Characteristics of Stationary Closed Streamlines in Stratified Fluids Part 1 : Theory of Laminar Flows

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Abstract

The Prandtl–Batchelor theorem for high Reynolds number flow of a homogeneous fluid is generalized to laminar flows of vorticies moving in stratified fluids. Density is homogenized in the vortices as well as the vorticity is. We pay attention to the mass flux by means of the basic equations governing the density distribution with/without diffusions. The theorem is also applicable to a flow with a passive scalar contaminant.

1. Introduction

Integral constraints are derived for steady flows with closed streamlines, arising from the action of a small amount of viscosity. In the two dimensional case it is shown that the vorticity is homogeneous in the closed nested streamlines in a homogeneous fluid. This result is well known as Prandtl–Batchelor theorem[1),2)] (see also Batchelor[3)], and Stuart[4)]).

Some extensions and generalizations of the theorem have been made. Blennerhassett[5)] obtained an exact integral condition relating the constant axial pressure gradient and the viscous terms for a class of flows with herical streamlines. And the axial velocity is proportional to the stream function for the motion in the plane normal to the axial velocity.

Grimshaw[6)] derived integral constraints for steady recirculating flows with the action of a small amount of viscosity and heat conduction. In two dimensional flows containing closed streamlines, the flows are isothermal and the vorticity is homogeneous. The constant values of temperature and vorticity are determined from boundary conditions by means of an approximate integration of the boundary layer equations. The result may be realized in the cat's eye flow patterns in the nonlinear critical layer of a slightly stratified shear flow[7)].

Recently, Prandtl–Batchelor theorem has been generalized for quasi–geostrophic flows in planetary fluids[8),9)]. Owing to the Ekman friction, laminar quasi–geostrophic flows are stagnant in a closed streamline. The result is confirmed with a numerical experiment of steady two dimensional flows around a circular cylinder on an f plane. In a turbulent flow, Yamagata and Matsuura[8)] obtained two different mean states. One state corresponds to the laminar flow in a limit and the other is a flow which has a uniform

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potential vorticity in closed geostrophic contours. The latter may be consistent with a result of Rhines and Young[9], though they used an assumption of gradient transport closure hypothesis.

Although the results derived from the theorem have been confirmed, they have been confined flows of closed streamlines in homogeneous rotating/nonrotating fluids or a thermally stratified fluid. Generalization of the theorem to the flow of stratified fluid or the flow with passive scalar contaminant, therefore, seem to be necessary in order to confirm the Prandtl–Batchelor's closed streamline theorem. We report the characteristics of stationary closed streamlines of stratified laminar or turbulent fluids. In Part 1 we derive the theorem for laminar stratified fluids. In Parts 2 and 3 we attempt to develop a deeper understanding of results of Part 1. In Part 2 we compare the theory of Part 1 with laboratory experimental data of the internal solitary vortex pair. Further, we generalize the theorem for a turbulent stratified fluid flow and compare the theorem with field data of aircraft–trailing vortices in stratified fluids in Part 3.

Firstly, in Part 1, we review the Prandtl–Batchelor theorem briefly in Section 2. In Section 3, the theorem is generalized with four types of equations of mass–conservation equations. Finally, in Section 4, we discuss the results.

2. Review of Prandtl–Batchelor Theorem

In this section we introduce the Prandtl–Batchelor theorem for steady flows with closed streamlines in a homogeneous fluid.

The vorticity equation governing the two dimensional laminar flows of a homogeneous incompressible fluid is, in a nondimensional form,

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \frac{1}{R_e} \Delta \omega, \tag{2.1}$$

where $\omega = (\Delta \times \boldsymbol{u}).\boldsymbol{k}$ is the vorticity of the two dimensional flow, $\boldsymbol{k}$ is the fundamental vector pointed out upward and is parallel to the gravity, $\boldsymbol{u} = \Delta \times \boldsymbol{B}$ the velocity vector, $\boldsymbol{B} = (0, 0, \psi)$ the vector potential, ψ the stream function, R_e the Reynolds number, Δ the Laplacian, and

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \tag{2.2}$$

the Jacobion.

The viscous force is measured by an inverse Reynolds number, $R_e^{-1} = \nu/UL$, where L is a typical length scale, U is a typical velocity scale, and ν is the kinematic viscosity. If the viscous force is small, an approximate solution is

$$\omega = \omega_0(\mathbf{x}) + R_e^{-1}\omega_1(\mathbf{x}, t) + O(R_e^{-2}), \tag{2.3}$$

where $\boldsymbol{x}$ is the position vector.

Now we introduce orthogonal curvilinear coordinates (ψ, ξ) as shown in Fig. 1. In a region of a closed streamline, there are no source and sink of the vorticity. The vorticity is independent of ξ (if ω is the function of ξ (i.e. $\partial \omega / \partial \xi \neq 0$), the vorticity diffuses along the streamline and the flow can not be steady). The vorticity, therefore, is the function of the stream function only :

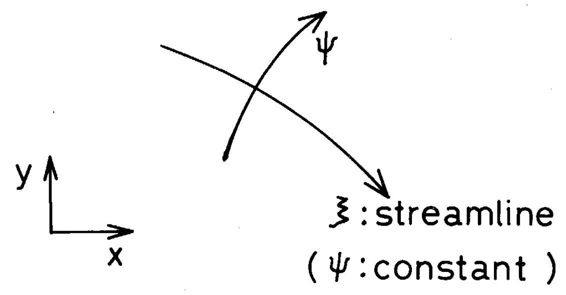

Fig. 1 Coordinate system.

$$\omega = \omega_0 (\psi). \tag{2.4}$$

This argument is analogous to a flow along a curved boundary (e.g. recirculating flow along a circular cylinder) : in the flow, the vorticity distribution is constant along the boundary and a streamline, and is a function of distance normal to the boundary. If the vorticity is not the function of the streamline only in the high Reynolds number flow, the error of the above argument may be estimated to be O (Re^{-1}). We can interpret the argument in the other way. Because the flow has high Reynolds number, we can divide the flow region into two parts : a boundary layer (e.g. Burggraf et al.[10], Belcher et al.[11]) and an inviscid flow region. Outside the boundary layer, Eq. (2.1) is

$$J(\omega_0, \psi) = 0\ . \tag{2.5}$$

We have, therefore, $\omega_0 = \omega_0 (\psi)$.

Integration of Eq. (2.1) about the area S within a closed steamline gives

$$\iint_S \frac{\partial \omega}{\partial t}\, \mathrm{d}S + \iint_S J(\omega, \psi)\, \mathrm{d}S = \frac{1}{R_e} \iint_S \Delta\, \omega\, dS. \tag{2.6}$$

The left hand side vanishes by use of the steadiness, the mass continuity and the divergence theorem. Then Eq. (2.6) reduces

$$\frac{1}{R_e}\iint_S \Delta \omega dS = \frac{1}{R_e}\oint_\xi \nabla \omega . n d\xi = 0, \tag{2.7}$$

where n is the unite vector normal to ξ and points outwards. Substitution of Eq. (2.3), or

$$\nabla \omega = \frac{d\omega_0(\psi)}{d\psi} \nabla\psi + R_e^{-1}\nabla\omega_1 + O(R_e^{-2}), \tag{2.8}$$

into Eq. (2.7) gives

$$\frac{1}{R_e}\frac{d\omega_0}{d\psi}\oint_\xi \nabla\psi . n d\xi = O(R_e^{-2}). \tag{2.9}$$

We have, then,

$$\frac{d\omega_0}{d\psi}\Gamma = O(R_e^{-1}), \tag{2.10}$$

where

$$\Gamma = \oint_\xi \nabla\psi . n d\xi = \oint_\xi u . d\xi \tag{2.11}$$

is the circulation. Therefore, if Γ dose not vanish in the closed streamline, we obtain

$$\frac{d\omega_0}{d\psi} = 0, \tag{2.12}$$

with the error of $O(Re^{-1})$: the vorticity is uniform in the closed region. This Prandtl–Batchelor theorem is a torque balance in a steady closed laminar flow with high Reynolds number.

3. Prandtl–Batchelor Theorem for Stratified Fluids

We derive the Prandtl–Batchelor theorem for stratified fluid flows using some types of mass–conservation equations which have been used in the studies of geophysical fluid dynamics.

At first we consider a case of a nearly incompressible fluid with a constant coefficient of thermal expansion γ, so that the equation of state is

$$\rho = \rho_{00} [1 - \gamma (T - T_{00})], \tag{3.1}$$

where ρ_{00} and T_{00} are constants, T the temperature and the density[12]. The equation of heat transfer is

$$\frac{DT}{Dt} = \kappa_1 \Delta T + \Phi, \tag{3.2}$$

where κ_1 is the coefficient of thermal conductivity and Φ the viscous dissipation function[13]. In view of Eq. (3.1), (3.2) is equivalent to

$$\frac{D\rho}{Dt} = \kappa_1 \Delta \rho \tag{3.3}$$

where we assume the Prandtl number, P, is small ; afterwards P will be assumed smaller than $O(Re^{-1})$ compared with the error in the Prandtl–Batchelor theorem.

The introduction of the term $\kappa_1 \Delta \rho$ in Eq. (3.3) has a problem[13),14]. The equation would mean at least a change in the definition of the basic quantities ; in particular, the velocity would no longer be the momentum of unit mass of fluid. We must, therefore, bear in mind that the mass flux, j, must always be the momentum of unit volume of fluid. The mass flux j should be defined by the mass–conservation equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0. \tag{3.4}$$

In our case, we must regard the mass flux as

$$\rho u - \kappa_1 \nabla \rho . \tag{3.5}$$

Next, consider a closed surface whose position is fixed relative to the coordinate axes and which encloses a volume (we can interpret it as the control volume) entirely occupied by a stratified fluid. We can derive a balance equation among the time derivative of the mass of fluid enclosed by the surface at any instant, the net rate at which mass is flowing outwards across the surface, and the net rate of diffusion of mass across the surface :

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho u) = \kappa_1 \Delta \rho . \tag{3.6}$$

This is a typical form of Eq. (3.4). Subtracting the equation of heat transfer, Eq. (3.3), from Eq. (3.6) gives the continuity equation

$$\nabla . u = 0. \tag{3.7}$$

Therefore, the continuity equation is not changed.

We rewrite Eq. (3.3) in a form

$$\frac{\partial \rho}{\partial t} + \mathrm{J}(\rho, \psi) = \kappa_1 \Delta \rho. \tag{3.8}$$

In a steady state, we have $\mathrm{J}(\rho, \psi) = 0$ so that ρ is a function of ψ :

$$\rho(\psi), \tag{3.9}$$

outside a boundary layer (e.g. thermal boundary layer). Using a similar manner in Section 2, we have a constraint

$$\kappa_1 \frac{\mathrm{d}\rho}{\mathrm{d}\psi} \oint_{\xi} u.\, \mathrm{d}\xi = O(\kappa_1^{\,2}), \tag{3.10}$$

and we get a result that

$$\rho = \text{costant} \tag{3.11}$$

with the error of $O(\kappa_1)$. This result means that the density is homogeneous in a closed streamline in a stratified fluid. The homogenization is due to the mixing of mass by the diffusivity. The result is consistent with a result about the temperature[6].

When the flow is stationary, e.g. a moving vortex in a stratified fluid[15], the above result is not changed with

$$\mathrm{J},\ \Delta \text{ and } \psi \text{ replacing } \mathrm{J}_{XY},\ \Delta_{XY} \text{ and } \Psi, \tag{3.12}$$

where $J_{XY}(a, b) = (\partial a / \partial X)(\partial b / \partial Y) - (\partial a / \partial Y)(\partial b / \partial X)$, $\Delta_{XY} = \partial^2 / \partial X^2 + \partial^2 / \partial Y^2$ and $\Psi = \psi + cy$ the steady streamline in a coordinate $(X, Y) = (x + ct, y)$ moving with a travelling speed, c, of the vortex.

Next we assume the hydrostatic condition, and consider a fluctuation, σ, about a mean state ρ_0 (y). From the equation (3.3), we have, in a nondimensional form,

$$\frac{\partial \sigma}{\partial t} + J(\sigma, \psi) + F^2 \frac{\partial \psi}{\partial x} = \kappa_1 \Delta \sigma, \tag{3.13}$$

where F is the Froude number $N_* L / U$, $N_*^{\,2} = -(g / \rho_*)\, d\rho_0 / dy$ and ρ_* is the reference density. In a stationary state, Eq. (3. 13) is

$$J_{XY}(\sigma - F^2 Y, \Psi) = \kappa_1 \Delta_{XY}(\sigma - F^2 Y). \tag{3.14}$$

In this case, on the other hand, the vorticity equation is

$$J_{XY}(\omega, \Psi) + \frac{\partial \sigma}{\partial x} = \frac{1}{R_e} \Delta_{XY} \omega. \qquad (3.15)$$

Using the similar manner of the derivation of the results (3.10) and (3.11), we get

$$\frac{\mathrm{d}(\sigma - F^2 Y)}{\mathrm{d}\Psi} \Gamma = O(\kappa_1), \qquad (3.16)$$

and

$$\sigma - F^2 Y = \text{constant}, \qquad (3.17)$$

with the error of $O(\kappa_1)$. Using the result (3.17), we have

$$\frac{\partial \sigma}{\partial Y} = F^2. \qquad (3.18)$$

This result (3.18) means that the absolute value of the vertical gradient of the fluctuating density distribution is the same as that of the gradient of the mean state and the sign is opposite. Therefore, the fluid in the vortex is homogeneous. It is consistent with the result (3.11). Using Eq. (3.17), the vorticity equation (3.15) is the same as the equation of homogeneous fluid.

When we also assume a linear damping term, $-\kappa_2 \sigma$, which is used as a Newtonian cooling in dynamic meteorology (e. g. Plumb [16]), Eq. (3.14) becomes

$$J_{XY}(\sigma - F^2 Y, \Psi) = -\kappa_2 \sigma + K_1 \Delta_{XY}(\sigma - F^2 Y). \qquad (3.19)$$

We get an integral constraint

$$\frac{\mathrm{d}(\sigma - F^2 Y)}{\mathrm{d}\Psi} = \frac{\kappa_2}{\kappa_1} \frac{\Sigma}{\Gamma}, \qquad (3.20)$$

where

$$\Sigma = \iint_S \sigma \, \mathrm{dS} \qquad (3.21)$$

is a total density anomaly from the mean state in a closed region. If the fluid has no diffusivity, $\kappa_1 = 0$, the total density anomaly must be zero. It means that the mixing of the density occurs inside the closed region and is independent of a region with open streamlines. We have the other limit : when the fluid has no Newtonian cooling, $\kappa_2 = 0$, the result is (3.18) and no motion, $\Gamma = 0$, may be able to occur. Therefore, above two results are different drastically. This effect of the Newtonian cooling is analogous to that in the ocean[17], although the oceanic current system has open streamlines and the earth's rotation. Their discussion[17] about the breakdown of the no motion theorem may be applied to the stratified fluid flows.

At last, the arguments in this section can be generalized to a fluid flow with a passive scalar contaminant. In this fluid flow a diffusion equation

$$\frac{\mathrm{D}C}{\mathrm{D}t} = K\Delta C \tag{3.22}$$

may be used. If C is a pollutant, the discussion about the theorem will be able to be applied to an environmental problem ; e. g., the pollutant will be mixed up in a closed region and be independent of a region with open streamlines by means of the discussion about the result (3.20).

4. Results

We derived the Prandtl–Batchelor theorem for laminar stratified fluid flows, using four types of mass–conservation equations with/without two types of diffusivity. In the fluid flows, the density is homogenized as well as the vorticity is. The homogenization is due to the mixing of mass by the diffusivity. It is analogous to the homogenization of the vorticity by the momentum transfer, which is due to the viscosity (i.e. momentum diffusion).

We also considered the difference of two types of the mass diffusivity. An analogy to the breakdown of the no motion theorem in the ocean was inferred.

An application of the theorem to the environmental problem are suggested.

In this Part 1, we have considered the theorem analytically. In order to confirm the theorem, efforts to devise suitable laboratory experiments and to consider the turbulent stratified fluid flows seem to be necessary.

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