Nonconservative Stability of the Columns with Thin Walled Open Cross-section

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Abstract

An equation of virtual work for geometrical non-linear vibration of a column with thin walled open cross-section, subjected to concentrated and distributed tangential follower loads, is derived by introducing the concept of initial stress, and stability problems of columns with channel section are investigated systematically and detailed by illustrating the space trajectories of eigenvalues. The five-term approximation of the extended Galerkin method is used, and the eigenfuctions of the flexural free vibration and the torsional one of a column are applied to the fundamental function which satisfies only the geometrical boundary conditions.

Introduction

Nonconservative stability problems of columns with double symmetric cross section, e.g., the stability of a clamped-free column subjected to a concentrated tangential loads at the free end and simply supported-simply supported, clamped-simply supported, and clamped-clamped columns subjected to uniformly and triangularly distributed tangential loads, have been investigated by many authors [1, 2, 3]. These researches have produced important results. Nonconservative problems of stability of columns with a single symmetric cross section or an unsymmetrical one, which involve bending and torsion coordinate coupling, are of great interest. Barsoum [4] applies the finite element method to the stability problems of columns subjected to a nonconservative load with thin walled open cross section and Mote and Matsumoto [5] examine the coupled nonconservative stability of I-beam and channel by the finite element method.

In this research, an equation of virtual work for geometrical non-linear vibration of a column is derived by introducing the concept of initial stress [6], and stability problems of columns with the channel section under the loading and boundary conditions shown in Fig. 1 are investigated more systematically and detailed than the work done by Mote and Matsumoto by obtaining the trajectories of eigenvalues. Loads considered in this paper are concentrated and distributed tangential follower compressive loads applied on the centroid of the cross section. The five-term approximation of the extended Galerkin method is used in the analysis, and the eigenfunctions of the flexural free vibration and the torsional one of a column are applied to the fundamental function which satisfies only the geometric boundary conditions.

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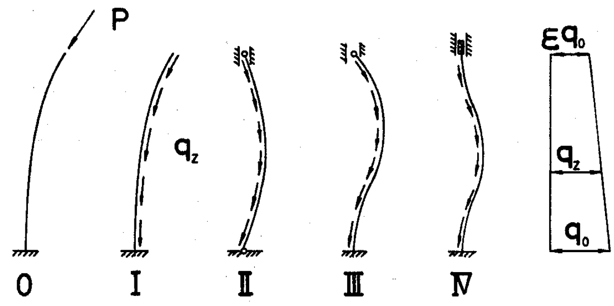

Fig. 1. Elastic columns subjected to tangential compressive loads.

Equation of Virtual Work

Let us consider a structural member in static equilibrium state, e.g., in initially stressed condition, in which the distributed external loads q_x^s, q_y^s, q_z, m_x, m_y, m_z^s and m_ω act as shown in Fig. 2. The shearing forces $Q_x^0(z)$, $Q_y^0(z)$, an axial force $Q_z^0(z)$, the bending moments $M_x^0(z)$, $M_y^0(z)$, a torsional moment $M_z^0(z)$, and a warping moment $M_\omega^0(z)$ are produced at an arbitrary section of the member in this state.

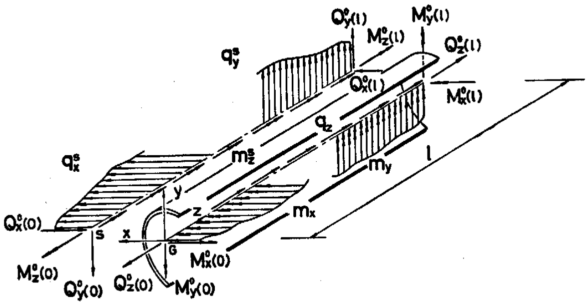

Fig. 2. Structural member with thin walled open cross-section.

Next, let us consider the geometrical non-linear vibration of a member due to a disturbance given to the static equilibrium state. It is assumed that the distributed external loads are follower loads except for the external warping moment m_ω with respect to the centroid.

The shearing forces $Q_x(z)$, $Q_y(z)$, an axial force $Q_z(z)$, the bedning moments $M_x(z)$, $M_y(z)$, a torsional moment $M_z(z)$, and a warping moments $M_\omega(z)$ produced in a state of vibration are added to them in the above mentioned static equilibrium state respectively. The end forces $Q_x(0)$, $Q_y(0)$, $Q_z(0)$ and the moments $M_x(0)$, $M_y(0)$, $M_z(0)$, $M_\omega(0)$ at the end i and the end forces $Q_x(1)$, $Q_y(1)$, $Q_z(1)$ and the moments $M_x(1)$, $M_y(1)$, $M_z(1)$, $M_\omega(1)$ at the end j of member ij in a state of vibration are also added to end forces and moments in static equilibrium as shown in Fig. 2. Displacements of shear center and cenoid at any section in a state of vibration are shown in Fig. 3.

An equation of virtual work is obtained due to the principle of virtual work and D'Alembert's principle as follows when the virtual displacements δu_S, δv_S, δw_S, $\delta\theta$, δ_G, δv_G and δw_G, which are kinematically admissible variations, are introduced at a

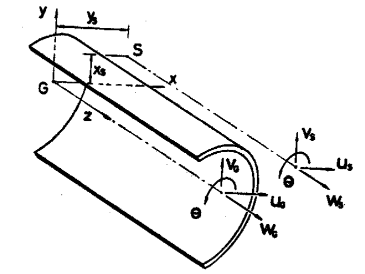

Fig. 3. Displacements of member with thin walled open cross-section.

certain time in a state of vibration.

(Virtual work of inertia forces)

$$W_I = -\int_0^l \{m(\ddot{u}_S + y_S\ddot{\theta})\delta u_S + m(\ddot{v}_S - x_S\ddot{\theta})\delta v_S + m\ddot{w}_G\delta w_G$$
$$+\mu I_{xx}\ddot{u}'_S\delta u'_S + \mu I_{yy}\ddot{v}'_S\delta v'_S + (\mu I_{ps}\ddot{\theta} + m y_S\ddot{u}_S$$
$$- m x_S\ddot{v}_S)\delta\theta + \mu I^s_\omega\ddot{\theta}'\delta\theta'\}dz \qquad (1)$$

with $(\dot{\ }) = \partial/\partial t$, $(\ ') = \partial/\partial z$, $m = \mu A$.

$$I_{xx} = \int_A x^2 dA, \quad I_{yy} = \int_A y^2 dA, \quad I^s_\omega = \int_A \omega_S dA,$$

and $I_{ps} = I_{xx} + I_{yy} + A(x_S^2 + y_S^2)$.

(Virtual work of internal forces)

Applying the nonlinear theory of elasticity, the virtual work of the internal forces is evaluated as follow:

$$W_S = -\delta\Pi_i^0 - \int_0^l [Q_x^0\{\beta_x(\theta'\delta\theta + \theta\delta\theta') + \theta\delta v'_S$$
$$+ v'_S\delta\theta\} + Q_y^S\{\beta_y(\theta'\delta\theta + \theta\delta\theta') - \theta\delta u'_S - u'_S\delta\theta\}$$
$$+ Q_z^0\{(u'_S + y_S\theta')(\delta u'_S + y_S\delta\theta') + (v'_S - x_S\theta')(\delta v'_S$$
$$- x_S\delta\theta') + r_0^2\theta'\delta\theta'\} + M_x^0(\theta'\delta u'_S + u'_S\delta\theta' - 2\beta_y\theta'\delta\theta')$$
$$- M_y^0(\theta'\delta v'_s + v'_S\delta\theta' + 2\beta_x\theta'\delta\theta') + 2M_\omega^{S0}\beta_\omega\theta'\delta\theta'$$
$$- M_z^{\omega 0}\beta_\omega(\theta'\delta\theta + \theta\delta\theta')]dz - \int_0^l (EAw'_G\delta w'_G$$
$$+ EI_{xx}u''_S\delta u''_S + EI_{yy}v''_S\delta v''_S + EI_\omega^S\theta''\delta\theta'' + GK\theta'\delta\theta')dz \qquad (2)$$

with $M_\omega^{S0} = M_\omega^0 + M_x^0 x_S + M_y^0 y_S$,

$$M_z^{\omega 0} = -dM_\omega^{S0}/dz, \; r_0^2 = (I_{xx} + I_{yy})/A,$$

$$\beta_x = -x_S + \left(\int_A xy^2 dA + \int_A x^3 dA\right)\Big/ 2I_{xx},$$

$$\beta_y = -y_S + \left(\int_A y^3 dA + \int_A x^2 y dA\right) \Big/ 2I_{yy},$$

$$\beta_\omega = \left(\int_A x^2 \omega_S dA + \int_A y^2 \omega_S dA\right) \Big/ 2I_\omega^S,$$

and $$K = \int_A [\{\partial\omega_S/\partial x - (y - y_S)\}^2 + \{\partial\omega_S/\partial y + (x - x_S)\}^2] dA.$$

and $\delta\Pi_i^0$ is the virtual work produced by initial stresses and linear virtual strains. The second term on the right side of Eq. (2) expresses the virtual work produced by initial stresses and nonlinear virtual strains and the third term expresses the virtual work produced by stresses due to the vibration and linear virtual strains.

(Virtual work of distributed external loads)

When shear center and centoid are rotated by $(-v_S', u_S', \theta)$ and $(-v_G', u_G', \theta)$ respectively with member deformation, the distributed follower loads become following loads as shown in Fig. 4.

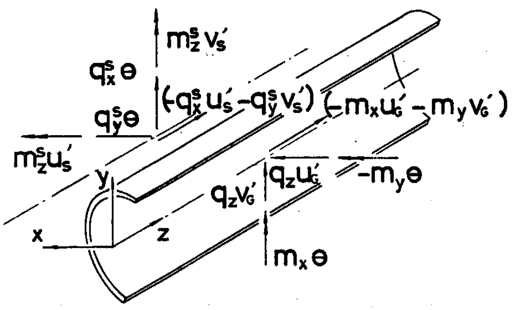

Fig. 4. Additional distributed loads by follow.

initial load		follower load
$(q_x^S, 0, 0)$	$\longrightarrow$	$(q_x^S, q_x^S\theta, -q_x^S u_S')$
$(0, q_y^S, 0)$	$\longrightarrow$	$(-q_y^S\theta, q_y^S -q_y^S v_S')$
$(0, 0, q_z)$	$\longrightarrow$	$(q_z u_G', q_z v_G', q_z)$
$(m_x, 0, 0)$	$\longrightarrow$	$(m_x, m_x\theta, -m_x u_G')$
$(0, m_y, 0)$	$\longrightarrow$	$(-m_y\theta, m_y, -m_y v_G')$
$(0, 0, m_z^S)$	$\longrightarrow$	$(m_z^S u_S', m_z^S v_S', m_z^S)$

The virtual work of the distributed follower loads becomes

$$W_D = \delta\Pi_{01}^0 + \int_0^l [\{-q_y^S\theta + q_z(u_S' + y_S\theta')\}\delta u_S + \{m_x\theta$$

$$+ m_z^S v_S' + x_S(q_x^S u_S' + q_y^S v_S')\}\delta u_S' + \{q_x^S\theta + q_z(v_S'$$

$$- x_S\theta)\}\delta v_S + \{m_y\theta - m_z^S u_S' + y_S(q_x^S u_S' + q_y^S v_S')\}\delta v_S'$$

$$-(q_x^S u_S' + q_y^S v_S')\delta w_G + \{y_S q_z(u_S' + y_S\theta') - x_S q_z(v_S'$$

$$- x_S\theta') - m_x(u_S' + y_S\theta) - m_y(v_S' - x_S\theta')\}\delta\theta + \{-\omega_n(x_S, y_S)$$

$$\times (q_x^S u_S' + q_y^S v_S') - m_y \theta x_S + m_x \theta y_S\} \delta\theta']dz \tag{3}$$

where $\delta \Pi_{01}^0$ is the virtual work done by the initial loads, i.e., the conservative parts of the follower loads, and the second term on the right side of Eq. (3) is the virtual work done by the nonconservative parts of the follower loads in which $\omega_n(x_S, y_S)$ is the value of ω_n at the shear center and m_ω^S is the external warping moment with respect to the shear center.

$$m_\omega^S = m_\omega + m_x x_S + m_y y_S.$$

(Virtual work of end forces and moments)

$$\begin{aligned} W_E = \delta \Pi_{02}^0 + \{ & Q_x(z)\delta u_S + Q_y(z)\delta v_S + Q_z(z)\delta w_G \\ & - M_x(z)\delta v_S' + M_y(z)\delta u_S' + M_z(z)\delta\theta + M_\omega^S(z)\delta\theta'\} |_0^l \end{aligned} \tag{4}$$

where $\delta \Pi_{02}^0$ is the virtual work done by end forces and moments in static equilibrium state and $M_\omega^S(z)$ is a warping moment with respect to the shear center.

$$M_\omega^S(z) = M_\omega(z) + M_x(z) x_S + M_y(z) y_S.$$

Using Eqs. (1), (2), (3) and (4), the equation for the virtual work of member ij

$$W_I + W_S + W_D + W_E = 0 \tag{5}$$

is obtained. Applying the equation for the virtual work in initial stress condition

$$-\delta \Pi_i^0 + \delta \Pi_{01}^0 + \delta \Pi_{02}^0 = 0 \tag{6}$$

integrating by parts and carrying out simple calculation, Eq. (5) for the virtual work is rewritten to

$$\begin{aligned} \int_0^l & [\![[-m(\ddot{u}_S + y_S\ddot{\theta}) + \mu I_{xx}\ddot{u}_S'' - EI_{xx}u_S^{('''')} + \{Q_z^0(u_S' + y_S\theta')\}' - (M_x^0\theta)'' - q_y^S\theta \\ & + q_z(u_S' + y_S\theta') - \{m_x\theta + m_z^S v_S' + x_S(q_x^S u_S' + q_y^S v_S')\}']\delta u_S + [-m(\ddot{v}_S - x_S\ddot{\theta}) + \mu I_{yy}\ddot{v}_S'' \\ & - EI_{yy}v_S^{('''')} + \{Q_z^0(v_S' - x_S\theta')\}' - (M_y^0\theta)'' + q_x^S\theta + q_z(v_S' - x_S\theta') - \{m_y\theta - m_z^S u_S' \\ & + y_S(q_x^S u_S' + q_y^S v_S')\}']\delta v_S + \{-m\ddot{w}_G + EAw_G'' - q_x^S u_S' - q_y^S v_S'\}\delta w_G + [-(\mu I_{ps}\ddot{\theta} + m y_S \ddot{u}_S \\ & - m x_S \ddot{v}_S) + \mu I_\omega^S \ddot{\theta}'' - EI_\omega^S \theta^{('''')} + GK\theta'' - \{-Q_z^0(y_S u_S' - x_S v_S') - Q_z^0 r_S^2 \theta' - 2(M_x^0\beta_y \\ & - M_x^0\beta_x + M_\omega^{S0}\beta_\omega)\theta'\}' - M_x^0 u_S'' - M_y^0 v_S'' + (Q_x^0\beta_x + Q_y^0\beta_y - M_z^{\omega 0}\beta_\omega)'\theta + y_S q_z(u_S' \\ & + y_S\theta') - x_S q_z(v_S' - x_S\theta') - m_x(u_S' + y_S\theta') - m_y(v_S' - x_S\theta') - \{\omega_n(x_S, y_S)(q_x^S u_S' \\ & + q_y^S v_S') + (m_x y_S - m_y x_S)\theta\}']\delta\theta]\!] \, dz + [\mu I_{xx}\ddot{u}_S' - EI_{xx}u_S^{(''')} + Q_z^0(u_S' + y_S\theta') - (M_x^0\theta') \\ & - \{m_x\theta + m_z^S v_S' + x_S(q_x^S u_S' + q_y^S v_S')\} - Q_x]\delta u_S|_0^l + (EI_{xx}u_S'' - M_y)\delta u_S'|_0^l + [\mu I_{yy}\ddot{v}_S' \\ & - EI_{yy}v_S^{(''')} + Q_z^0(v_S' - x_S\theta') - (M_y^0\theta)' - \{m_y\theta - m_z^S u_S' + y_S(q_x^S u_S' + q_y^S v_S')\} - Q_y]\delta v_S|_0^l \end{aligned}$$

$$+(EI_{yy}v_S''+M_x)\delta v_S'|_0^l+(EAw_G'-Q_z)\delta w_G|_0^l+[\mu I_\omega^S\ddot{\theta}'-EI_\omega^S\theta^{('''})}+GK\theta'+Q_z^0(y_Su_S'$$
$$-x_Sv_S'+r_S^2\theta')-M_x^0u_S'-M_y^0v_S'+2(M_x^0\beta_y-M_y^0\beta_x+M_\omega^{S0}\beta_\omega)\theta'+(Q_x^0\beta_x+Q_y^0\beta_y$$
$$-M_z^{\omega 0}\beta_\omega)\theta-\{-\omega_n(x_S,\ y_S)(q_x^Su_S'+q_y^Sv_S')+(m_xy_S-m_yx_S)\theta-M_z]\delta\theta|_0^l$$
$$+(EI_\omega^S\theta''-M_\omega^S)\delta\theta'|_0^l=0 \qquad (7)$$

with $\quad r_S^2=r_0^2+x_S^2+y_S^2$

Characteristic Equations of Columns with Channel Section, Subjected to Tangential Follower Compressive Loads.

The terms $\mu I_{xx}\ddot{u}_S''$, $\mu I_{xx}\ddot{u}_S'$, $\mu I_{yy}\ddot{v}_S''$, $\mu I_{yy}\ddot{v}_S'$, $\mu I_\omega^S\ddot{\theta}''$, and $\mu I_\omega^S\ddot{\theta}'$ in Eq. (7) are disregarded and the effect of M_ω^{S0} and $M_z^{\omega 0}$ is neglected in this research. $Q_x^0=Q_y^0=M_x^0=M_y^0=0$ hold true and Q_z^0 only remains because only a concentrated or distributed load which acts on to the centroid is considered.

Let us set up the characteristic equations for five cases as shown in Fig. 1. In these problems, the initial stress Q_z^0 is

$$Q_z^0=-P \qquad \text{for case 0}$$

and $Q_z^0=-q_0\{(l-z)^2+\varepsilon(l^2-z^2)\}/2l$ for the cases I, II, III, and IV. The boundary condition are

$$\delta u_S=\delta u_S'=\delta v_S=\delta v_S'=\delta\theta=\delta\theta'=0 \quad \text{at} \quad z=0,$$
$$Q_x=-P(u_S'+y_S\theta'),\quad Q_y=-P(v_S'-x_S\theta'),$$
$$M_x=M_y=M_z=M_\omega^S=0 \quad \text{at } z=1 \text{ for case 0},$$

and

$$Q_x=Q_y=M_x=M_y=M_z=M_\omega^S=0 \quad \text{at the free end},$$
$$\delta u_S=\delta v_S=\delta\theta=M_x=M_y=M_\omega^S=0 \quad \text{at the simple supported end},$$
$$\delta u_S=\delta u_S'=\delta v_S=\delta v_S'=\delta\theta=\delta\theta'=0 \quad \text{at the clamped end}$$

for the cases I, II, III, and IV.

In the analysis, u_S, v_S, θ, and z are replaced by

$$\left.\begin{array}{ll} u_S=e^{\lambda t}U(\zeta), & v_S=e^{\lambda t}V(\zeta), \\ \theta=e^{\lambda t}\Theta(\zeta), & z=\zeta\cdot l \end{array}\right\} \qquad (8)$$

respectively, where λ is a characteristic number and $U(\zeta)$, $V(\zeta)$, and $\Theta(\zeta)$ are unknown coordinated functions, and the extended Galerkin method is applied by expressing $U(\zeta)$, $V(\zeta)$, and $\Theta(\zeta)$ with series

Table 1. Eigenfunction and frequency equations of the flexural vibration of columns.

Bowndary Condition		Normal Function u_{Sk} (v_{Sk})		Frequency Equation
CASE II	$u_{Sk}(0)=u''_{Sk}(0)=0$ $u_{Sk}(1)=u''_{Sk}(1)=0$	$u_{Sk}=\sin\omega_k\zeta$		$\sin\omega_k=0$
CASE I	$u_{Sk}(0)=u'_{Sk}(0)=0$ $u''_{Sk}(1)=u'''_{Sk}(1)=0$	$u_{Sk}=\cos\omega_k\zeta-\cosh\omega_k\zeta$ $+A_k(\sin\omega_k\zeta-\sinh\omega_k\zeta)$	$A_k=\dfrac{\sin\omega_k-\sinh\omega_k}{\cos\omega_k+\cosh\omega_k}$	$\cos\omega_k\cdot\cosh\omega_k+1=0$
CASE III	$u_{Sk}(0)=u'_{Sk}(0)=0$ $u_{Sk}(1)=u''_{Sk}(1)=0$		$A_k=-\dfrac{\cos\omega_k-\cosh\omega_k}{\sin\omega_k-\sinh\omega_k}$	$\cos\omega_k\cdot\sinh\omega_k$ $-\cosh\omega_k\cdot\sin\omega_k=0$
CASE IV	$u_{Sk}(0)=u'_{Sk}(0)=0$ $u_{Sk}(1)=u'_{Sk}(1)=0$			$\cos\omega_k\cdot\cosh\omega_k-1=0$

Table 2. Eigenfunction and frequency equations of the torsional vibration of columns.

Bowndary Condition		Normal Function θ_k		Frequency Equation
CASE II	$\theta_k(0)=\theta''_k(0)=0$ $\theta_k(1)=\theta''_k(1)=0$	$\theta_k=\sin\omega_{1k}\zeta$		$\sin\omega_{1k}=0$
CASE I	$\theta_k(0)=\theta'_k(0)=0$ $\theta''_k(1)=\theta'''_k(1)-\chi^2\theta'_k(1)=0$	$\theta_k=\cos\omega_{1k}\zeta$ $-\cosh\omega_{2k}\zeta$ $+B_k(\omega_{2k}\sin\omega_{1k}\zeta$ $-\omega_{1k}\sinh\omega_{2k}\zeta)$	$B_k=\dfrac{\omega_{1k}^2\cos\omega_{1k}+\omega_{2k}^2\cosh\omega_{2k}}{\omega_{1k}\omega_{2k}(\omega_{1k}\sin\omega_{1k}+\omega_{2k}\sinh\omega_{2k})}$	$(\omega_{1k}^2\cos\omega_{1k}+\omega_{2k}^2\cosh\omega_{2k})$ $\times\{(\omega_{1k}^2+\chi^2\cos\omega_{1k}+(\omega_{2k}^2-\chi^2)\cosh\omega_{2k}\}$ $+(\omega_{1k}\sin\omega_{1k}+\omega_{2k}\sinh\omega_{2k})$ $\times\{\omega_{1k}(\omega_{1k}^2+\chi^2)\sin\omega_{1k}-\omega_{2k}(\omega_{2k}^2-\chi^2)\sinh\omega_{2k}\}$ $=0$
CASE III	$\theta_k(0)=\theta'_k(0)=0$ $\theta_k(1)=\theta''_k(1)=0$		$B_k=\dfrac{\cos\omega_{1k}-\cosh\omega_{2k}}{\omega_{2k}\sin\omega_{1k}-\omega_{1k}\sinh\omega_{2k}}$	$\omega_{1k}\omega_{2k}(\cos\omega_{1k}-\cosh\omega_{2k})$ $\times(\omega_{1k}\sin\omega_{1k}+\omega_{2k}\sinh\omega_{2k})$ $-(\omega_{2k}\sin\omega_{1k}-\omega_{1k}\sinh\omega_{2k})$ $\times(\omega_{1k}^2\cos\omega_{1k}+\omega_{2k}^2\cosh\omega_{2k})$ $=0$
CASE IV	$\theta_k(0)=\theta'_k(0)=0$ $\theta_k(1)=\theta'_k(1)=0$			$\omega_{1k}\omega_{2k}(\cos\omega_{1k}-\cosh\omega_{2k})^2$ $+(\omega_{2k}\sin\omega_{1k}-\omega_{1k}\sinh\omega_{2k})$ $\times(\omega_{1k}\sin\omega_{1k}+\omega_{2k}\sinh\omega_{2k})$ $=0$

$\chi^2=GKl^2/EI_\omega^S$

$$\left.\begin{aligned} &U(\zeta)=\sum_{k=1}^{N}\rho_k u_{Sk}(\zeta), \quad V(\zeta)=\sum_{k=1}^{N}\sigma_k v_{Sk}(\zeta),\\ &\Theta(\zeta)=\sum_{k=1}^{N}\tau_k\theta_k(\zeta). \end{aligned}\right\} \tag{9}$$

where ρ_k, σ_k and τ_k are expansion coefficients, $u_{Sk}(\zeta)$, $v_{Sk}(\zeta)$ are kth eigenfunctions of the flexural free vibration of an unloaded column, and $\theta_k(\zeta)$ is a kth eigenfunction of the torsional free vibration of the same column. Eigenfunction and frequency equation are written in Tables 1 and 2.

Rearranging Eq. (7) by substituting Eqs. (8) and (9) into it and introducing the notations

$$\eta_{Px}=Pl^2/EI_{xx}, \quad \eta_{qx}=q_0l^3/EI_{xx},$$

$$\xi_x=m\lambda^2l^4/EI_{xx}, \quad \alpha=I_{yy}/I_{xx}, \quad \beta=GK/EI_{xx}$$

$$\gamma=I_\omega^s/h^2I_{xx}, \quad \kappa=I_{PS}/Ah^2, \quad \bar{x}_S=x_S/h,$$

$$\bar{y}_S=y_S/h, \quad \bar{r}_0^2=r_0^2/h^2, \quad \bar{r}_S^2=(r_0^2+x_S^2+y_S^2)/h^2$$

and

$$\bar{l}=l/h.$$

the following two system of equations are obtatined because $\delta\rho_k$, $\delta\sigma_k$ and $\delta\tau_k$ are arbitrary:

$$\left.\begin{aligned} &\int_0^1\sum_{k=1}^{N}\{\xi_x(\rho_k u_{Sk}+\bar{y}_S\tau_k\theta_k)+\rho_k u_{Sk}^{('''')}+\eta_{Px}(\rho_k u''_{Sk}+\bar{y}_S\theta''_k)\}u_{Si}d\zeta=0,\\ &\int_0^1\sum_{k=1}^{N}\{\xi_x(\sigma_k v_{Sk}-\bar{x}_S\tau_k\theta_k)+\alpha\sigma_k v_{Sk}^{('''')}+\eta_{Px}(\sigma_k v''_{Sk}-\bar{x}_S\theta''_k)\}v_{Si}d\zeta=0,\\ &\int_0^1\sum_{k=1}^{N}\{\xi_x(\kappa\tau_k\theta_k+\bar{y}_S\rho_k u_{Sk}-\bar{x}_S\sigma_k v_{Sk})+\gamma\tau_k\theta_k^{('''')}-\beta\bar{l}^2\tau_k\theta''\\ &\quad-\eta_{px}(\bar{y}_S\rho_k u''_{sk}+\bar{x}_S\sigma_k v''_{sk}-\bar{r}_S^2\tau_k\theta''_k\}\theta_i d\zeta-\eta_{Px}\bar{r}_0^2(\sum_{k=1}^{N}\tau_k\theta'_k)\theta_i|_1=0,\\ &\qquad\qquad i=1,2,\ldots,N, \quad \text{for case 0,} \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} &\int_0^1\sum_{k=1}^{N}\Big[\xi_x(\rho_k u_{Sk}+\bar{y}_S\tau_k\theta_k)+\rho_k u_{Sk}^{('''')}+\frac{1}{2}\eta_{qx}\{(1-\zeta)^2\\ &\quad+\varepsilon(1-\zeta^2)\}(\rho_k u''_{sk}+\bar{y}\tau_k\theta''_k)\Big]u_{Si}d\zeta=0,\\ &\int_0^1\sum_{k=1}^{N}\Big[\xi_x(\sigma_k v''_{Sk}-\bar{x}_S\tau_k\theta''_k)+\alpha\sigma_k v_{Sk}^{('''')}+\frac{1}{2}\eta_{qx}\{(1-\zeta)^2\\ &\quad+\varepsilon(1-\zeta)^2\}(\sigma_k v''_{Sk}-\bar{x}_S\tau_k\theta''_k)\Big]v_{Si}d\zeta=0,\\ &\int_0^1\sum_{k=1}^{N}\Big[\xi_x(\kappa\tau_k\theta_k+\bar{y}_S\rho_k u_{Sk}-\bar{x}_S\sigma_k v_{Sk})+\gamma\tau_k\theta_k^{('''')}-\beta\bar{l}^2\tau_k\theta''_k \end{aligned}\right\} \tag{11}$$

$$+\eta_{qx}\bar{y}_S\{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}\rho_k u''_{Sk}-\eta_{qx}\bar{x}_S\{(1-\zeta)^2$$

$$+\varepsilon(1-\zeta^2)\}\sigma_k v''_{Sk}-\eta_{qx}\bar{r}_0^2\{1-(1-\varepsilon)\zeta\}\tau_k\theta'_k+\frac{1}{2}\eta_{qx}\bar{r}_S^2\{(1-\zeta)^2$$

$$+\varepsilon(1-\zeta^2)\}\tau_k\theta''_k\Big]\theta_i d\zeta=0, \quad i=1,\ 2,\ldots,\ N, \Bigg)$$

for cases I, II, III and IV.

The determinant of these system of equations must be zero if the homogeneous systems are to have the unique solution η_k, σ_k and τ_k. These conditions are formulated in matrix form as follow:

$$|\xi_x\mathbf{A}+\mathbf{B}+\eta_{Px}\mathbf{C}|=0 \tag{12}$$

and

$$|\xi_x\mathbf{A}+\mathbf{B}+\eta_{qx}\mathbf{D}|=0 \tag{13}$$

with $\mathbf{A}$ = mass matrix, $\mathbf{B}$ = stiffness matrix,
$\mathbf{C}$ = initial stress matrix under concentrated follower load,
and $\mathbf{D}$ = initial stress matrix under distributed follower load.

The structure of these marices is

$$\mathbf{A}=\begin{pmatrix}[a_{1ik}], & [a_{2ik}], & [a_{3ik}]\\ [a_{4ik}], & [a_{5ik}], & [a_{6ik}]\\ [a_{7ik}], & [a_{8ik}], & [a_{9ik}]\end{pmatrix} \qquad (i,\ k=1,\ 2,\ldots,\ N)$$

with $a_{1ik}=a_{5ik}=\delta_{ik}$, $a_{2ik}=a_{4ik}=0$,

$$a_{3ik}=a_{7ik}=\bar{y}_S\int_0^1\theta_k u_{Si}d\zeta,$$

$$a_{6ik}=a_{8ik}=-\bar{x}_S\int_0^1\theta_k v_{Si}d\zeta,$$

$a_{9ik}=\kappa\delta_{ik}$, and δ_{ik} = Kronecker's symbol.

$$\mathbf{B}=\begin{pmatrix}[b_{1ik}], & [b_{2ik}], & [b_{3ik}]\\ [b_{4ik}], & [b_{5ik}], & [b_{6ik}]\\ [b_{7ik}], & [b_{8ik}], & [b_{9ik}]\end{pmatrix} \qquad (i,\ k=1,\ 2,\ldots,\ N),$$

$$b_{1ik}=\lambda_i^4\delta_{ik}, \quad b_{5ik}=\alpha\lambda_i^4\delta_{ik},$$

$$b_{2ik}=b_{3ik}=b_{4ik}=b_{6ik}=b_{7ik}=b_{8ik}=0,$$

$$b_{9ik}=\gamma\int_0^1\theta_k^{('''')}\theta_i d\zeta-\beta\bar{l}^2\int_0^1\theta''_k\theta_i d\zeta,$$

$$\mathbf{C}=\begin{pmatrix} [c_{1ik}], & [c_{2ik}], & [c_{3ik}] \\ [c_{4ik}], & [c_{5ik}], & [c_{6ik}] \\ [c_{7ik}], & [c_{8ik}], & [c_{9ik}] \end{pmatrix} \qquad (i,\ k=1,\ 2,\dots,\ N)$$

$$c_{1ik}=\int_0^1 u_{Sk}u_{Si}d\zeta, \quad c_{2ik}=c_{4ik}=0,$$

$$c_{3ik}=\bar{y}_S\int_0^1 \theta_k'' u_{Si}d\zeta, \quad c_{6ik}=-\bar{x}_S\int_0^1 \theta_k'' v_{Si}d\zeta,$$

$$c_{7ik}=\bar{y}_S\int_0^1 u_{Sk}''\theta_i d\zeta, \quad c_{8ik}=-\bar{x}_S\int_0^1 v_{Sk}''\theta_i d\zeta,$$

$$c_{9ik}=\bar{r}_S^2\int_0^1 \theta_k''\theta_i d\zeta-\bar{r}_0^2\theta_k'\theta_i|_1,$$

$$\mathbf{D}=\begin{pmatrix} [d_{1ik}], & [d_{2ik}], & [d_{3ik}] \\ [d_{4ik}], & [d_{5ik}], & [d_{6ik}] \\ [d_{7ik}], & [d_{8ik}], & [d_{9ik}] \end{pmatrix} \qquad (i,\ k=1,\ 2,\dots,\ N),$$

$$d_{1ik}=\frac{1}{2}\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}u_{Sk}''u_{Si}d\zeta,$$

$$d_{2ik}=d_{4ik}=0,$$

$$d_{3ik}=\bar{y}_S\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}\theta_k''u_{Si}d\zeta,$$

$$d_{5ik}=\frac{1}{2}\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}v_{Sk}''v_{Si}d\zeta,$$

$$d_{6ik}=-\bar{x}_S\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}\theta_k''v_{Si}d\zeta,$$

$$d_{7ik}=\bar{y}_S\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}u_{Sk}''\theta_i d\zeta,$$

$$d_{8ik}=-\bar{x}_S\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}v_{Sk}''\theta_i d\zeta,$$

$$d_{9ik}=\frac{1}{2}\bar{r}_S^2\int_0^1 \{(1-\zeta)^2+\varepsilon(1-\zeta^2)\}\theta_k''\theta_i d\zeta$$

$$-\bar{r}_0^2\int_0^1 \{1-(1-\varepsilon)\zeta\}\theta_k'\theta_i d\zeta,$$

Eqs. (12) and (13) are the characteristic equations.

Eigenvalue Curves of Columns

Stability problems of following column with channel section, shown in Fig. 5, are examined by obtaining trajectories of eigenvalues: (a) a column clamped at one end and subjected to a concentrated tangential load (case 0) and (b) four typical columns as shown in Fig. 1, subjected to uniformly, triangularly, and trapezoidally distributed tangential loads (cases I, II, III, and IV).

(A column subjected to a concentrated tangential load...case 0)

Space eigenvalue curves of a column with $f=b/h=0.75$ are illustrated in Fig. 6, where η_{Px} is a nondimensional load and $R_e\xi_x$ and $I_m\xi_x$ are real and imaginary parts of $\xi_x=\lambda\tau^2\sqrt{m/EI_{xx}}$, denoting the nondimensional eigenvalue. The curve AIE is an

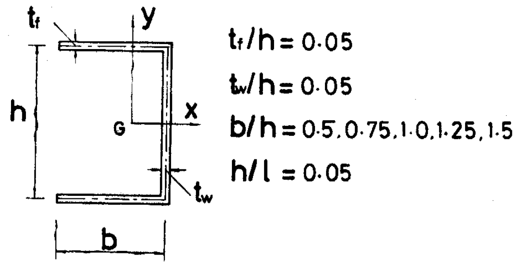

Fig. 5. Channel section.

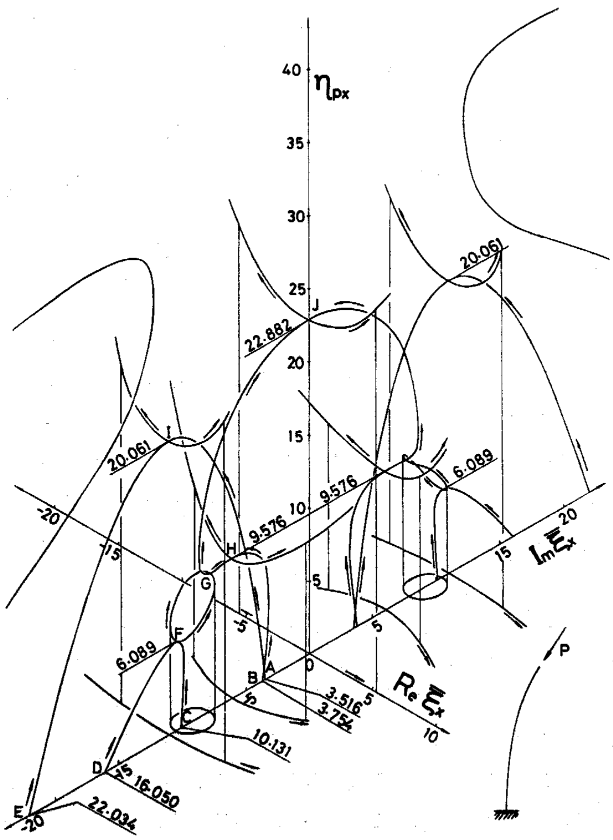

Fig. 6. Eigenvalue trajectories of a column with channel section ($b/h=0.75$), which is clamped at one end and free at other end and subjected to a concentrated load.

eigenvalue curve for uncoupled flexural vibration with respect to the principal axis y of the column section, of which $\eta_{Px}=20.061$ at the top I corresponds to Beck's solution $\eta_{Px}=20.05$, and the other curves are the trajectories of the eigenvalue of the coupled vibration of bending with respect to axies x and torsion about axis z.

When f, the ratio of b to h, is increased from 0.5 to 1.5, the behavior of the trajectories of 1st, 2nd and 3rd eigenvalues of the coupled vibration, which corresponds to curves BHGJ and CFD in Fig. 6, is shown in Fig. 7. Comparing the critical values in Fig. 7 with the critical value of the uncoupled flexural vibration $\eta_{Py}=20.05$, which is estimated under the assumption that bending and torsion are uncoupled, it is seen that the critical value is lowered by coupling with torsion.

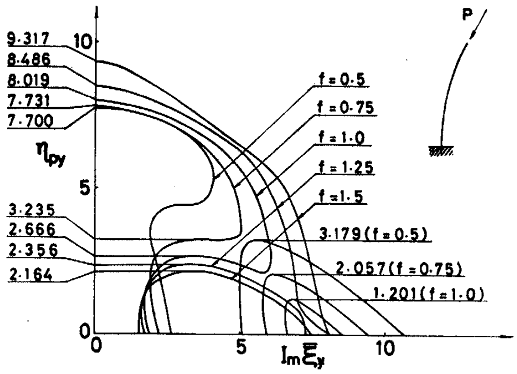

Fig. 7. The change of behavior of eigenvalue trajectories of a column of Case 0 for change of ratio $f=b/h$.

The critical value decreases as f increases from 0.5 to 1.0 and from 1.25 to 1.5, but it increases for f between 1.0 and 1.25, and there will be a jump in the critical value.

An unstable phenomenon of the flutter type occurs in the above region of f.

(Columns subjected to a uniformly distributed tangential load)

This problem corresponds to the case where ε is equal to 1.0 in the shape of loading shown in Fig. 1. Space Eigenvalue curves of a column with $f=0.75$, which is clamped at one end and free at the other, are illustrated in Fig. 8.

The curve AIE is the eigenvalue curve for uncoupled flexural vibration with respect to the principal axis y, where the nondimensional load $\eta_{qx}=40.083$ at the top I corresponds to Leipholz's solution $\eta_{qx}=40.05$ [7], and the other curves are the trajectories of eigenvalues for coupled vibration of bending and torsion. The change of behavior of the eigenvalue curves for coupled vibration with increase of f is shown in Fig. 9.

It is seen that the behavior of the eigenvalue curves is different from that shown in Fig. 7, but instability of the flutter type occurs for every value of the ratio f between 0.5 and 1.5 as shown in Fig. 7.

Eigenvalue curves of four typical columns with $f=0.75$, where case I shows a column clamped at one end, case II a column simply supported at both ends, case III a column clamped at one end and simply supported at the other end, and case IV a column clamped at both ends as show in Fig. 1, are shown in Fig. 10, where the curve groups on left and right side respectively correspond to coupled vibration of bending with respect to the axis x and torsion and uncoupled flexural vibration with respect to the axis y. The curves and critical values for uncoupled vibration agree with the results [3, 7,

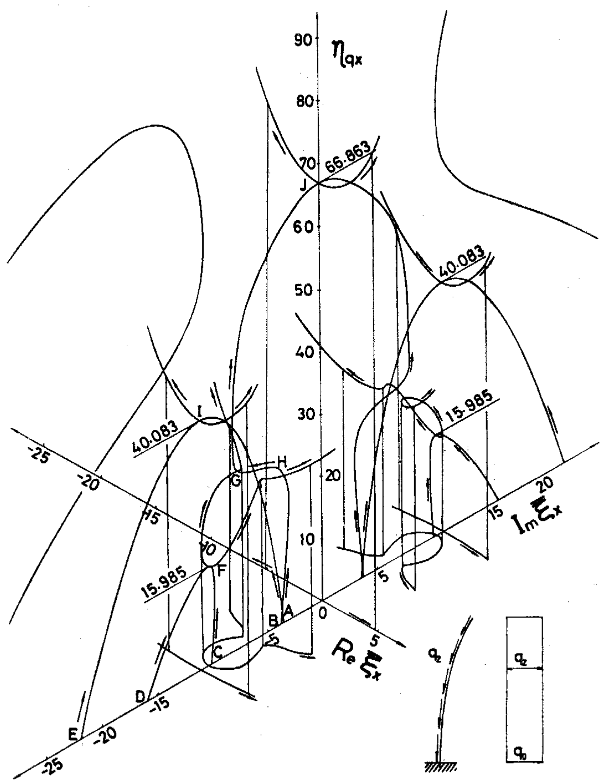

Fig. 8. Eigenvalue trajectories of a column with channel section ($b/h=0.75$), which is subjected to uniformly distributed tangential load.

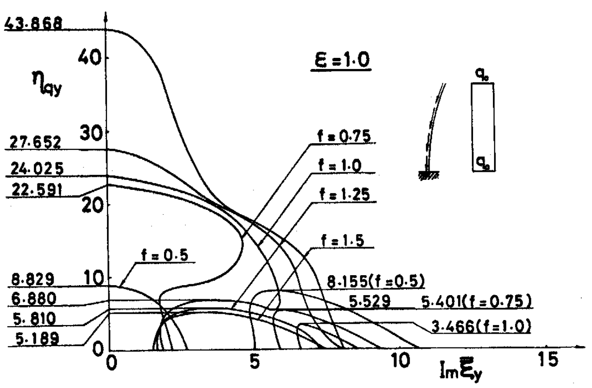

Fig. 9. The change of behavior of eigenvalue trajectories of a column, subjected to uniformly distributed tangential load, when a ratio $f=b/h$ varies.

8] investigated by many researchers up to date.

The eigenvalue curves for coupled vibration are similar to those for uncoupled vibration, and instability of the divergence type occurs in the same way as for uncoupled vibration except for case I.

(Columns subjected to triangulaly and trapezoidally distributed tangential loads)

The results of the investigation of the problems where ε is equal to 0.0, 0.5 and 1.5 are shown subsequently.

Space eigenvalue curves of a column with $f=0.75$, clamped at one end and subjected to a triangularly distributed load, are illustrated in Fig. 11. The curve AIE is an eigenvalue curve for the same uncoupled flexural vibration as the one in Fig. 8, and the nondimensional load $\eta_{qx}=150.97$ at the top I of the same curve corresponds to

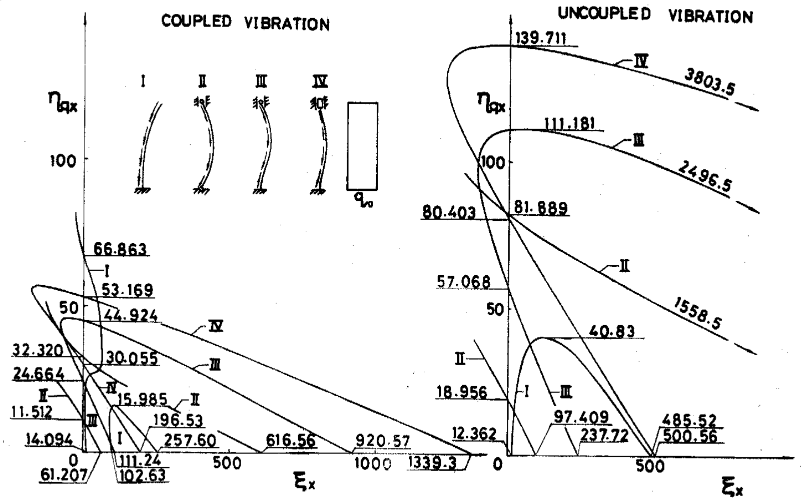

Fig. 10. Eigenvalue curves of columns with channel section (f=0.75), which are subjected to uniformly distributed tangential load.

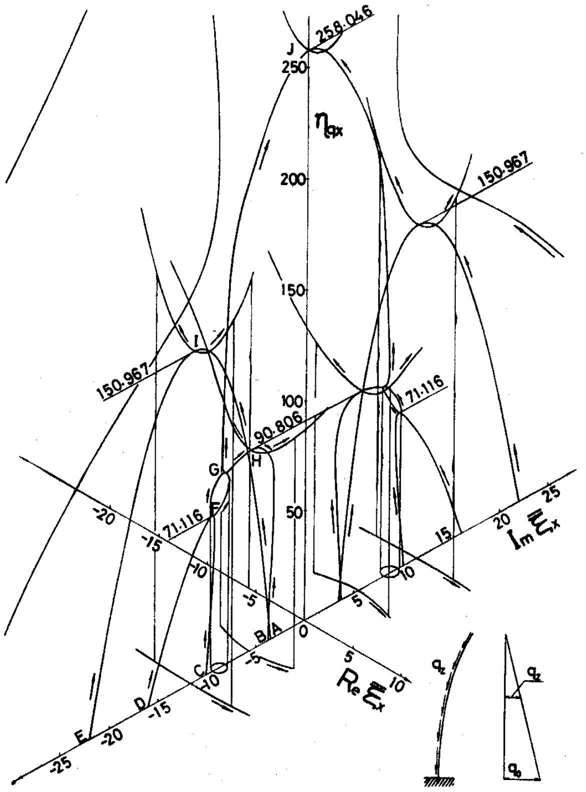

Fig. 11. Eigenvalue trajectories of a column with channel section (b/h=0.75), which is subjected to triangularly distributed tangential load.

Hauger's soulution $\eta_{qx}=158.2$ [3]. The other curves are eigenvalue curves for coupled vibration of bending and torsion. Eigenvalue curves of coumns for the cases II, III, and IV are shown in Fig. 12 together with those for case I in the same manner as in Fig. 10.

The critical values for uncoupled vibration agree with the results [3, 8] calculated up to date. Instability of the flutter type occurs for case I and instability of the divergence type occurs for the other cases in either coupled or uncouped vibration.

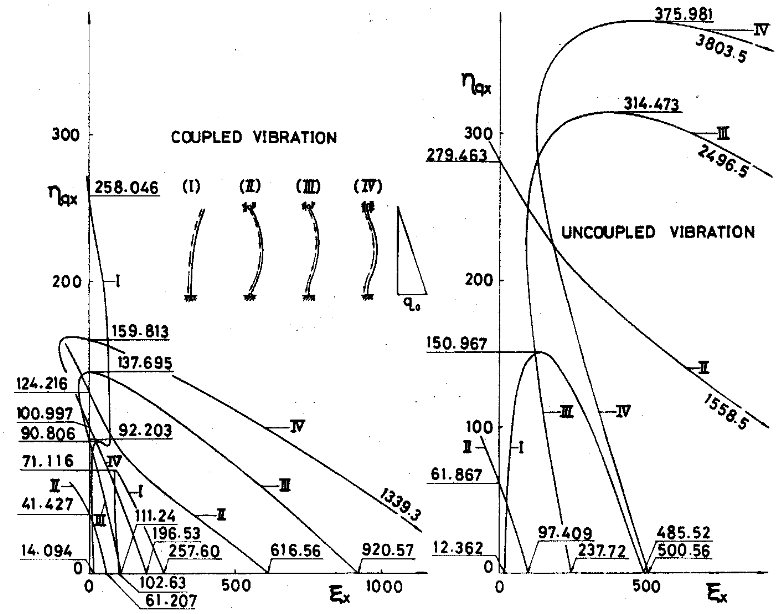

Fig. 12. Eigenvalue curves of columns with channel section ($b/h=0.75$), which are subjected to triangularly distributed tangential load.

The change of behavior of the trajectories of 1st, 2nd, and 3rd eigenvalues for coupled vibration of a column clamped at one end is shown in Figs. 13, 14, and 15 for change of f from 0.5 to 1.5, where Fig. 13 corresponds to the shape of loading with $\varepsilon=0.0$, that is triangularly distributed loading, and Figs. 14 and 15 correspond to the shape of loading with $\varepsilon=0.5$ and 1.5 respectively, that is trapezoidally distributed loading. Comparing with Figs. 13, 14, 9, and 15, it is seen that the behavior of the eigenvalue cueves for coupled vibration is similar when ε changes from 0.0 to 1.5, and the critical values decrease as ε increases.

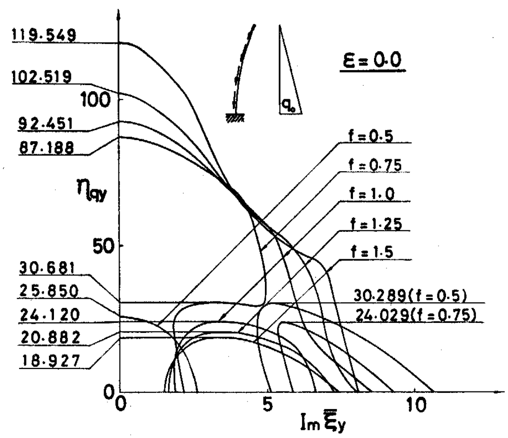

Fig. 13. The change of behavior of eigenvalue trajectories of a column, subjected to triangularly distributed tangential load, when a ratio $f=b/h$ varies.

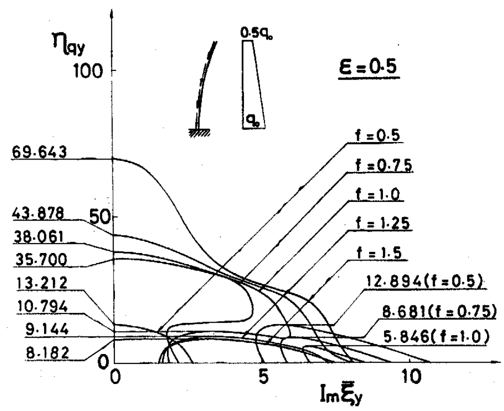

Fig. 14. The change of behavior of eigenvalue trajectories of a column, subjected to trapezoidally distributed tangential load with $\varepsilon=0.5$, when a ratio $f=b/h$ varies.

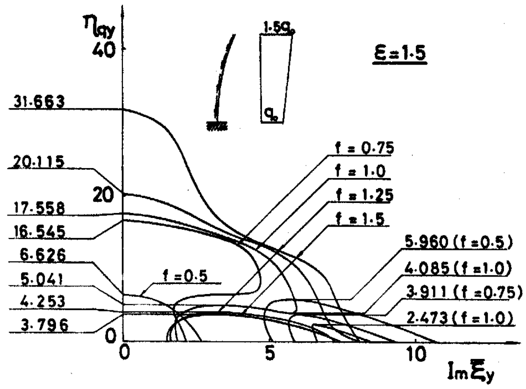

Fig. 15. The change of behavior of eigenvalue trajectories of a column, subjected to trapezoidally distributed tangential load with $\varepsilon=1.5$, when a ratio $f=b/h$ varies.

Conclusion

The trajectories of the eigenvalues of columns with channel section, subjected to concentrated and distributed tangential loads, have been obtained by applying the five-term approximation of the extended Galerkin method, and their instabilities have been investigated. The following facts have been confirmed as results; (1) coupled vibration of column clamped at one end and subjected to a concentrated tangential load and uniformly and trapezoidally distributed loads, where ε is equal to 0.5, 1.0, and 1.5 in the shape of loading, show only unstable phenomena of the flutter type as f increases from 0.5 to 1.5. However, the same columns subjected to a triangulaly distributed load show flutter for f between 0.75 and 1.5 and divergence for $f=0.5$, (2) when a column with $f=0.75$ is subjected to unformly and triangularly distributed loads, coupled and uncoupled vibrations cause flutter for case and divergence for the cases II, III, and IV, and (3) the critical load is lowered by coupling, as reported by Mote and Matsumoto in their publication [5].

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