Characteristics of Stationary Closed Streamlines in Stratified Fluids Part 3 : An Exact Solution of the Vortex Pair and Theory of A Turbulent Flow

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Abstract

An exact solution of a nonlinear equation which governs the internal solitary vortex pair is derived. The solution seems to reveal the flow pattern of the internal solitary bulge. The Prandtl–Batchelor theorem is generalized for a turbulent stratified fluid flow. The theorem is compared with field data of aircraft–trailing vorticies in stratified fluids.

1. Introduction

As for the vortex pair in a stratified fluid, analytical studies of the flow seem to be necessary, although the analytical studies have been done extensively for the vortex pairs in a homogeneous fluid (e.g. Batchelor[1]) and in a rotating fluid (e.g. Stern[2], and Larichev and Reznik[3]). Therefore, one of two aims of this paper is a derivation of an exact solution of nonlinear equations for the vortex pair in a stratified fluid flow. We will compare the flow pattern drawn by use of the solution with that of laboratory experiment of the internal solitary bulge.

On the other hand, the Prandtl–Batchelor theorem has been generalized to turbulent flows which system contains the Coriolis force (Yamagata and Matsuura[4]; Rhines and Young[5]). They have new results about the Prandtl–Batchelor theorem as momentum mixing by turbulence as well as viscosity. We will, therefore, generalize the Prandtl–Batchelor theorem to a turbulent stratified fluid flow. The theorem also will be compared with field data of aircraft–trailing vortices in stratified fluids.

In Section 2 we derive an exact solution and discuss its characteristics. The Prandtl–Batchelor theorem will be generalized to a turbulent stratified fluid flow in Section 3. Finally, we discuss the characteristics of the closed regions in the turbulent stratified fluid in Section 4.

2. Exact Solution of Internal Solitary Bulge

Basic equations of mass conservation and vorticity of the non–diffusive inviscid fluid are

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$$\frac{\partial \sigma}{\partial t} + J(\sigma, \psi) + F^2 \frac{\partial \psi}{\partial x} = 0 , \tag{2.1}$$

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) + \frac{\partial \sigma}{\partial x} = 0 , \tag{2.2}$$

where σ is a fluctuation about a density of a mean state, J the Jacobian, ψ the stream function, F the Froud number, and ω the vorticity. Combining these equations gives

$$\frac{\partial A}{\partial t} + J(A, \psi) + \beta \frac{\partial \psi}{\partial x} = 0 , \tag{2.3}$$

where $A = \sigma + \omega$, $\beta = F^2 - \lambda_\rho^2$ and $\sigma = - \lambda_\rho^2 \psi$. Using this type of the equation with diffusivity and viscosity we have one form of the Prandtl—Batchelor theorem which is omitted in the Part 1 (see Kamachi[6] for detail). Equation (2.3) is analogous to the vorticity equation on a beta plane in a rotating fluid[3]. Therefore, the quantity A, which we call internal—vorticity, may be considered as the potential vorticity in a nonrotating stratified fluid (cf. a similar form is Eq. (5.3.6) in Batchelor's text[1]). Using a coordinate $(X, Y) = (x - c_e t, y)$ moving with the vortex pair, we have

$$J_{XY}(A - \beta Y, \psi - c_e Y) = 0 , \tag{2.4}$$

where $\psi - c_e Y$ is the steady stream function observed in the coordinate with the vortex pair, $c_e = -F^2/\lambda_\rho^2$, and J_{XY} is the Jacobian. In and around the vortex pair exact solutions of Eq. (2.4) are

$$A - \beta Y = K^2(\psi - c_e Y), \qquad r \leq a , \tag{2.5}$$

$$A - \beta Y = -L^2(\psi - c_e Y), \qquad r \geq a , \tag{2.6}$$

where K^2 and $-L^2$ are constants, and a is the amplitude of the vortex pair. Although Eq. (2.4) has a different form of the solution which is proportional to $(\psi - c_e Y)^n$, $n > 1$, we dose not consider the solution. The solution shows a flow is not a dipole type but contains many closed regions. This type of the flow may not be realized, although Yih[7] derived a similar type of the solution and the stability of the solution has not been considered.

Using a cylindrical coordinate $X = r \cos \theta$, $Y = r \sin \theta$, and putting $\psi = R(r) \sin \theta$, we have

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left\{(\lambda_\rho^2 + K^2) - \frac{1}{r^2}\right\}\right] R = (L^2 c_e - \beta)\, r, \; r \leq a, \tag{2.7}$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \{(\lambda_\rho^2 - L^2) - \frac{1}{r^2}\}\right] R = -(L^2 c_e + \beta)\, r,\ r \geq a, \tag{2.8}$$

Using a boundary condition at infinity, $R \to 0$ as r tends to infinity, we have a relation $L^2 c_e + \beta = 0$. We get also $L^2 < \lambda_\rho^2$ with the above condition. We have therefore, solutions of equations (2.7) and (2.8) as follows

$$R(r) = C_I J_1(k_e r) + \frac{K^2 c_e - \beta}{\lambda_\rho^2 + K^2}\, r, \qquad r \leq a, \tag{2.9}$$

$$R(r) = C_O J_1(l_e r) + D_O Y_1(l_e r), \qquad r \geq a, \tag{2.10}$$

where J_1, Y_1 are the first and the second kind of the Bessel functions, respectively, and the wavenumbers are

$$k_e = (\lambda_\rho^2 + K^2)^{1/2},\ l_e = (\lambda_\rho^2 - L^2)^{1/2}.$$

At the boundary of the vortex pair, $r = a$, boundary conditions are continuations of the total pressure, an azimuthal velocity, and the vorticity. Using these boundary conditions we get the coefficients:

$$C_I = a c_e \left(\frac{l_e}{k_e}\right)^2 \frac{1}{J_1(k_e a)}, \tag{2.11}$$

$$C_O = \frac{\pi a}{2}\left[a c_e l_e Y_0(l_e a) - \{2 c_e \frac{k_e^2 - l_e^2}{k_e^2} + a c_e \left(\frac{l_e}{k_e}\right)^2 \frac{J_0(k_e a)}{J_1(k_e a)}\} Y_1(l_e a)\right], \tag{2.12}$$

$$D_O = \frac{\pi a}{2}\left[-a c_e l_e J_0(l_e a) + \{2 c_e \frac{k_e^2 - l_e^2}{k_e^2} + a c_e \left(\frac{l_e}{k_e}\right)^2 \frac{J_0(k_e a)}{J_1(k_e a)}\} J_1(l_e a)\right], \tag{2.13}$$

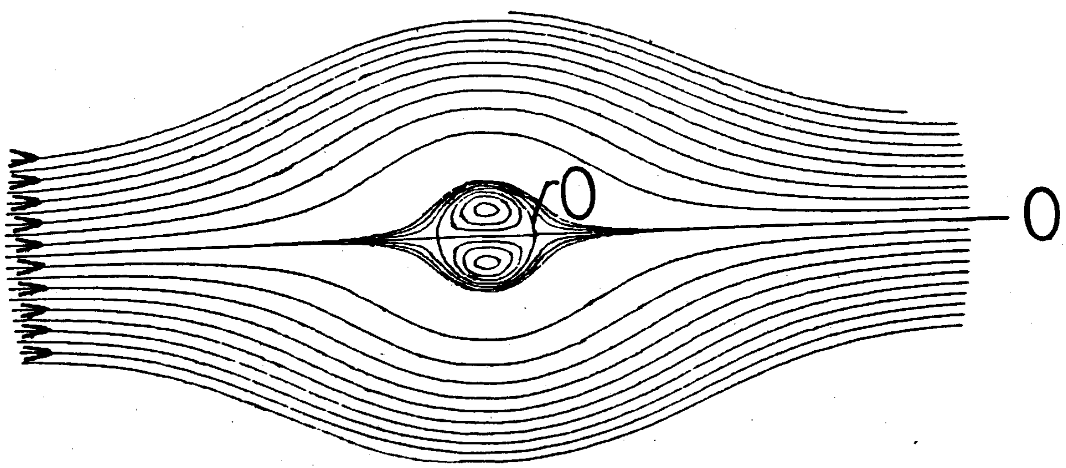

Fig. 1 An example of flow pattern. $a = 1.0$, $F^2 = 1.0$, $\lambda_\rho^2 = 0.5$, $k_e = 2.0$, and $l_e = 0.5$.

A typical flow pattern is shown in Fig. 1. In this figure, steps of the values of the stream function are 0.01 and 0.5 in and around the vortex pair, respectively. The flow pattern is similar to that of the internal solitary bulge[8]. The solution may show the internal solitary vortex pair. Because the functions of the solutions are the first and the second kind of the Bessel functions, J_1 and Y_1, internal waves which are similar to the lee waves are induced around the vortex pair ; although the figure does not show the lee waves clearly. These lee waves may radiate the energy of the internal solitary vortex pair. Radiation damping will be occure and the phenomena may not be steady by means of these lee waves. However, the internal solitary vortex pair can be stationary, when a time scale of the internal solitary vortex pair used the amplitude and the azimuthal velocity is smaller than the inverse of an angular frequency of the lee waves.

In the upstream region of the internal solitary vortex pair. the waves around it may disappear, when the boundary condition at infinity in the upstream is changed. Under the upstream condition, $O\ (r^{-3/2})$, the solution around the vortex pair may be the same as the solution of the lee waves theory.

If the depth of the fluid is finite, $2h$, the lee waves do not appear under a condition that the propagation speed of the vortex pair is larger than the phase speed of the long nondispersive internal wave. In this case the boundary condition should be used as $\partial \psi / \partial x = 0$ at $y = r \sin \theta = h$. In our analysis, the condition is $dR/dr + (\sin \theta /h)R = 0$ at $y = h$. When the type of the density distribution is not a linear function of the vertical coordinate but the hyperbolic–tangent type used in the Part 2, the above argument can be applied. In this case, we can estimate the depth as the thickness of the transition layer of the density. When the propagation velocity of the vortex pair is larger than the phase speed of the long nondispersive internal waves with the thickness, the lee waves do non appear. On the oter hand, when the propagation velocity is smaller than the maximum phase speed of the internal wave, the lee wave is not also induced actually by means of almost homogeneous fulid in the upper and lower outside of the vortex pair. If the density distribution is an exponential type, the above argument also can be applied. In this fluid flow, the finite depth may be estimated as the scale height $N^2/2g$. For this fluid flow we can derive the two–dimensional evolution equation which will be a generalized form of the well known one–dimensional KdV equation, although the characteristics of the equation and the solution have not been considered yet.

3. Analysis of Turbulent Stratified Flows

In this section we will generalized the Prandtl–Batchelor theorem to turbulent stratified fluid flows.

We use Eq. (2. 3) with viscosity as a basic equation :

$$\frac{Dq}{Dt} = \frac{1}{R_e} \Delta q, \tag{3.1}$$

where $q = \underline{A - \beta} Y$. We separate the motion into two parts, that is, the time–mean field denoted by ($\overline{\quad}$) and the disturbance field denoted by ('). We assume two–dimensional disturbances. Under this assumption, we omit the production of the enstrophy due to the stretching of the vortex. Then a part of Eq. (3. 1) governing the disturbance fild is

$$\frac{\partial q'}{\partial t}+u'\frac{\partial \bar{q}}{\partial x}+v'\frac{\partial \bar{q}}{\partial y}+\bar{u}\frac{\partial q'}{\partial x}+\bar{v}\frac{\partial q'}{\partial y}=\frac{1}{R_e}\Delta q'. \tag{3.2}$$

The equation for time–mean field is

$$\bar{u}\frac{\partial \bar{q}}{\partial x}+\bar{v}\frac{\partial \bar{q}}{\partial y}+\overline{u'\frac{\partial q'}{\partial x}}+\overline{v'\frac{\partial q'}{\partial y}}=\frac{1}{R_e}\Delta \bar{q}, \tag{3.3}$$

where we assumed the time–mean field is statistically steady.

We consider the steady two–dimensional, high Reynolds number flows. We consider the region without a turbulent boundary layer, and in the region we have

$$\frac{\mathrm{D}q}{\mathrm{D}t}=0 \tag{3.4}$$

correct to $O(R_e^{-1})$. Now it is seen from Eq. (3.4) that in the flow q associated with a material element is constant ; and in the steady flow th pants of meterial elements are streamline. Hence q has the same value at all points of a streamline, and can evidently be written as a function of ψ alone. The streamlines, therefore, coincide with the equi–q line. Hereafter we call the equi–q line as equi–internal–vorticity line.

If $\bar{q}$ has a closed equi–internal–vorticity line ξ, then we have, utilizing Eq. (3.3),

$$\oint_{\xi}\overline{q'u'}.\, n d\xi=\frac{1}{R_e}\oint_{\xi}\nabla\bar{q}.\, n d\xi, \tag{3.5}$$

where n is the unit vector which is normal to ξ and directed outwards. Equation (3.5) show that flux of the 'internal vorticity', q, due to the disturbance u' across the closed contour ξ is balanced with the viscous torque which operates on the mean motion.

Multiplying Eq. (3.2) by q' and averaging, we obtain a equation governing the enstrophy of the disturbance as

$$\overline{u'q'}\frac{\partial \bar{q}}{\partial y}+\overline{v'q'}\frac{\partial \bar{q}}{\partial y}+\bar{u}\frac{\partial}{\partial x}(\frac{1}{2}\overline{q'^2})+\bar{v}\frac{\partial}{\partial y}(\frac{1}{2}\overline{q'^2})=\frac{1}{R_e}\overline{q'\Delta q'}, \tag{3.6}$$

where we used the statistical steadiness of the mean field. If we further assume the homogeneous disturbance, we obtain

$$\overline{r'u'}.\,\nabla\bar{q}=\frac{1}{R_e}\overline{q'\Delta q'}. \tag{3.7}$$

Because ξ is a closed equi–internal–vorticity line, we have the relation

$$n = \pm \nabla \bar{q} / |\nabla \bar{q}| , \tag{3.8}$$

where the upper (lower) branch of the double signs is chosen when $\bar{q}$ increases (decreases) outwards. Substituting equations (3.7) and (3.8) into (3.5) gives

$$\pm \oint_{\xi} \frac{\frac{1}{R_e}\overline{(q' \Delta q')}}{|\nabla \bar{q}|} d\xi = \frac{1}{R_e} \oint_{\xi} \nabla \bar{q} . n d\xi . \tag{3.9}$$

Using the assumption of the homogeneous disturbance, we have

$$\mp \oint_{\xi} \frac{\overline{(\nabla q')^2}}{|\nabla \bar{q}|} d\xi = \oint_{\xi} \nabla \bar{q} . d\xi . \tag{3.10}$$

We do not assume that the disturbance is weak in Eq. (3.10). We notice that the parameter R_e has dissapeared in the result (3.10). Viscosity is necessary in order that the relation of Eq. (3.10) is hold only, and the value is not necessary. The result coincides the result of the acoustic streaming by Lord Rayleigh. If the mean field is large contribution to Eq. (3.10) we have

$$|\nabla \bar{q}| \sim 0 \qquad on\ \xi , \tag{3.11}$$

and the result is similar to the result in the roating fluid flow (see Yamagata and Matsuura[4], Rhines and Young[5]). The above asymptotic relation shows that the mean field of the internal vorticity is homogeneous in the statistically steady closed region. We also notice that the above asymptotic relation is due to the momentum transfer by the Reynolds stress or the Radiation stress.

If the flow is laminar, the result. (3.10) coincides that in Part 1.

If the fluid is homogeneous (*i. e.* $F = 0$ and $\beta = 0$), $\bar{q} = \bar{\omega}$, $\overline{q' \omega'}$. Equation (3.10) becomes

$$\mp \oint_{\xi} \frac{\overline{(\nabla \omega')^2}}{|\nabla \bar{\omega}|} d\xi = \oint_{\xi} \Delta \bar{q} . n d\xi . \tag{3.12}$$

and in the case of weak disturbance we have

$$|\nabla \bar{\omega}| \sim 0 \qquad on\ \xi . \tag{3.13}$$

This asymptotic relation means that the flow in the statistically steady circular closed region is the rigid–body rotation.

4. Results and Discussions

We considered the characteristics of the stationary closed streamline in turbulent fluid flows. The result is that the internal vorticity is statistically homogeneous in the closed region bounded by the equi–internal–vorticity line. In our analysis the disturbances, as well as the viscosity in the laminar flow, transfer the momentum in the statically steady closed region. In a special case, the vorticity is homogeneous in the region.

Burnham et al.[9] reported about the trailng vorticbs behind an aircraft in stratified fluids (see also Hechet et al.[10] for numerical analysis of the vortex). Using their result about a tangentiai velocity vs the vortex radius, we assumed that the shape of the vortex is a circle and evaluated the vorticity as the velocity divided by the radius. The relation between the tangential velocity and the vortex radius is shown in Fig. 2. In a core region of the vortex, where r smaller than $3m$, the flow may be a rigid–body rotation. The turbulent version of the Prandtl–Batchelor theorem for stratified fluids, therefore, may be realized in the region, although Hechet et al.[10] considered all regions of the vortex in

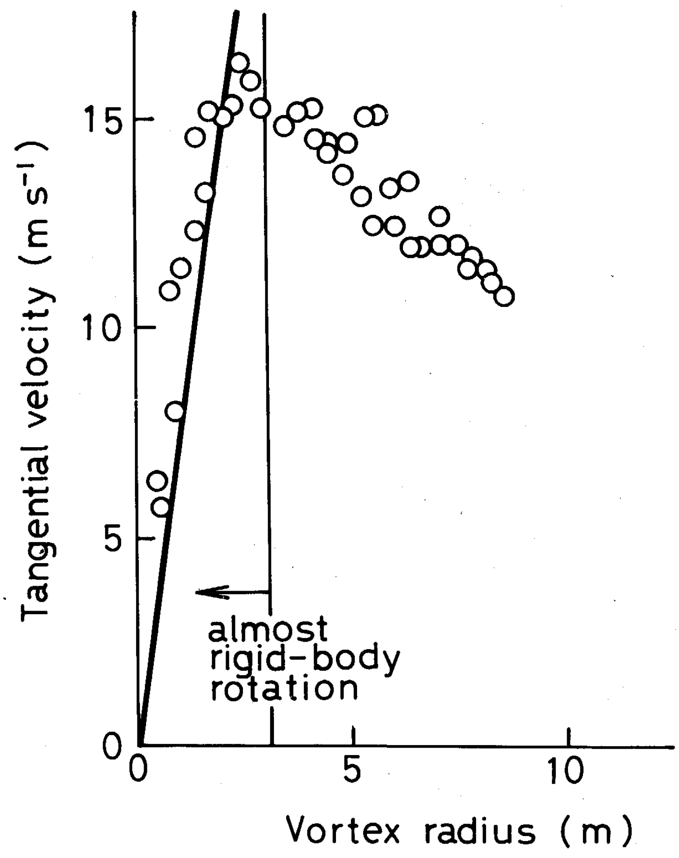

Fig. 2 Measured tangential velocity vs vortex radius. The solid line is $v = 6.879\ r$ obtained from fifteen data in $r \leq 3m$ by use of the least square method. We may interpret that the flow is a rigid–body rotation and the Prandtl–Batchelor theorem is realized in $r \leq 3m$.

In our analysis, we used two assumptionp : (1) two–dimensional disturbances, (2) homogeneous and isotropic disturbances, although the disturbance is not weak. Under the first assumption, the stretching of the mean and perturbed vortex does not produse the vorticity, and enstrophy. When the fluid is confined in a narrow channel, the above effect does not take an important role. However the effect becomes important when the width of the channel is large compared with the amplitude of the vortex (e.g. phenemena in the ocean and the atmosphere). In actual turbulent stratified fluid flows, the disturbance may be inhomogeneous and anisotropic. The theorem, therefore, seems to be necessary to be generalized to the flows.

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