One Method for Problem-Solving by the form of Quantum Mechanics

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Abstract

For the problem-solving in the state-space, this article investigates the effectiveness of the method which makes use of quantum mechanics. First this article restricts the state-space to vector-space, and shows that quantum mechanics has more general property than state-space approach in the vector space. For the general problem, this article then fromalizes the relation of between problem and solution by the method of quantum mechanics, and provides the urgent subject. Finally, this article provides the laws which exist under the conditions of linearity and Hermiticity, and investigates for the key to the next step.

1. Introduction

In the past it has being done various trial for the problem-solving in the field of Artificial Intelligence. In these trial, it has played large part that is cold state-space. There are many expressions of state-space. And it is possible to express as vector many of them. Then it is considered the concept of operator against vector.

By the way, it can be seen another theories which use vector and operator. One is so cold system theory. And in physics, there is quantum mechanics (abbreviated by Q.M.) that is strictly formalized by von Neumann. Thus these theories have similarity, but there is one difference between Q.M. and other theories. In other theories, operator is only means, but in Q.M. it is object itself. Then it rised one problem what vector means in Q.M. But it has been 60 years, and it has never found out the experimental data which deny Q.M., then there is no room for doubt that Q.M. is collect as physical law. There is possibility that Q.M. has some universality. Here, holding the mathematical form, when expand its object from physical phenomena to general problem, it is interesting to consider what property the theory has.

2. Basic Consideraton

There are a lot of forms which express state-space. This article restricts it vector-space. Therefore its object is restricted which is able to be expressed numerically its conditions for the problem-solving and solutions of problem. And it can be dealed with similarly which can be transformed to vector that is in another forms.

Frist we compare state-space approach and Q.M. on the formal side. In the state-

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space approach, if the vector which expresses initial condition is $\mathbf{x}$, and the vector which expresses final conditiion is $\mathbf{g}$, it is subject to find out the operator which fills following equation.

$$A\mathbf{x}=\mathbf{g} \tag{1}$$

On the other side in Q.M., it is similar on the point that uses vector and its operator, but their roles are different. Against the operator B which expresses one physical value, by solving eigenvalue equation of B

$$B\Psi_n=b_n\Psi_n \tag{2}$$

it gets eigenvalue b_n and eigenvector Ψ_n. On the equation, eigenvalue b_n is the value which can be actually, and eigenvector Ψ_n is the state which is attendent on the eigenvalue.

Now, by the view of problem-solving, we consider of the formal difference between both theories. First I set up problem G against initial condition which is expressed by the vector $\mathbf{x}$. $\mathbf{g}_n$ is the solution of G, and it exists plural generally. As the initial condition $\mathbf{x}$ is included by the problem G, $\mathbf{x}$ can be expressed $G(\mathbf{x})$. In Eq.(1), operator A which shifts initial condition $\mathbf{x}$ to $\mathbf{g}_n$ that is the solution of problem $G(\mathbf{x})$ is made by the problem $G(\mathbf{x})$. So operator A is expressed $A(G(\mathbf{x}))$. Now we call $\mathbf{g}$ which $\mathbf{g}_o$, because $\mathbf{g}$ is one of plural $\mathbf{g}_n$. Then Eq.(1) is expressed as follows.

$$A(G(\mathbf{x}))\mathbf{x}=\mathbf{g}_o \tag{1'}$$

In the Eq.(2), "The value which one physical value can be actually" corresponds to problem $G(\mathbf{x})$. And it exists that corresponds to initial condition $\mathbf{x}$.(for example, potential function etc.) In this case also, operator B is defined by the problem $G(\mathbf{x})$. As the solution $\mathbf{g}_n$ corresponds to $b_n\Psi_n$ if Ψ_n is orthonormal vector, Eq.(2) is expressed as follows.

notes] There is voluntariness about the choice of $\{\Psi_n\}$, it can be chosen orthonormal base.

$$B(G(\mathbf{x}))\Psi_n=\mathbf{g}_n \tag{2'}$$

Comparing Eq.(1') and Eq.(2'), in Eq.(1') matrix A is settled only for the special solution $\mathbf{g}_n$. On the other side in Eq.(2'), $\mathbf{g}_n$ which are all solutions of problem $G(\mathbf{x})$ are lead by the operator B. That is to say, in the state-space approach for getting all solutions of problem $G(\mathbf{x})$, it must make up operator A_n which fills following Eq.

$$A_n\mathbf{x}=\mathbf{g}_n$$

for each $\mathbf{g}_n$. Besides in Eq.(2), $\mathbf{g}_n$ is separated to eigenvalue b_n and eigenvector Ψ_n and it gets eigenvector system $\{\Psi_n\}$ that is expression which includs basic significance.

As is mentioned above, it is showed that Q.M. has more general property than state-space approach. Therefore, by extending the area of its application from "Physical value" to general "problem", and by making up the operator which corresponds to the problem, new problem-solving method will be established. By this method, when one problem has the difficulty that is to say "Combinational Explosion" on the problem-solving, it is guessed that there is one possibility to get solutions analytically.

3. Formalization

According to the method of Q.M., it is formalized the problem-solving as follows. As it is stated in the section 2., in Q.M. to solve problem is to let given problem arrive to its eigenvalue problem of one operator.

When one problem G(**x**) is given to r-demensional vector **x**, ($\mathbf{x} \in C^r$: C^r is r-dimensional complex vector space, after this we call **x** state-vector), we suppose that it is eigen square matrix which expressed problem G(**x**). (After, we call it operator-matrix). Then eigenvalue equation of A is following.

$$A_n \mathbf{f}_n = a_n \mathbf{f}_n \ (a_n \in C,\ \mathbf{f}_n \in H) \tag{3}$$

Eigenvalue a_n that fills Eq.(3) are solutions of G(**x**). And $\mathbf{f}_n$ are vectors on the Hilbert-Space. As the eigenvalue a_n is the solution of characteristic equation of A,

$$|A - aI| = 0 \quad \text{(I is unit matrix)} \tag{4}$$

it is complex number generally. And the elements of eigenvector $\mathbf{f}_n$ are complex number. The number of solutions is the number of dimension including multiplue root. Whether the solution except real number will have the meaning or not, it depends on each case. When all elements of **x** is real and there is no real solution, it is guessed that problem G has no solution.

There is not such condition about operator matrix that is linear and Hermite operator such like Q.M., it is impossible the normlization of eigenvector and it does not fill the completeness of eigenvector.

4. Subjects

To Complete this method is to find out the process to get operator matrix A(G(**x**)). In other words, it is guessed that it needs basic process corresponds to the Schrödinger's replacement in Q.M. For getting this process it needs to classify problems according to its structure, and it needs to research the basic operator. Also, it is considered another following subject.

1. By solving the characteristic equation, eigenvalue is generally complex number, then it needs to consider the meaning of solutions of imaginary number. There is possibility to have some meaning about some **x**.

2. When one operator consists of more basic plural operators, it is the subject to consider whether it's possible to get eigenvalue from these commutation relation or not.

5. Expansive Consideration

As we argued in Section 4., in this method most of laws of Q.M. do not exist, because there is no condition about operator-matrix A in Eq.(3). Now we set two conditions about A. One is linearity.

$$A(c_1\mathbf{f}_2+c_2\mathbf{g})=c_1A\mathbf{f}+c_2A\mathbf{g}$$

[A is operator matrix, **f**,**g** are optinal vectors, c_1, c_2 are optinal scalar coefficients]
The other is Hermiticity.

$$A^{+}=A,\ a^{*}=a$$

[A is operator matrix, a is eigenvalue of A^{+}, means Hermite conjugate,* means complex conjugate]

By above two conditions, it is got following laws. As it means a guide after, we consider without strict proof. (Bcause these are same contens as Q.M.)

Notes] Saying about the condition of linearity, if the elements of operator matrix A are all scalar quantities, matrix is linear operator itself basically. However being clear in Q.M., elements of operator matrix A of Eq.(3) do not always consist of simple scalar quantity.

(1) Two forms

The contents of this article stated so far, accorded to the Heisenberg's form in Q.M. Next we consider about the relation to the Schrödinger's form. When it is given on state **x** and problem G, we suppose operator $A_{op}(G(\mathbf{x}))$. The solution of problem $G(\mathbf{x})$ on the state **x** is given as eigenvalue a_n of eigenvalue equation (5).

$$A_{op}f_n(r)=a_nf_n(r)\quad (r\in R) \tag{5}$$

[Notes: $f_n(r)$ is scalar function, and $\mathbf{f}_n$ is vector]
Eq.(5) is that Eq.(3) is transformed to another expression, and it is equivalent to Eqt. (3) mathematically.

Comparing Eq.(3) and Eq.(5), it is recognized that $f_n(r)$ corresponds to $\mathbf{f}_n$ and A_{op} corresponds to A.

Relation between A and A_{op} is

$$(A)_{rr'}=A_{op}\delta(r-r')\quad [\delta(r-r')\text{ is Dirac's }\delta\text{-function}]$$

In this equation, A_{op} operates to only r. Relation between $\mathbf{f}_n$ and $f_n(r)$ is

$(\mathbf{f}_n)_r = \mathbf{f}_n(r)$

Because

$$(A\mathbf{f}_n)_r = \int A_{op}\delta(r-r')\mathbf{f}_n(r')dr'$$
$$= A_{op}\delta(r-r')\mathbf{f}_n(r')dr' \quad [\because A_{op} \text{ is linear operator}]$$
$$= A_{op}\mathbf{f}_n(r) = (a_n\mathbf{f}_n)_r$$

(2) Unitary Transformation

As operator matrix A is linear and Hermite matrix, $(\mathbf{f}_n)$ can be choosed orthonormal system. Consequently

$$\mathbf{f}_n{}^t\mathbf{f}_m = \delta_{nm} \quad [\delta_{nm} \text{ is Kronecker's } \delta] \tag{6}$$

Next, unitary matrix U is defined from $\mathbf{f}_n$ as follows.

$$U \triangleq [\mathbf{f}_1, \mathbf{f}_2 \cdots\cdots\cdots \mathbf{f}_n \cdots\cdots\cdots]^t \tag{7}$$

We aquire matrix A' and vector f' are made by unitary transformation.

$$A' \triangleq UAU^t, \quad \mathbf{f}'_n \triangleq U\mathbf{f}_n \tag{8}$$

an element of A', is (by Eqs.(6), (7), (8))

$$(A')_{nm} = (UAU^t)_{nm} = (UA\ [\mathbf{f}_1, \mathbf{f}_2, \cdots\cdots\cdots \mathbf{f}_n, \cdots\cdots]_{nm}$$
$$= (U\ [a_1\mathbf{f}_1, a_2\mathbf{f}_2, \cdots\cdots\cdots, a_n\mathbf{f}_n \cdots\cdots\cdots]_{nm})$$
$$= a_n\mathbf{f}^t\mathbf{f}_m = a_m\delta_{nm}$$

an element of $\mathbf{f}'_n$ is (by Eqs.(6), (7), (8))

$$(\mathbf{f}'_n)_m = (U\mathbf{f}_n)_m = \mathbf{f}_m{}^t\mathbf{f}_n$$
$$= \delta_{nm}$$

Therefore, A' is diagonal matrix whose diagonal elements are eigenvalues, $\mathbf{f}'_n$ is a orthonormal vector whose No.n element is 1 and another elements are 0.

$$A' = \begin{bmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & a_3 & & \\ & & & \cdot & \\ 0 & & & & \cdot \end{bmatrix} \qquad \mathbf{f}_n' = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \end{bmatrix} \leftarrow \text{No.n element}$$

By this process, against given problem G(x), it is given normal form.

(3) Commutation Relation

Against two states x_1, x_2, and two problems $G_1(x_1)$, $G_2(x_2)$, we suppose operator matrices A,B which correspond to each problem. At this time, to exist eigenvector f which fills eigenvalue equations Af=af, Bf=bf at the same time, it needs that A and B are commutative, and f can exist only that time.

Therefore, in commutative case,

$$AB=BA$$

it can exists such state f that $G_1(x_1)$ and $G_2(x_2)$ have solutions at the same time. Consequently it is recognized that two problems can have solutions a,b at the same time. And in uncommutative case,

$$AB \neq BA$$

it does not exist such state f that $G_1(x_1)$ and $G_2(x_2)$ have solutions at the same time. Consequently it is recognized that two problems can not have solutions at the same time.

7. Conclusion

This article mentioned above about effectiveness of the method by the form of Q.M. It is not ready conditions to complete the subject which is showed in section 4. But this consideration shows that the form Q.M. has possibility of development of more general theory, compared to the theory which was investigated in the area of Artificial Intelligence. The result of consideration in section 5-(2), that the conditions of linearity about operator make possible to normalize problem, is impotant. After, it may need minute considerstion for the normalization of problem.

References

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