Shock Wave Propagation Analysis in Coordinated Signal Systems by Kinematic Wave Theory

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Abstract

This paper describes a model which, by applying the kinematic wave theory, draws the trajectories of shock waves propagating from link to link of coordinated traffic signal systems. A BASIC program is developed to draw the trajectories on the time-space diagrams on the personal computer display in turn under arbitrary street, traffic and signal conditions. Using this model, the delay in each link and inflow traffic from main and cross streets at each intersection can easily be calculated. This paper expands the model by Michalopoulos et al., which target is confined to a single signalized link, so as to be able to apply to arterial streets made up of a given number of links.

1. Introduction

In coordinated traffic signal systems under the oversaturated or near-saturated traffic conditions, it is important that the control is optimized, considering the existence of queues at intersections and how they change dynamically. This requires building a highly operable traffic flow model that expresses the dynamic behavior of queues and the propagation of shock waves. One such model is the model based on the kinematic wave theory. In this research we have formulated, according to wave theory, a model that can express the trajectories of traffic waves in a unified way that propagate through links of coordinated signal systems.

Much research has been done in wave theory since Lighthill and Whithman[1] for the streets, Stephanopoulos et al as well as Michalopoulos et al have proposed analytical models that focus on single link,[2][3] and Ikenoue has attempted to develop a model that can be applied to oversaturated multiple links.[4][5] But the model of Michalopoulos et al covers only single link and unless modified cannot be applied to multiple links. And the Ikenoue model, although it can be applied to multiple links, handles only the oversaturated case and unless modified cannot be applied to the undersaturated case. Fukuyama et al do an approximative analysis of unsteady states for multiple links.[6][7] And for expressways, Okutani and Inoue do an analysis concerning congestion due to traffic accidents.[8][9]

Hisai and Tamura, following mainly the research of Michalopoulos et al and of

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Ikenoue, have arranged wave propagation models, have rebuilt an analytical model that can express the phenomenon of shock wave propagation through links of coordinated signal systems consisting of an arbitrary number of links, and have applied this model to undersaturated and oversaturated steady-state coordinated signal systems.[10) In this research, we have further developed the research by Hisai and Tamura and have made it possible to

(1) handle both the case when there is left- and right-turn inflow from cross streets and the case when there is no such turning traffic, and

(2) handle both the case when the most-upstream intersection of study section is oversaturated and the case when it is undersaturated.

Specifically, using a personal computer the behavior of queues at intersections and the propagation of shock waves from links to links are computed for each link and displayed graphically on a time-space diagram in an easy-to-understand visual way.

2. Study section and the assumptions made in building the model

The streets considered in this research are the coordinated control streets consisting of an arbitrary number of links, with the streets as a whole either oversaturated or undersaturated; we do not consider streets in which both oversaturated and undersaturated links coexist.

The traffic flow is assumed to be a compressible fluid, and the relationship between traffic density and speed is assumed to be given by the Greenshields formula, which is linear. Because of this, the traffic volume-density curve is quadratic

The inflow traffic from upstream to downstream at an intersection consists of inflow due to through traffic from upstream of the main street, and inflow due to left and right turns into the main street from the cross streets. However, the inflow from cross streets is assumed to be a uniform flow of density $K_2(K_2 \geqq 0)$ and is assumed to be the same for all intersections. If there is no inflow, $K_2=0$.

Michalopoulos et al and Ikenoue make the assumption that the inflow from an upstream intersection of the main street during green time is a uniform flow of density $K_1(K_1>K_2)$,[3)5)] but here, because our purpose is to handle traffic flow between multiple links, we assume that the rate of through traffic of the main street is 100%, and that when the signal changes from red to green, the traffic density instantaneously changes from K_j(jam density) to K_m(critical density) upstream of the intersection and from K_2 to K_m downstream of the intersection, as shown in Fig.1(a).[10)] Thus at the beginning of green signal, instantaneously there is an arbitrary density in the range $K_j \rightarrow K_m$ upstream of the stop line and in the range $K_2 \rightarrow K_m$ downstream of it. Thus iso-density lines that represent density in the range $K_2 \rightarrow K_j$ can be drawn on the time-space diagram from the beginning of green time, as shown in Fig.1(b). In the diagram, LINE1 is the starting wave, and LINE4 is the iso-density line for K_2 that radiates from the point when green starts. As shown in Fig.1(c), the density at points A → B → C before and after the stop line changes linealy $K_2 \rightarrow K_m \rightarrow K_j$.

The density K_S at the stop line is equal to K_m, and this can be confirmed as follows:

$$K_S=\frac{h(K_2)t}{h(K_2)t-h(K_j)t}(K_j-K_2)+K_2. \qquad (1)$$

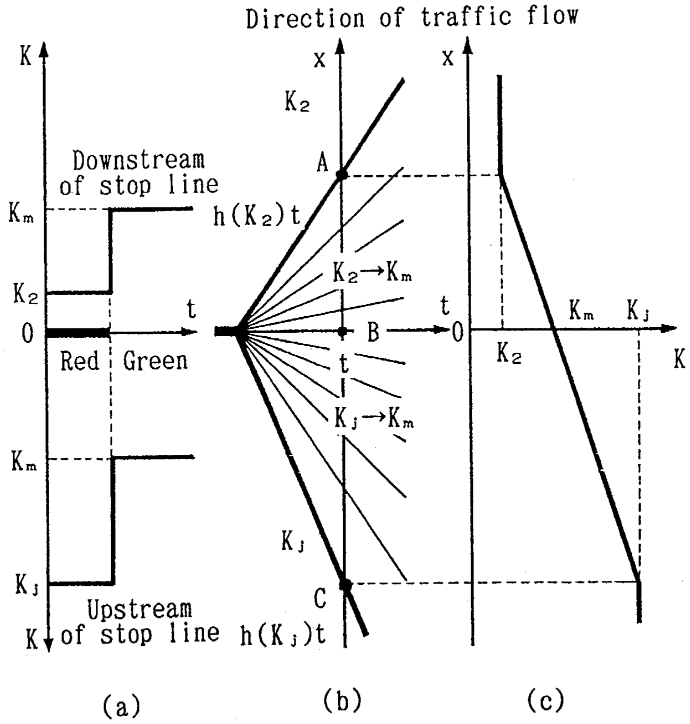

Fig. 1 Density on the main street near stop line at the beginning of green time

Here h(K) is the slope of the tangent to the volume-density curve, that is, h(K) is the propagation speed of the density K.

$$h(K)=u_f\left(1-\frac{2K}{K_j}\right) \tag{2}$$

and therefore

$$\begin{aligned} Ks &= \frac{u_f(1-2K_2/K_j)}{u_f(1-2K_2/K_j)-u_f(1-2K_j/K_j)}(K_j-K_2)+K_2 \\ &= \frac{1}{2}K_j=K_m. \end{aligned} \tag{3}$$

Therefore the density at the stop line is K_m.

The calculation can be done under the conditions given arbitrarily as follows:

1) street conditions such as number of links, link lengths, etc.

2) traffic conditions such as the critical density, jam density, free speed, arrival density at the most-upstream intersection, inflow density from cross streets, etc.

3) signal conditions such as cycle lengths, green times, offsets, etc. Thus it is possible to study how the control parameters affect the traffic flow under various street and traffic conditions.

3. Theory of shock wave propagation

A shock wave is the point where the traffic density changes discontinuously on a street. The propagation speed u_W of a shock wave is given by the following equation:

$$u_W = \frac{K_d u_d - K_u u_u}{K_d - K_u} \tag{4}$$

where K_u : traffic density on the upstream side

u_u : speed of the traffic flow on the upstream side

K_d : traffic density on the downstream side

u_d : speed of the traffic flow on the downstream side.

If it is assumed that the relationship between the speed u and the density K is given by

$$u = u_f\left(1 - \frac{K}{K_j}\right) \tag{5}$$

where u_f is the free speed, the propagation speed is given by the following equation:

$$u_W = u_f \left\{1 - \left(\frac{K_d}{K_j} + \frac{K_u}{K_j}\right)\right\}. \tag{6}$$

Equation (6) can be written as

$$\frac{dx}{dt} = \frac{u_f}{K_j}(K_j - K_d - K_u) \tag{7}$$

where x is the distance measured in the direction of traffic flow and t is time.

Next, we present the basic approach of Stephanopoulos et al for a solution for the shock wave propagation trajectories.[2] First we consider the continuity equation

$$\frac{\partial K}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{8}$$

where the traffic volume q is given by $q = Ku$. If we assume that $u = f(K)$, equation (8) becomes

$$\frac{\partial K}{\partial t} + \left[f(K) + K\frac{df}{dk}\right]\frac{\partial K}{\partial x} = 0. \tag{9}$$

Because K is a function of t and x, the change of the total differential of K becomes

$$\frac{dx}{dt} = \frac{\partial K}{\partial t} + \frac{dx}{dt}\frac{\partial K}{\partial x}. \tag{10}$$

Comparing equation (9) with equation (10) shows[11] that the change dK/dt of K is 0 at points that move at speed

$$\frac{dx}{dt} = f(K) + K\frac{df}{dK} \equiv h(K). \tag{11}$$

That is, on the curve represented by equation (11) the density is constant, and therefore equation (11) is for an iso-density line. Equation (11) is a curve that expresses the propagation speed of a turbulence wave which is equal to the slope dq/dK of the tangent of the $q-K$ curve , and this is called a characteristic curve (in this case, a straight line)in wave differential equations. When two characteristic lines intersect in the $t-x$ plane, then at the intersecting point there are two different densities. In other words, at that intersecting point the density changes discontinuously. Therefore, this intersecting point expresses a shock wave.

In the time-space diagram there are only five kinds of density regions: the K_1 region, the K_2 region, the K_j region, the $K_j \rightarrow K_m$ transition region, and the $K_2 \rightarrow K_m$ transition region. These density regions are determined by the arrival conditions at the most -upstream intersection and the boundary conditions at the intersections. If different density regions overlap, which leads to the generation of shock waves.

4. Solution for shock wave propagation trajectories

Michalopoulos et al determine analytically the propagation trajectories of shock waves assuming single link.[3] Here we determine general equations corresponding to multiple links. First, if the density is constant both upstream and downstream of the shock wave, it can be determined by simply integrating equation (7). The integration constant is determined by the starting-point coordinates (t_0,x_0) of the shock wave. If either the upstream, downstream, or both side of the shock wave is in the density transition region, the trajectory is determined by solving simultaneous differential equations. This solution is illustrated by WAVE60.

With shock wave WAVE60, the density K_u on the upstream side is K_2, a constant, and the density on the downstream side is in the $K_2 \rightarrow K_m$ transition region. The density K_d is determined as follows. That is, taking K_d as the density of the characteristic line, since its slope is $h(K_d)$, the equation for the characteristic line is as follows:

$$x=h(K_d)(t-t_{f1})+D_1 \tag{12}$$

where (t_{f1},D_1) are the coordinates of the beginning of green time (radiation point of the characteristic line) of a certain intersection.

Substituting $h(K_d)=u_f(1-2K_d/K_j)$, we get

$$x=u_f\left(1-\frac{2K_d}{K_j}\right)(t-t_{f1})+D_1. \tag{13}$$

Solving for K_d, we get

$$K_d=\frac{Kj}{2}-\frac{K_j(x-D_1)}{2u_f(t-t_{f1})}. \tag{14}$$

Substituting the above equation and $K_u=K_2$ into equation (7) and rearranging, we get

$$\frac{dx}{dt}=\frac{x-D_1}{2(t-t_{f1})}+\frac{1}{2}h(K_2). \tag{15}$$

Setting

$$w=(x-D_1)/(t-t_{f1}) \tag{16}$$

and solving this homogeneous differential equation, we get the equation for WAVE60. The derivation for the other shock waves is similar. The results are listed in Table 1.

There are 12 types of shock waves and characteristic lines, and when they are classified by content, we get the following.

A. queue building waves････････････QBW2, QBW3
B. queue dissipation waves･･･････QDW2, QDW3
C. shock waves toward upstream･･････WAVE20, WAVE10
D. shock waves toward downstream････WAVE5, WAVE60, WAVE70, WAVE8
E. characteristic lines･････LINE1, LINE4

QBW2, QBW3, QDW2, and QDW3 are shock waves representing the tail of a queue of vehicles. The C-group shock waves are found only on oversaturated streets, and the D-group shock waves are found only on undersaturated streets.

WAVE10 is the same as what Ikenoue proposed, WAVE8 is newly proposed in this research, and the other shock waves correspond to single-link shock waves proposed by Michalopoulos et al.

Table 1 Equations of shock waves and characteristic lines

Name	Generalized equation
LINE1	$x=x_0-u_f(t-t_0)$
WAVE20	$x=z_5(t-t_{f3})^{1/2}-u_f(t-t_{f3})+D_3$ $z_5=[u_f+\frac{x_0-D_3}{t_0-t_{f3}}](t_0-t_{f3})^{1/2}$
WAVE10	$x=z_3(t-t_{f1})^{1/2}(t-t_{f3})^{1/2}+\frac{D_3(t-t_{f1})}{t_{f3}-t_{f1}}+\frac{D_1(t-t_{f3})}{t_{f1}-t_{f3}}$ $z_3=\frac{x_0-\frac{D_3(t_0-t_{f1})}{t_{f3}-t_{f1}}-\frac{D_1(t_0-t_{f3})}{t_{f1}-t_{f3}}}{(t_0-t_{f1})^{1/2}(t_0-t_{f3})^{1/2}}$ if $t_0=t_{f1}, x_0=D_1$ or $t_0=t_{f3}, x_0=D_3$, $x=\frac{D_3(t-t_{f1})}{t_{f3}-t_{f1}}+\frac{D_1(t-t_{f3})}{t_{f1}-t_{f3}}$ if $t_{f1}=t_{f3}$, $x=z_4(t-t_{f3})+\frac{1}{2}(D_3+D_1)$ $z_4=[x_0-\frac{1}{2}(D_3+D_1)]/(t_0-t_{f3})$
LINE4	$x=x_0+h(K_2)(t-t_0)$
WAVE5	$x=x_0+h(K_1,K_2)(t-t_0)$
WAVE60	$x=z_6(t-t_{f1})^{1/2}+h(K_2)(t-t_{f1})+D_1$ $z_6=-[h(K_2)-\frac{x_0-D_1}{t_0-t_{f1}}](t_0-t_{f1})^{1/2}$
WAVE70	the same as WAVE10
WAVE8	$x=z_7(t-t_{f1})^{1/2}+h(K_1)(t-t_{f1})+D_1$ $z_7=-[h(K_1)-\frac{x_0-D_1}{t_0-t_{f1}}](t_0-t_{f1})^{1/2}$
QBW2	$x=x_0-\frac{u_fK_2(t-t_0)}{K_J}$
QBW3	$x=z_1(t-t_{f1})^{1/2}-u_f(t-t_{f1})+D_1$ $z_1=[u_f+\frac{x_0-D_1}{t_0-t_{f1}}](t_0-t_{f1})^{1/2}$
QDW2	$x=z_2(t-t_{f3})^{1/2}+h(K_2)(t-t_{f3})+D_3$ $z_2=-[h(K_2)-\frac{x_0-D_3}{t_0-t_{f3}}](t_0-t_{f3})^{1/2}$
QDW3	the same as WAVE10

5. Shock wave matrix

When a shock wave intersects with another shock wave or with a characteristic line on the time-space diagram, the original shock waves vanish and a new shock wave originates from the intersecting point. This has been summarized by Michalopoulos et al as a single-link shock wave matrix.[3] In our research we similarly created multiple -link shock wave matrix; they are listed in Table 2.

6. Examples of computation

Using the above-determined equations for shock waves and characteristic lines and shock wave matrix, we determine a wave structural diagram for coordinated signal streets. In this case, the inflow traffic of the most-upstream intersection of the study section is arranged so as to be able to handle both the undersaturated case and the oversaturated case.

In the undersaturated case, traffic density K_1 arrives at the most-upstream intersection, and the wave structural diagram is as shown in Fig.2. That is, a queue building wave QBW2, a starting wave LINE1, and a queue dissipation wave QDW2 are produced upstream of the stop line, and a characteristic line LINE4 and shock wave WAVE5 are produced downstream. After QDW2 moves downstream of the intersec-

Table 2 Shock wave matrix

Upstream \ Downstream	LINE1	WAVE20	WAVE10	LINE4	WAVE5	WAVE60	WAVE70	WAVE8	QBW2	QBW3	QDW2	QDW3
LINE1												
WAVE20	WAVE10											
WAVE10		WAVE20	WAVE10									
LINE4						WAVE70			QBW3		QDW3	
WAVE5								WAVE60	QBW2		QDW2	
WAVE60							WAVE60			QBW2		QDW2
WAVE70							WAVE70			QBW3		QDW3
WAVE8							WAVE8			QBW2		QDW2
QBW2	QDW2											
QBW3	QDW3											
QDW2		QBW2	QDW2									
QDW3		QBW3	QDW3									

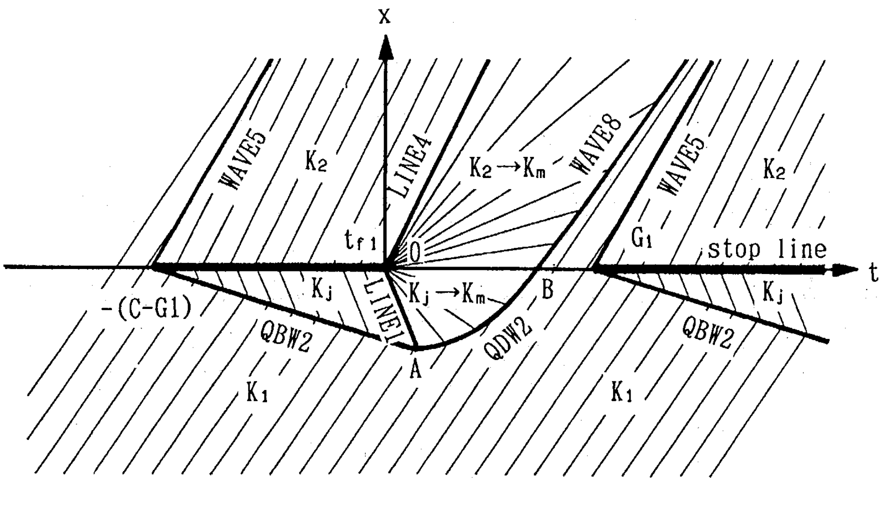

Fig. 2 Shock waves near the most-upstream intersection under the condition of undersaturation

tion, its name is changed to shock wave WAVE8. Here we determine the point B at which the saturated flow terminates. To do this, we first determine the point A where QBW2 and LINE1 intersect. For this, we substitute $x_0=0$, $t_0=-(C-G_1)$ into QBW2 in Table 1, substitute $x_0=0$, $t_0=0$ into LINE1, and solve the simultaneous equations to get

$$t_A=\frac{K_1(C-G_1)}{K_j-K_1},\ x_A=-\frac{u_f K_1(C-G_1)}{K_j-K_1}. \tag{17}$$

Then we substitute (t_A, x_A) into (t_0, x_0) of QDW2 in Table 1, set $x=0$, and solve to determine the time t_B at point B as

$$t_B=\left[\frac{z_2}{h(K_1)}\right]^2=\frac{q_1(C-G_1)}{(q_m-q_1)}. \tag{18}$$

In the oversaturated case, the wave structural diagram of the most-upstream intersection is as shown in Fig.3. That is, shock wave WAVE20 and characteristic line LINE1 are generated upstream of the stop line, while characteristic line LINE4 and shock wave WAVE60 are generated downstream. In this case the inflow traffic of the main street for the green time is a saturated flow.

A wave structural diagram is drawn graphically as follows: time is plotted along the horizontal axis, distance is plotted along the vertical axis, and the direction of traffic flow is taken upward. On an undersaturated street the computations are made toward downstream from the most-upstream link, and on an oversaturated street the computations are made toward upstream from the most-downstream link. For the wave structure, a computation is made at each link. When doing so, the shock wave that propagates to the next link and shock wave that newly originates at the next link are pre-determined.

On a wave structual diagram, first the coordinate axes of a time-space diagram are drawn, then the trajectories of the shock waves are drawn. The distance coordinates of the shock waves are computed, and plotted on the time-space diagram, for every small time interval $\triangle t$ according to the equations previously determined analytically as a function of the time t as in Table 1. The wave structural diagram is drawn by repeating this over the cycle length.

Fig.4 and Fig.5 are examples of computation for undersaturated streets. Fig.4 is for the case when there is left- and right-turn inflow($K_2>0$), and Fig.5 is for the case when

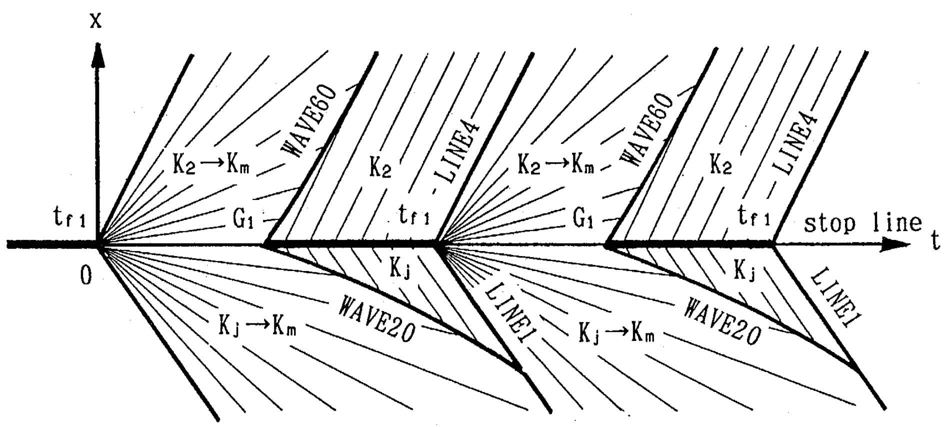

Fig. 3 Shock waves near the most-upstream intersection under the condition of oversaturation

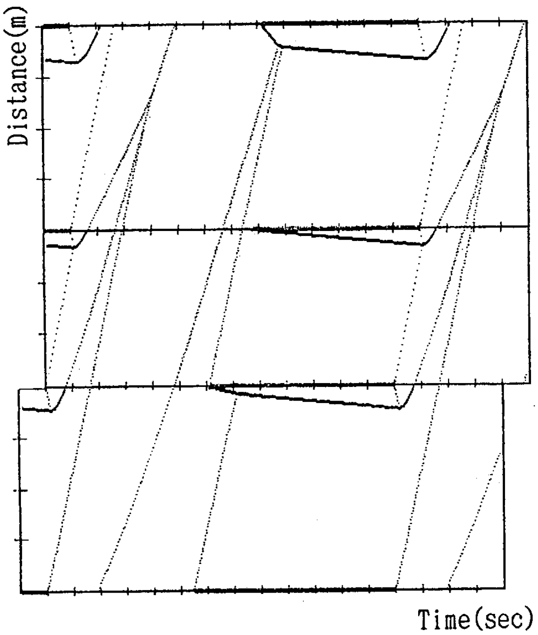

Fig. 4 Wave structure of undersaturated street in case of K2>0

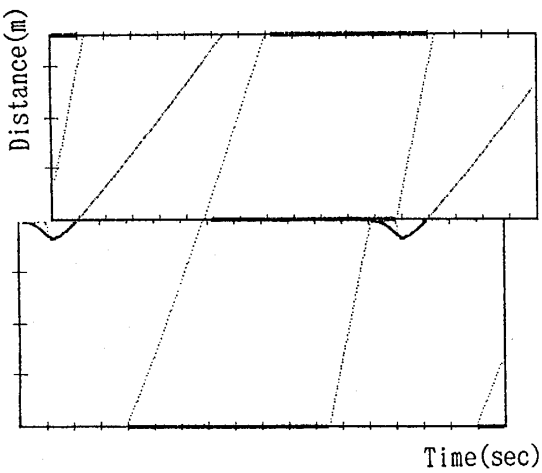

Fig. 5 Wave structure of undersaturated street in case of K2=0

there is no left- or right-turn inflow($K_2=0$). Fig.6 is an example for an oversaturated street. These diagrams were made by copying graphically displayed wave structural diagrams and joining them together.

The computation conditions used here were: free speed $u_f=12.5$(m/sec), jam density $K_j=0.16$(veh/m), saturation flow rate $q_m=0.5$(veh/sec).

7. Summary

In this research we have formulated, based on wave theory, a model that can express the behavior of queues in coordinated signal systems as well as the traffic

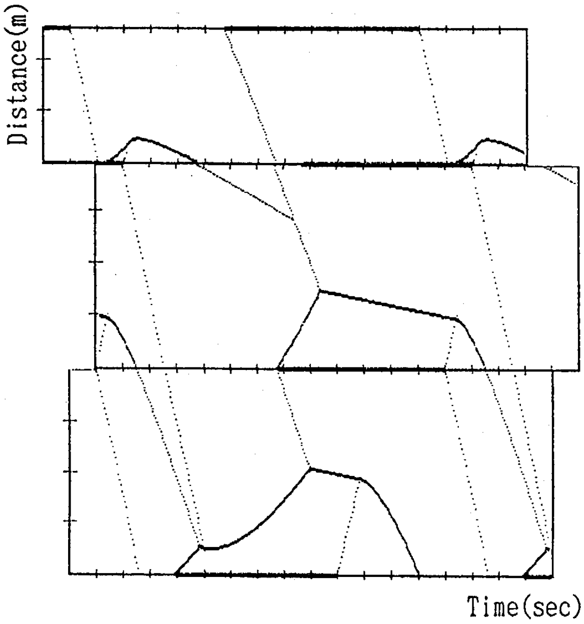

Fig. 6 Wave structure of oversaturated street

wave motion that propagates through links, and we have portrayed in a time-space diagram on a personal computer screen the phenomenon in which each shock wave is propagated successively from link to link. This has made it possible to visually grasp the phenomenon of shock wave propagation under various street, traffic, and signal conditions, including both the undersaturated and oversaturated cases. Using this model, it will also be possible to study optimization of signal control. In the future we would like to make it possible to also handle cases in which there is left- and right-turn outflow and cases of other density-speed relationships.

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