by any 3-DTM^c $(L_1(n))$.

Let A' be a 3-DTM^c which constructs the space function $L_2(n)$. If the set of input symbols of A' is Σ', then that of A is $\Sigma = \Sigma' \times \{0,1\}$. In short, the symbols, which mark each element of Σ' with the suffix 0, 1, are used. A first simulates A' withont paying attention to the suffix 0, 1 until A' halts. If A' does not halt, A also does not. Given some three-dimensional inlput tape $x \in \Sigma^{(3)}$ with $l_1(x) = l_2(x) = l_3(x) = n$, A can mark exactly $L_2(n)$ cells of the storage tape. After this action on x, if A is obliged to use more cells than $L_2(n)$ cells, then A will halt and reject x.

A takes notice of the suffixes of the input symbols of x, and systematically reads x from the first plane to the n-th plane, from the first column to the n-th column in a plane and from the first row to the n-th row in a column, as a binary number. If the binary number is j, A writes the code of the j-th 3-DTM^c M_j on the storage tape. Therefore, given a sufficiently large three-dimensional input tape on which the binary number j is written, A can write the code of M_j by using at most $L_2(n_i)$ cells of the storage tape.

If A can write the code of M_j, then A simulates the action of M_j on the input of A by using the code. If M_j happens to be the $L_1(n)$ space- bounded 3-DTM^c, A needs $cL_1(n)$ cells of the storage tape to simuilate M_j, where $c > 0$ is a constant depending on the number of storage tape symbols of M_j. By the way, condition (1) holds for the sequence $\{n_i\}$, so A can simulate M_j if a suitable three-dimensional input tape x with $l_1(x) = l_2(x) = l_3(x) = n_i$ is given for sufficiently large i. At every step of simulation, A checks whether or not M_j accepts the three- dimensional input tape. If M_j accepts, then A halts without accepting. If M_j halts without accepting the input word, then A

accepts and halts. Moreover, A counts the number of steps of M_j by using the other track of the storage tape to check whether or not M_j enters a loop. Let s and t be the numbers of states (of the finite control) and storage tape symbols of M_j, respectively. Let $c(n_i)$ be the number of possible configurations of M_j on the tapes of sidelength n_i. Then, we get the inequality

$$c(n_i) \leq sn_i^3 L_1(n_i) t^{L_1(n_i)}.$$

If M_j does not halt within $c(n_i)$ steps, then A can conclude that M_j is looping. Here, let r be the number of symbols on the track to use for the purpose of counting the number of steps of M_j. Then, A can count up to the number $r^{L_2(n)}$. If r satisfies $\log r > 3/k$, it follows from conditions (1) and (2) that

$$lim_{i \to \infty} sn_i^3 L_1(n_i) t^{L_1(n_i)} / r^{L_2(n_i)} = 0.$$

Therefore, for sufficiently large i, A can also check whether or not M_j is looping in this case. If M_j loops, then A accepts the input tape and halts. Suppose that an $L_1(n)$ space -bounded 3-DTM^c B accepts the set T accepted by A which is constructed as mentioned above. Then, for sufficiently large i, if a three-dimensional input tape x with $l_1(x) = l_2(x) = l_3(x) = n_i$ whose binary number is the number of B is given as the input of B, then a contradiction occurs. Thus, T is not accepted by any $L_1(n)$ space -bounded 3-DTM^c. *Q.E.D.*

Recently, it was shown in [9,20] that, for each space construtible function $L(n) \geq \log n$, the class of sets accepted by $L(n)$ space-bounded one- dimensional nondeterministic Turing machines is closed under complementation. This result can be extended to the three-dimensional case. By using these facts, we can extend Theorem 3.1 to the nondeterministic case [5]. That is, we have

Theorem 3.2. Let $L_1(n)$ and $L_2(n)$ be any three-dimensionally space constructible functions such that

(1) $\lim_{i \to \infty} L_1(n_i)/L_2(n_i) = 0$, and

(2) $L_2(n_i)/logn_i > k\,(i = 1,2,\cdots)$

for some increasing sequence of natural numbers $\{n_i\}$ and for some constant $k > 0$. Then there exists a language T such that $T \in £[3\text{-}NTM^c(L_2(n))]$ but $T \notin £[3\text{-}NTM^c(L_1(n))]$.

4 Hierarchy Theorem for $< \log n$

Next, we consider the case that the space function $L(n)$ grows more slowly than the order of log n.

As a preliminary to get the desired results, we need the idea of chunks. The idea of a *chunk* was introduced by Blum and Hewitt to investigate the acceptabilities of two -dimensional finite antomata [1]. In this paper, we expand this idea for 3-DTM^c's.

Definition 4.1. Let Σ be a finite set of symbols, and let $n \in N$. An element of $\Sigma^{(n,n,n)}$ is called an *n-chunk* over Σ. Let c_1, c_2 be two n-chunks, and let M be a 3-DTM^c. If the numbers of states and storage tape symbols of M are s and t, respectively, then the number of possible storage states of M when M uses at most ℓ cells of the storage tape is $s\ell t^{\ell}$. Thus, when M uses at most ℓ cells of the storage tape, the number of ways for M to enter an n-chunk is $6n^2slt^l$. For each case of $6n^2slt^l$ ways for M to enter an n-chnuk, there are at most $(6n^2slt^l+s+1)$ ways for M to exit the chunk ($6n^2slt^l$ ways of exitting the chunk, s ways of halting in the chunk and 1 way of looping). Two n-chunks c_1 and c_2 are said to be *(M,l)-equivalent,* if the entrance-exit relations of M to c_1 and c_2 when M uses at most ℓ cells of the storage tape are the same for all the ways of entering the n-chunks. Then, we can easily get the following lemma.

Lemma 4.1. Let M be a 3-DTM^c. There are at most

$$(u+s+1)^u$$

(M,l)-equivalence classes of n-chunks, where $u = 6n^2slt^l$, s is the number of states of the finite control of M, and t is the number of storage tape symbols of M.

We can also get the following lemma.

Lemma 4.2. Let M be a 3-DTM^c. Let $\{(x_i,l_i)\}$ be a sequence of pairs of nonnegative integers that satisfies

(1) $\lim_{i\to\infty} l_i/\log x_i=0$, and

(2) $\lim_{i\to\infty} x_i=\infty$,

and let $\{D_i\}$ ($D_i \subseteq \Sigma^{(x_i,x_i,x_i)}$) be a sequence of sets of chunks that satisfies

(3) $|D_i| > r^{x_i^3}$

for some constant r>1. Then, there exists some integer $i_o > 0$ such that there exist two different (M,l_i)-equivalent x_i-chunks $c_i,c_i' \in D_i$ for every $i > i_o$.

Proof, Let s and t be the numbers of states and storag tape symbols of M, respectively. From Lemma 4.1, there are at most $(u_i+s+1)^{u_i}$ (M,l_i)-equivalence classes of x_i-chunks in D_i, where $u_i=6n_i^2sl_it^{l_i}$. Here, we denote

$f_i=(u_i+s+1)^{u_i}/r^{x_i^3}$.

From conditions (1) and (2), we can derive $lim_i\to\infty f_i=0$. Then, from condition (3), there must exist some io such that for every $i>i_0$,

$$|D_i| \geq r^{x_i^3} > (u_i+s+1)^{u_i},$$

and thus, there are two different (M_i,l_i)-equivalent x_i-chunks $c_i,c_i' \in D_i$. *Q.E.D.*

From Lemma 4.2, we can get the following theorem.

Theorem 4.1. Let $L_2(n)$ be three-dimensionally space constrctible function of 3-DTM^c. Suppose that

(1) $\lim_{i\to\infty} L_1(n_i)/L_2(n_i)=0$,

(2) $\lim_{i\to\infty} L_2(n_i)=\infty$, and

(3) $L_2(n_i) < k\log n_i (i=1,2,\cdots)$

for some increasing sequence of natural numbers $\{n_i\}$ and for some constant $k > 0$. Then, there exists a set T such that $T \in$ £[3-$DTM^c(L_2(n))$] but $T \notin$ £[3-DTM^c $(L_1(n))$].

Proof. We will construct a 3-DTM^c $(L_2(n))$ A which accepts the language T not accepted by any 3-$DTM^c(L_1(n))$. Let A' be a 3-DTM^c which constructs the space function $L_2(n)$. If the input alphabet of A' is Σ', then that of A is $\Sigma=\Sigma'\times\{0,1\}$. Let the mapping $\tilde{h}_1 : \Sigma^{(3)}\to\Sigma'^{(3)}$ ($\tilde{h}_2 : \Sigma^{(3)}\to\{0,1\}^{(3)}$) be the projection obtained by extending the mapping $h_1 : \Sigma\to\Sigma'$ ($h_2 : \Sigma\to\{0,1\}$, where $h_1((a,j))=a$ and $h_2((a,j))=j$ for any $(a,j) \in \Sigma=\Sigma'\times\{0,1\}$. If an input word $w \in \Sigma^{(3)}$ is given to A, A first simulates the movements of A' on the input $\tilde{h}_1(w)$ until it halts. Let l be the number of cells of the storage tape which A' has used during its simulation. If each sidelength of w is n, then $l \leq L_2(n)$. However, if an suitable input w is given, then $l = L_2(n)$.

Now we consider two *d-chunks* w_a and w_b, Where $d = 2^{\lceil \ell/k \rceil - 1}$, at the north-west corner of upper planes of w as shown in Fig.2. If the sidelength n of w happens to satisfy '$n = n_i$' for some i, we can take such d-chunks w_a, w_b on w due to condition (3), If not (that is, $n < 2d$), we cannot. Then, A halts without accepting the input w. Let $r = 2^{\lceil l/k \rceil + 1}$. Then d can be easily written on at most l cells of the storage tape using *r-ary* number. In order to check whether $n < 2d$ or not, we have only to move the input head along the first row on the top plane from north to south while subtracting d written on the storage tape from n one by one.

Next, A checks whether $\tilde{h}_2(w_a) = \tilde{h}_2(w_b)$ or not. A can easily do this by using l cells of the storage tape. Then, if $\tilde{h}_2(w_a) = \tilde{h}_2(w_b)$, A accepts w and halts. If not, A halts without accepting w.

Now let T be the language accepted by A which moves like the above, and we suppose that there exists a 3-$DTM^c(L_1(n))$ B which accepts T. Let $\{(d_i, l_i)\}$ be a sequence of pairs of nonnegative integers such that

$$d_i = 2^{\lceil L_2(n_i)/k \rceil - 1}, \text{ and}$$
$$l_i = L_1(n_i).$$

Let v_i be a cubic word in $\Sigma^{(n_i, n_i, n_i)}$ which makes A' use exactly $L_2(n_i)$ cells of the storage tape. Let v_{ia} and v_{ib} be two d_i-chunks taken on v_i at the same position as in Fig.2. Let $\{D_i\}$ be a sequence of sets of chunks such that

$$D_i = \{c \mid c \in \Sigma^{(d_i, d_i, d_i)} \text{ and } \tilde{h}_1((c) = v_{ia}\}.$$

Then, $\{(d_i, l_i)\}$ and $\{D_i\}$ satisfy conditions (1),(2),(3) of Lemma 4.2, and thus, there exists some integer $i_0 > 0$ such that there are two different (B, ℓ_i)-equivalent di-chunks c_i, $c_i' \in Di$ for every $i > i_0$. Here, for every $i > i_0$, we consider two cubic words $w_i, w_i' \in \Sigma^{(3)}$ that satisfy the following conditions : Let $w_{ia}, w_{ib}, w_{ia}', w_{ib}' \in \Sigma^{(d_i, d_i, d_i)}$ be d_i-chunks taken on w_i and w_i' at the same position as in Fig.2. w_i and w_i' are the same except d_i-chunks w_{ia} and w'_{ia} and they satisfy $\tilde{h}_1(w_i) = \tilde{h}_1(w_i') = v_i$, $\tilde{h}_2(w_{ib}) = \tilde{h}_2(w_{ib}') = \tilde{h}_2(c_i)$, $w_{ia} = c_i$, *and* $w_{ia}' = c_i'$.

Clearly, $w_i \in T$ and w_i is accepted by B. On the other hand, w_{ia} and w_{ia}' are (B, l_i)-equivalent. Thus, w_i' is also accepted by B, which contradicts the fact that $w_i' \notin T$. This completes the proof. *Q.E.D.*

We next present a nondeterministic version of Theorem 4.1.

We first give several preliminaries to get the desired result. For each $m \geq 2$ and each $1 \leq n \leq m - 1$, an (m,n)-chunk is a three-dimensional pattern over Σ as shown in Fig. 3 [13,16].

Let M be a 3-$NTM^c(l)$. Note that if the numbers of states and storage tape symbolts of M are s and t, respectively, then the number of possible storage states of M is slt^l. Let Σ be the input alphabet of M, and let # be the boundary symbol of M. For any (m, n)-chunk x over Σ, we denote by $x(\#)$ the pattern (obtained by surrounding x by #'s) as shown in Fig.4. Below, we will assume without loss of generality that M enters or exits the pattern $x(\#)$ only at the face designated by the bold line in Fig.4.

Thus, the number of the entrance points to $x(\#)$ [or the exit points from $x(\#)$] for

M is $4n+8$. We suppose that these entrance points (or exit points) are numbered 1, 2,···, $4n+8$ in an appropriate way. Let $P=\{1,2,\cdots,4n+8\}$ be the set of these entrance points (or exit points). Let $C=\{q_1,q_2,\cdots,q_u\}$ be the set of possible storage states of M, where $u=slt^l$. For each $i \in P$ and $q \in C$, let $M_{(i,q)}(x(\#))$ be a subset of $P \times C \cup \{L\}$ which is defined as follows (L is a new symbol):

(1) $(j,p) \in M_{(i,q)}(x(\#))$

$\leftrightarrow$ when M enters the pattern $x(\#)$ in storage state q and at point i, it may eventually exit $x(\#)$ in storage state p and at point j.

(2) $L \in M_{(i,q)}(x(\#))$

$\leftrightarrow$ when M enters the pattern $x(\#)$ in storage state q and at point i, it may not exit $x(\#)$ at all.

Let x, y be any two (m,n)-chunks over Σ, we say that x and y are *M-equivalent* if for any $(i,q) \in P \times C$, $M_{(i,q)}(x(\#)) = M_{(i,q)}(y(\#))$. Thus, M cannot distinguish between two (m,n)-chunks that are M-equivalent. Clearly, M-equivalence is an equivalence relation on (m,n)-chunks, and we get the following lemma.

Lemma 4.3. Let M be a 3-$NTM^c(l)$. There are at most

$$(2^{(4n+8)u+1})^{(4n+8)u}$$

M-equivalence classes of (m,n)-chunks over Σ, where $u=slt^l$, s is the number of states of the finite control of M, and t is the number of storage tape symbols of M.

Proof. The proof is similar to that of Lemma 4.3 in [11]. *Q.E.D.*

We are now ready to prove the following theorem.

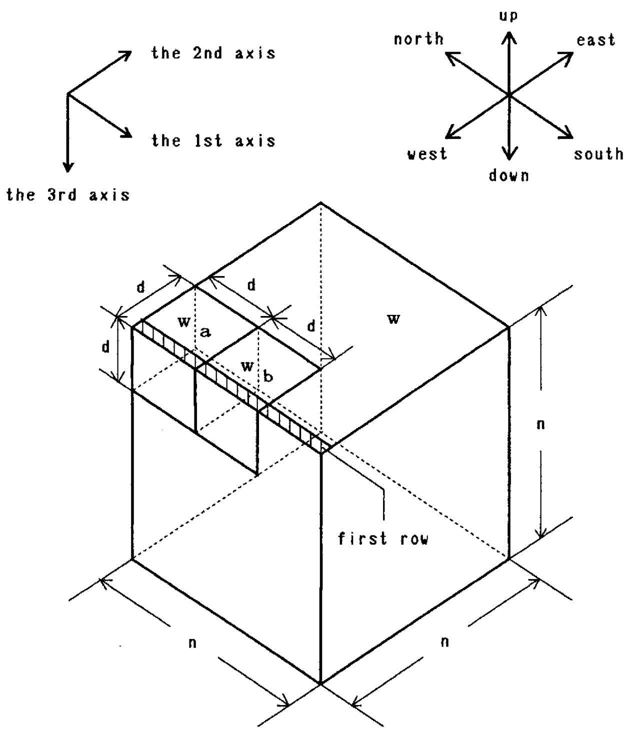

Fig.2. Two d-chunks w_a and w_b on an input w.

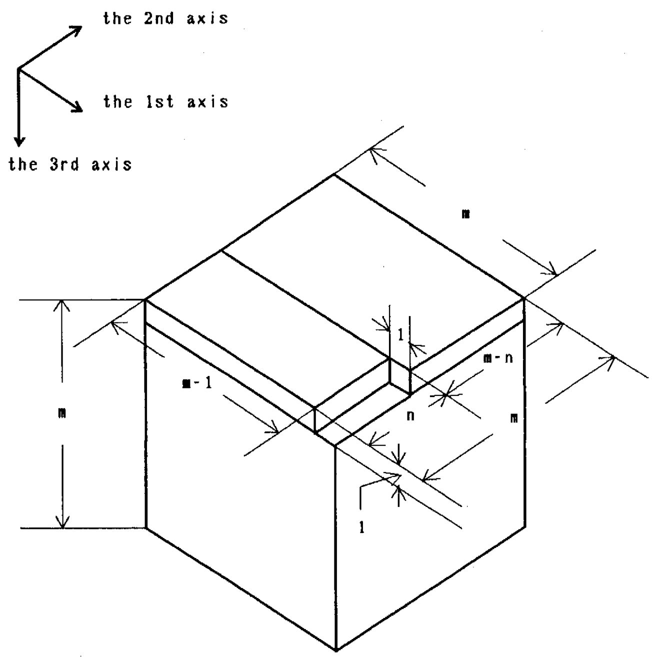

Fig.3. An (m,n)-chunk.

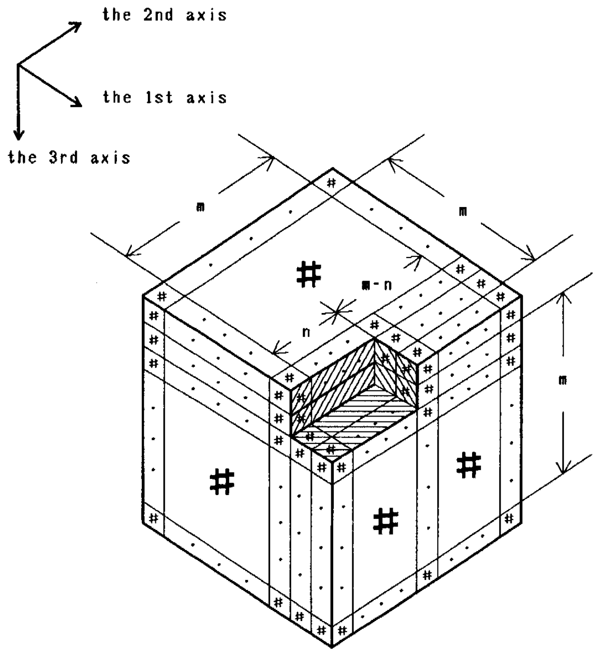

Fig.4. x(#).

Theorem 4.2. Let $L_2(n)$ be a three-dimensionally space constrnctible function such that $L_2(n) \leq \log n$. Suppose that $\lim_{n\to\infty} L_1(n)/L_2(n)=0$.Then there exists a set in £$[3\text{-}NTM^c(L_2(n))]$ (in fact, £$[3\text{-}DTM^c(L_2(n))]$), but not in £$[3\text{-}NTM^c(L_1(n))]$.

Proof. Let M be a 3-DTM^c which constructs the function L_2, and let $T[L_2,M]$ be the following set, which depends on L_2 and M. $T[L_2,M]=\{x \in (\Sigma\times\{0,1\})^{(3)} \mid \exists\, n\geq 2[l_1(x)=l_2(x)=l_3(x)=n\ \&\ \exists\, r(r\leq\ulcorner L_2(n)\urcorner)$ [when the tape $\tilde{h}_1(x)$ is presented to M, M

marks off r cells of the storage tape and then halts] & $\exists\, i(1 \leq i \leq n-1)$ $[\tilde{h}_2(x[(i,1,1),(i,r,1)]) = \tilde{h}_2(x[(n,1,1),(n,r,1)])]\}$, where Σ is the input alphabet of M, and $\tilde{h}_1$ $(\tilde{h}_2)$ is the projection which is obtained by extending the mapping $h_1 : \Sigma \times \{0,1\} \rightarrow \Sigma$ $(h_2 : \Sigma \times \{0,1\} \rightarrow \{0,1\})$ such that for any $c=(a,b) \in \Sigma \times \{0,1\}, h_1(c) = a\,(h_2(c) = b)$.

(1) : We first show that the set $T[L_2, M]$ is accepted by a 3-$DTM^c(L_2(n))$ M_1 which acts as follows. Suppose that a three-dimensional input talpe x with $l_1(x) = l_2(x) = l_3(x) = n\,(n>2)$ is presented to M_1. First, M_1 directly simulates the action of M on $\tilde{h}_1(x)$. (If M does not halt, then M_1 also does not halt, and will not accept x.) If M_1 finds out that M halts (in this case, note that M_1 has marked off at most $\lceil L_2(n) \rceil$ cells of the storage tape, because M constructs the function L_2), then M_1 stores the segment $\tilde{h}_2(x[n,1,1),(n,r,1)])$ on the storage tape, where r is the number of cells (of the storage tape) marked off by M_1. After that, M_1 simply checks that for some $i(1 \leq i \leq n-1), \tilde{h}_2(x[i,1,1),(i,r,1)])$ is identical with $\tilde{h}_2(x[(n,1,1),(n,r,1)])$ stored on the storage tape, and M_1 accepts the input x if this check is successfui. It will be obvious that $T[M_1] = T[L_2, M]$.

(2) : We next show that the set $T[L_2, M]$ is not in £$[3\text{-}NTM^c(L_1(n))]$. Suppose that there is a 3-$NTM^c(L_1(n))$ M_2 accepting $T[L_2, M]$, where $\lim_{n\to\infty}[L_1(n)/L_2(n)] = 0$ (note that $L_2(n) \leq \log n\,(n \geq 1)$. Let s and t be the numbers of states of the finite control and storage tape symbols of M_2, respectively. We assume without loss of generality that M_2 starts on position $(l_1(x),1,1)$ of x, and that when M_2 accepts an input tape x in $T[L_2, M]$, it halts on position $(l_1(x),1,1)$ of x (these assumptions are concerned with the shape of chunks described just before Lemma 4.3), and that M_2 never falls off an input tape out of the boundary symbol #. For each $n \geq 2$, let $z(n) \in \Sigma^{(3)}$ be a fixed tape such that (i) $l_1(z(n)) = l_2(z(n)) = l_3(z(n)) = n$ and (ii) when $z(n)$ is presented to M, it marks off exactly $\lceil L_2(n) \rceil$ cells of the storage tape and halts. (Note that for each $n \geq 2$, there exists such a tape $z(n)$ because M constructs the function L_2.) For cach $n \geq 2$, let

$$V(n) = \{x \in (\Sigma \times \{0,1\})^{(3)} \mid l_1(x) = l_2(x) = l_3(x) = n \;\&\; \tilde{h}_2(x[(1,1,1),(n, \lceil L_2(n) \rceil, 1)]) \in \{0,1\}^{(3)} \;\&\; (\text{the other part of } \tilde{h}_2(x) \text{ consists of 0's}) \;\&\; \tilde{h}_1(x) = z(n)\},$$

$$Y(n) = \{y \in \{0,1\}^{(3)} \mid l_1(y) = 1 \;\&\; l_2(y) = \lceil L_2(n) \rceil \;\&\; l_3(y) = 1\},$$

and

$$R(n) = \{row(x) \mid x \in V(n)\},$$

where for each x in $V(n)$, $row(x) = \{y \in Y(n) \mid y = \tilde{h}_2(x[(i,1,1),(i, \lceil L_2(n) \rceil, 1)])$ for some $i(1 \leq i \leq n-1)\}$. Since $|Y(n)| = 2^{\lceil L_2(n) \rceil}$, it follows that

$$|R(n)| = \begin{cases} \binom{2^{\lceil L_2(n) \rceil}}{1} + \cdots + \binom{2^{\lceil L_2(n) \rceil}}{n-1} & \text{if } 2^{\lceil L_2(n) \rceil} \geq n-1\,; \\ \binom{2^{\lceil L_2(n) \rceil}}{1} + \cdots + \binom{2^{\lceil L_2(n) \rceil}}{2^{\lceil L_2(n) \rceil}} = 2^{2^{\lceil L_2(n) \rceil}} - 1, & \text{otherwise.} \end{cases}$$

Note that $B = \{p \mid$ for some x in $V(n)$, P is the pattern obtained from x by cutting the part $x[(n,1,1),(n, \lceil L_2(n) \rceil, 1)$ off$\}$ is a set of all $(n, \lceil L_2(n) \rceil)$-chunks over $\Sigma \times \{0,1\}$. Since M_2 can use at most $L_1(n)$ cells of the storage tape when M_2 reads a tape in $V(n)$, from Lemma 4.3, there are at most

$$E(n) = (2^{(4\lceil L_2(n)\rceil+8)u[n]+1})^{(4\lceil L_2(n)\rceil+8)u[n]}$$

M_2-equivalence classes of $(n,\lceil L_2(n)\rceil)$-chunks (over $\Sigma\times\{0,1\}$) in B, where $u[n] = sL_1(n)t^{L_1(n)}$. We denote these M_2-equivalence classes by $C_1, C_2, \cdots, C_{E(n)}$. Since $L_2(n) \leq \log n$ and $\lim_{n\to\infty}[L_1(n)/L_2(n)] = 0$ (by assumption), it follows that for large n, $|R(n)| > E(n)$. For such n, there must be some $Q, Q'(Q = Q')$ in $R(n)$ and some $C_i(1 \leq i \leq E(n))$ such that the following statement holds :

"There exist two tapes x,y in $V(n)$ such that

(i) $x[(n,1,1),(n,\lceil L_2(n)\rceil,1)] = y[(n,1,1),(n,\lceil L_2(n)\rceil,1)]$ and $\tilde{h}_2(x[(n,1,1),(n,\lceil L_2(n)\rceil,1)]) = h_2(y[(n,1,1),(n,\lceil L_2(n)\rceil,1)]) = \rho$ for some ρ in Q but not in Q',

(ii) $row(x) = Q$ and $row(y) = Q'$, and

(iii) both p_x and p_y are in C_i, where $p_x(p_y)$ is the $(n,\lceil L_2(n)\rceil)$-chunk over $\Sigma\times\{0,1\}$ obtained from x (from y) by cutting the part $x[(n,1,1),(n,\lceil L_2(n)\rceil,1)]$ (the part $y[(n,1,1),(n,\lceil L_2(n)\rceil,1)]$) off". As is easily seen, x is in $T[L_2,M]$, and so x is accepted by M_2. It follows that y is also accepted by M_2, which is a contradiction. (Note that y is not in $T[L_2,M]$. This completes the proof of '$T[L_2,M] \notin £[3\text{-}NTM^c(L_1(n))]$.' Q.E.D.

Let $\log^{(k)} n$ be defined in the following way :

$$\log^{(0)} n = n,$$
$$\log^{(k)} n = \log(\log^{(k-1)} n,), \text{ for } k > 1.$$

It is shown in [18] that for each $k > 1$, the function $\log^{(k)} n$ is three-dimensionally space constructible (in fact, three-dimensionally fully space constructible). From this and Theorem 4.2, we have the following corollary.

Corollary 4.1. For any constant $c > 0$, each $k \in$ N, and each $X \in \{D,N\}$, $£[3\text{-}XFA^c] = £[3\text{-}XTM^c(c)] \subsetneq \cdots \subsetneq £[3\text{-}XTM^c(\log^{(k+1)} n)] \subsetneq £[3\text{-}XTM^c(\log^{(k)} n)] \cdots$.

5 Conclusion

In this paper, we have investigated the space complexity hierarchies of $L(n)$ space-bounded three-dimensional deterministic or nondeterministic Turing machines whose inputs are restricted to cubic ones, and we have shown that there exists an infinite hicrarchy of acceptabilities among these machines.

It will be interesting to investigate whether or not there exists an infinite hierarchy of acceptabilities for space-bounded three-dimensional alternating Turing machine [16].

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