Heat Conduction Combined with Thermal Radiation and Convection

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Abstract

The application of the Jacobi elliptic functions to heat transfer problems is proposed for the first time by the authors.

The applications to a flat plate and a cylinder, which have already been solved, are assured firstly. The authors apply elliptic functions to heat conduction problem combined with thermal radiation and convection, which has not been solved so far. The order of accuracy of numerical calculations are O(h^5). It is shown that the non-linear boundary condition of the second kind is reduced to the linear boundary condition of the first kind which is very simple, by the application of the Jacobi elliptic functions.

NOMENCLATURE

cn,	the elliptic function
dn,	the elliptic function
h,	$\frac{(r_2-r_1)}{\lambda}\cdot\alpha$
K,	the complete elliptic integral
k,	modulus of elliptic function
k',	$(k')^2+k^2=1$
N,	$\frac{\varepsilon\cdot\sigma\cdot T_0^3\cdot(r_2-r_1)}{\lambda}$
r,	radial coordinate
r_1,	inner radius ($x=0$)
r_2,	outer radius ($x=1$)
sn,	the elliptic function
T,	absolute temperature
T_0,	constant temperature
t,	$\frac{\kappa\cdot\tau}{(r_2-r_1)^2}$
v,	$\frac{\partial\theta}{\partial x}+h\theta+N\theta^4$, (38)
$X(t)$,	$(\theta)_{x=1}$
x,	$\frac{(r-r_1)}{(r_2-r_1)}$
Y,	$\frac{\partial cn}{\partial k}$, (26)
y,	$\frac{\partial sn}{\partial k}$, (11)

Greek symbols

α,	heat transfer coefficient
ε,	emissivity
σ,	Stefan-Boltzmann constant
κ,	thermal diffusivity
λ,	thermal conductivity
θ,	$\frac{(T-T_0)}{T_0}$
τ,	time

Subscripts

0,	constant
1,	inner radius r_1
2,	outer radius r_2

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1. Introduction

The problems of unsteady temperature profiles arise in considering thermal stress, thermal shock, thermal fatigue, etc.

Studies of heat conduction were, in the past, restricted almost exculsively to linear boundary conditions and constant properties. We may not be able to solve mathematically a fundamental equation of unsteady heat conduction under the non-linear boundary condition, in which the heat transfer combined with thermal radiation and convection arises on the boundary surface. Only recent developments in electronic computers may make it possible to solve under such situations.

Consequently, researches on the numerical analysis of unsteady heat conduction in case of linear boundary conditions have been published in various magazines.

The present paper describes the numerical investigations on the unsteady heat conduction of a pipe with the circular cross section under the boundary conditions that the temperature profiles of the inside surface periodically vary with time and the outside surface is subjected to the heat transfer combined with thermal radiation and convection.

The numerical solutions are obtained by use of the Jacobi elliptic functions.

The heat conduction of the flat plate with radiation on one face and perfect insulation on the other is calculated. After careful confirmation that the results coincides very well with the numerical solution by B. Gay [1], the Jacobi elliptic functions are introduced into the problem of heat conduction of a pipe under the linear boundary condition, and what is more, the problem of non-linear boundary condition with the heat transfer combined with thermal radiation and convection, which has not been solved so far. The calculated values on the temperature profiles on the outside surface of a circular hollow pipe as the function of time are shown in the next figures.

2. Principle of the method

In Fig. 1, the curve of A shows the exact solution, and the curve of S shows the

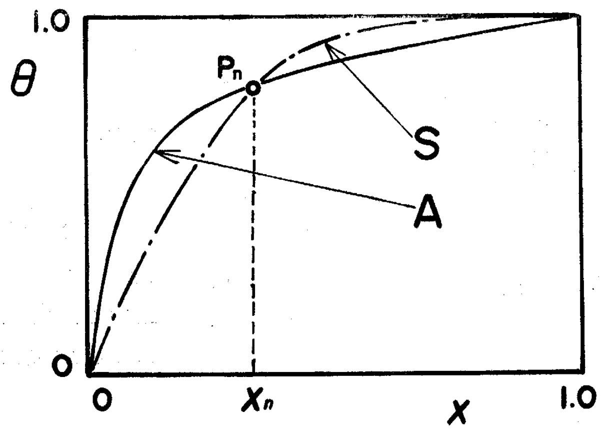

Fig. 1 Principle.
A=Exact Solution S=Sn Curve

Jacobi elliptic function Sn according to modulus k.

In Fig. 2, if the values of x_n are changed, P_n and k_n are changed accordingly.

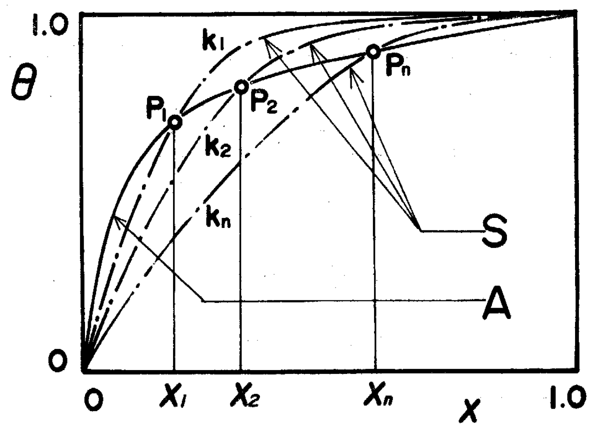

Fig. 2 Detail of Principle.
A = Exact Solution $S = Sn$ Curve

When we apply this principle to heat conduction, heat transfer and thermal radiation, the complicated boundary conditions must be reduced to the simplified boundary conditions which are satisfied by the elliptic functions. This may be able to do, in many cases of the heat transfer combined with thermal radiation and convection.

3. Flat plate

The authors apply firstly this principle to the problem on the thermal radiation and heat conduction which was already solved. The result is compared with the problem which was already solved.

Equation of heat conduction

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2} \tag{1}$$

Initial condition

$$t=0, \quad \theta=1 \tag{2}$$

Boundary conditions

$$x=0, \quad \frac{\partial\theta}{\partial x}=N\theta^4 \tag{3}$$

$$x=1, \quad \frac{\partial\theta}{\partial x}=0 \tag{4}$$

The boundary condition (3) is nonlinear and difficult to solve mathematically because of $N\theta^4$ in the right hand side of (3). This problem was solved by Gay et al. [1]. To solve (1), (2), (3) and (4) numerically, put

$$V=N\cdot v=-\frac{\partial\theta}{\partial x}+N\theta^4 \tag{5}$$

The equation and conditions (1), (2), (3) and (4) are reduced to the next equations.

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-12N\theta^2\left(\frac{\partial\theta}{\partial x}\right)^2 \tag{6}$$

$$t=0,\quad v=1 \tag{7}$$

$$x=0,\quad v=0 \tag{8}$$

$$x=1,\quad v=k(t) \tag{9}$$

To solve these equations, put

$$v=k(t)\cdot sn\{Kx,\ k(t)\},\quad k(0)=1 \tag{10}$$

From the formulas [2] on the elliptic function,

$$\varphi=\sin^{-1}sn=\int_0^x dn\,dx,$$

$$dn^2x=1-k^2sn^2x,$$

we obtain

$$y=\frac{\partial sn}{\partial k},$$

$$y=-kcn\int_0^x\left(\frac{sn^2}{dn}\right)\cdot dx-k^2\cdot cn\int_0^x\left(\frac{sn}{dn}\right)\cdot y\cdot dx \tag{11}$$

The equation for y (11) is the Volterra's integral equation.

If we consider the temperature at $x=1$ only, by use of (11), we obtain the next equation.

$$t=\int_k^1\frac{dk}{[K^2k(k^2-1)]} \tag{12}$$

The Simpson's 1/3 rule $0(h^5)$ is used for the numerical integration of (12). The order of accuracy of this numerical integration is the order of h^5 $\{=0(h^5)\}$. h is the increment of co-ordinate.

In Fig. 3, the result of numerical integration by the authors coincides very well with the result by Gay et al. [1].

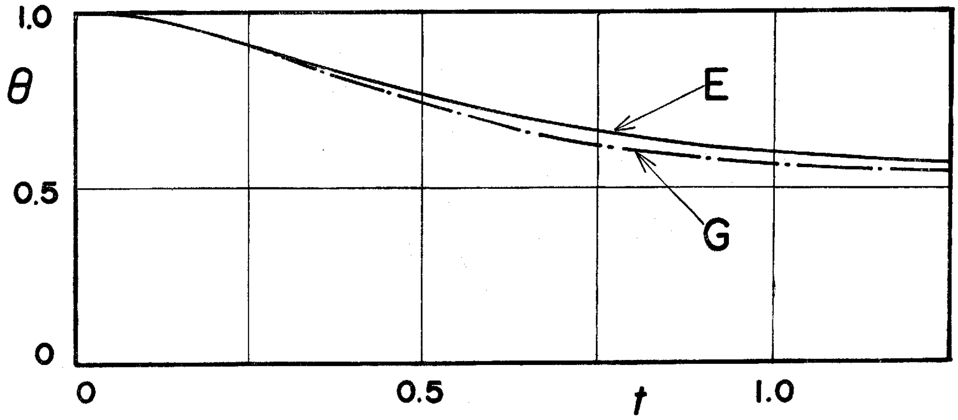

Fig. 3 Comparison.
$X=1.0$, $N=10$, E=Solution by Elliptic Function, G=Solution by B. Gay

4. Circular pipe with convection only

The authors apply secondly this principle to the problem on the heat conduction of a circular pipe which is easily solved by the finite-difference method, or with Bessel functions mathematically, but by the finite difference method or Bessel function method, the high accuracy cannot be expected, because, in case that high accuracy is demanded, the values of many terms are necessary by these two methods, but the values of many terms are generally unknown exactly. In case that high accuracy is demanded, usual methods are inconvenient. Therefore, the method to apply the elliptic functions is proposed by the authors, for the first time.

In Fig. 4, in case we use the following nondimensional forms,

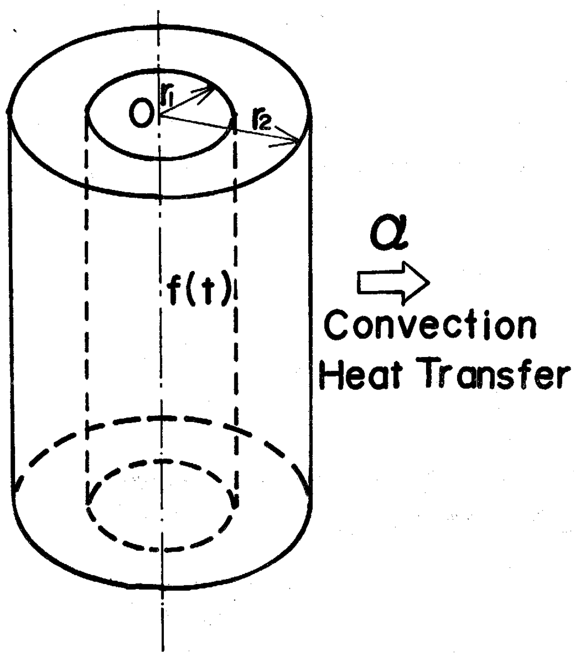

Fig. 4 Cylindrical Coordinate.

$$\theta=\frac{(T-T_0)}{T_0},\ x=\frac{(r-r_1)}{(r_2-r_1)},\ t=\frac{\kappa\tau}{(r_2-r_1)^2},\ h=\frac{(r_2-r_1)\alpha}{\lambda},$$

we obtain the next equation and boundary conditions.

Equation of heat conduction

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}+\frac{1}{(x+R_1)}\frac{\partial\theta}{\partial x},\ R_1=\frac{r_1}{r_2-r_1} \tag{13}$$

Initial condition

$$t=0,\quad \theta=0 \tag{14}$$

Boundary conditions

$$x=0,\quad \theta=f(t)=C\cdot\sin\omega t \tag{15}$$

$$x=1,\quad -\left(\frac{\partial\theta}{\partial x}\right)=h\cdot\theta \tag{16}$$

To simplify the boundary conditions, using

$$v=\frac{\partial\theta}{\partial x}+h\cdot\theta,\quad \left(\frac{\partial\theta}{\partial x}\right)_{x=0}=0, \tag{17}$$

we obtain the next equation and boundary conditions.

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\frac{1}{(x+R_1)}\cdot\frac{\partial v}{\partial x} \tag{18}$$

$$t=0,\quad v=0 \tag{19}$$

$$x=0,\quad v=\phi(t)=c\cdot h\cdot\sin\omega t \tag{20}$$

$$x=1,\quad v=0 \tag{21}$$

The Jacobi elliptic function cn is used to solve (18), (19), (20) and (21), putting in the forms

$$v=\phi(t)\cdot cn\{Kx,\ k(t)\} \tag{22}$$

$$k(0)=1 \tag{23}$$

If we consider the temperature at $x=1$ only, from (18), (19), (20), (21), (22) and (23), we obtain the next equation.

$$t=\int_k^1\frac{(1+R_1)\cdot Y}{\{K\cdot k'\}}\cdot dk \tag{24}$$

$$Y=\left(\frac{\partial cn}{\partial k}\right)$$

is found from (11) and

$$sn^2+cn^2=1 \tag{25}$$

$$Y=k\cdot sn\int_0^x\left(\frac{sn^2}{dn}\right)dx+k^2\cdot sn\int_0^x\left(\frac{cn}{dn}\right)\cdot Y\cdot dx \tag{26}$$

The equation for Y (26) is the Volterra's integral equation.

The Simpson's 1/3 rule is used for the numerical integration of (24), and we obtain the relation of t and k.

Next, $\theta(x)=\theta\cdot(1-a)$ is expanded into Taylor series at the point of $(x=1)$,

$$\theta(x)=(\theta)_{x=1}-a\cdot(\theta')_{x=1}+a^2\frac{1}{2!}(\theta'')_{x=1}-\cdots \tag{27}$$

$$\frac{\partial\theta}{\partial t}=\left\{\frac{\partial\theta(1-a)}{\partial t}\right\}_{x=1,a\to0}=\frac{dX}{dt} \tag{28}$$

$$\frac{\partial\theta}{\partial x}=\left\{\frac{\partial\theta(1-a)}{\partial a}\right\}_{x=1,a\to0}=h\cdot X \tag{29}$$

$$\frac{\partial^2\theta}{\partial x^2}=\left\{\frac{\partial^2\theta(1-a)}{\partial a^2}\right\}_{x=1,a\to0}=-\phi\cdot K\cdot k'+h^2\cdot X \tag{30}$$

$$X=(\theta)_{x=1} \tag{31}$$

From (13) and (24), we obtain the next Volterra's integral equation on X.

$$X=\int_k^1\left[\left\{\frac{1}{(1+R_1)}+h\right\}\frac{h(1+R_1)Y}{Kk'}\cdot X-\phi(1+R_1)\cdot Y\right]\cdot dk \tag{32}$$

X (32) is solved numerically by the Simpson's 1/3 rule on the method of numerical integration, and we obtain the relation between X and k, therefore the relation of X and t is obtained from (24).

To show the results of numerical calculations of the temperature on the outside surface $(x=1)$ of the circular hollow pipe of iron, the authors use the next thermo-physical property and constants [4],

$c=1,\quad \omega=2\pi,\quad$ thermal diffusivity $\kappa=0.036\ (\mathrm{m^2/h})$.

The results of numerical calculations with the elliptic functions are compared with the results by the finite difference method. In the finite-difference method on (13), $\frac{\partial\theta}{\partial t}$ is replaced by the forward difference quotient [3], and $\frac{\partial^2\theta}{\partial x^2}$ and $\frac{\partial\theta}{\partial x}$ are replaced by the central difference quotient by Collatz [3].

In Figs. 5 and 6, the result $0(h^5)$ of the numerical integration by the method to apply the elliptic functions proposed by the authors coincides very well with the result by the finite-difference method.

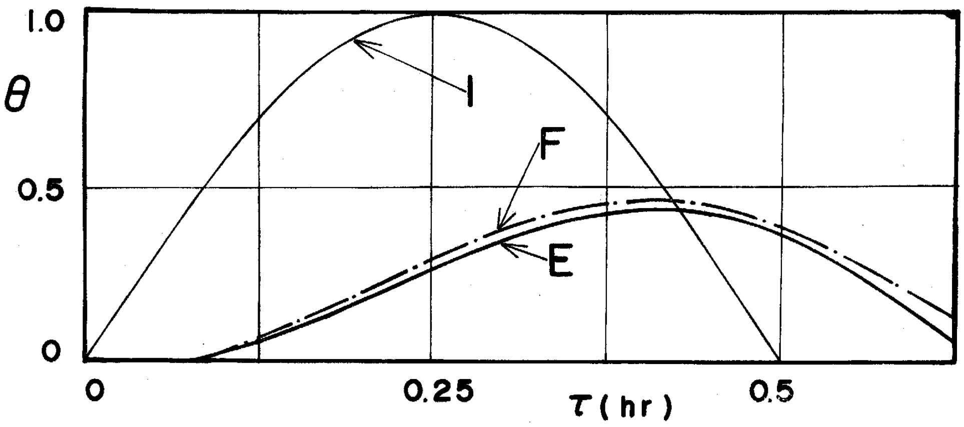

Fig. 5 Comparison.
$r_1=0.05$ m, $r_2=0.20$ m, $h=1.0$, $I(r=r_1)$, $E(r=r_2)$=Solution by Elliptic Function, $F(r=r_2)$=Solution by Finite-Difference Method

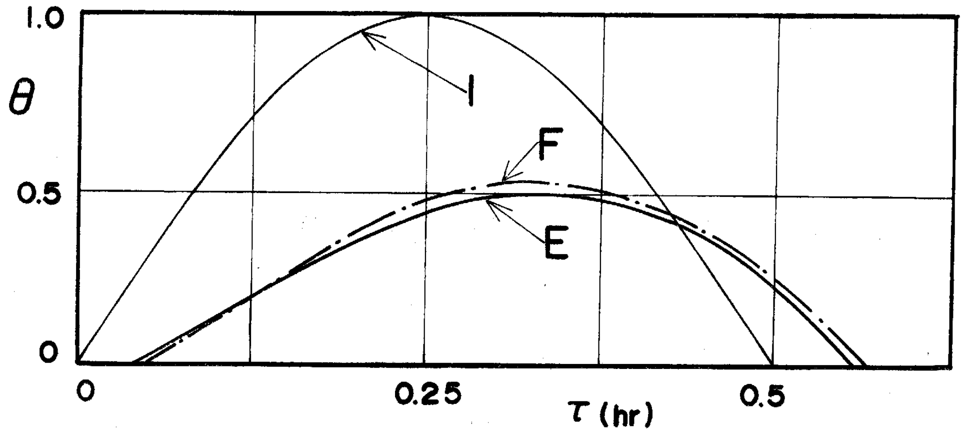

Fig. 6 Comparison.
$r_1=0.05$ m, $r_2=0.15$ m, $h=1.0$, $I(r=r_1)$, $E(r=r_2)$=Solution by Elliptic Function, $F(r=r_2)$=Solution by Finite-Difference Method

5. The combined thermal radiation and convection on the outer surface of a circular pipe

This problem has not yet been treated by anyone. The authors propose the method to solve the problem by the Jacobi elliptic function.

Equation of heat conduction

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}+\frac{1}{(x+R_1)}\cdot\frac{\partial\theta}{\partial x} \tag{33}$$

Initial condition

$$t=0,\quad \theta=0 \tag{34}$$

Boundary conditions

$$x=0, \quad \theta=f(t)=c\cdot\sin\omega t \tag{35}$$

$$x=1, \quad -\left(\frac{\partial\theta}{\partial x}\right)=h\cdot\theta+N\theta^4 \tag{36}$$

$$h=\frac{\alpha(r_2-r_1)}{\lambda}, \quad N=\frac{\varepsilon\cdot\sigma\cdot T_0^3\cdot(r_2-r_1)}{\lambda} \tag{37}$$

To simplify the non-linear boundary condition of the second kind (36), the authors propose for the first time the next transformation.

$$v=\frac{\partial\theta}{\partial x}+h\theta+N\theta^4 \tag{38}$$

The equations (33), (34), (35) and (36) are reduced to the next equations.

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\frac{1}{(1+R_1)}\cdot\frac{\partial v}{\partial x}-12N\theta^2\cdot\left(\frac{\partial\theta}{\partial x}\right)^2 \tag{39}$$

$$t=0, \quad v=0 \tag{40}$$

$$x=0, \quad v=\psi(t)=c\cdot h\cdot\sin\omega t+C^4\cdot N\cdot\sin^4\omega t \tag{41}$$

$$x=1, \quad v=0 \tag{42}$$

To solve the equations (39), (40), (41) and (42), use

$$v=\psi(t)\cdot cn\{Kx,\ k(t)\}, \quad k(0)=1 \tag{43}$$

To find $X(t)=\theta\ (x=1)$ on the outer surface of a pipe, use

$$Y=\frac{\partial cn}{\partial k}.$$

The equation (39) is reduced to the next equation (44).

$$t=\int_k^1\frac{\psi\cdot Y\cdot dk}{\left\{\frac{\psi\cdot K\cdot k'}{(1+R_1)}+12NX^4(h+NX^3)^2\right\}} \tag{44}$$

Next, at $x=1$, $\theta(x)=\theta(1-a)$ is expanded into Taylor series. The next equation (45) is obtained.

$$X=\int_k^1\frac{\psi Y\left\{-\psi Kk'-(h+4NX^3)+\frac{1}{(1+R_1)}X(h+NX^3)\right\}dk}{\left\{\frac{\psi Kk'}{(1+R_1)}+12NX^4(h+12NX^3)^2\right\}} \tag{45}$$

The equations (44) and (45) are the simultaneous equations of non-linear Volterra's integral equation with two unknown functions of t and X. The simultaneous equations are solved by the methods of numerical integartion, applying the Simpson's 1/3 rule $O(h^5)$.

The numerical calculation is accomplished on the outer surface ($r=r_2$) of a pipe subjected to combined thermal radiation and convection.

This problem has not been solved so far, and is treated for the first time by the authors.

$$c=1, \quad \omega=2\pi \quad \text{(frequency)},$$

$\kappa=0.036\,(m^2/h)$ of iron [4] is used in the numerical calculation, and the result of the numerical calculation $0(h^5)$ is shown in Fig. 7 and compared with the case of convection only.

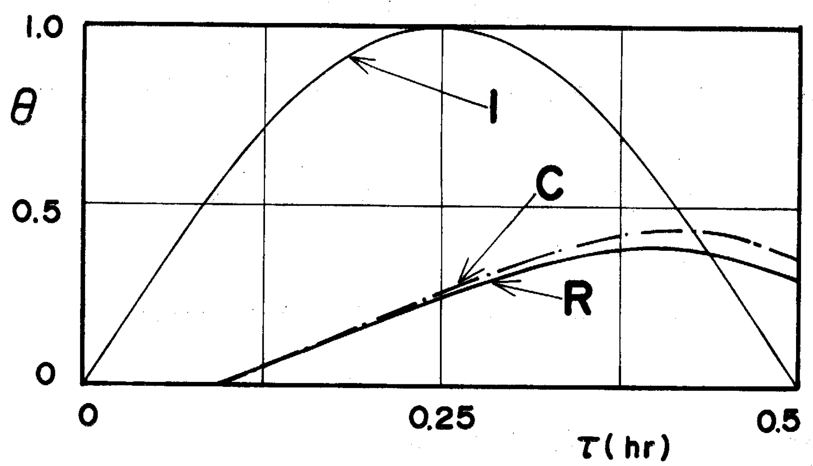

Fig. 7 Comparison.
I ($r=r_1$), C ($r=r_2$)=Convection only, R ($r=r_2$)=Thermal Radiation and Convection, $r_1=0.05$ m, $r_2=0.20$ m, $h=1.0$, $N=10$

The above-mentioned problems of non-linear boundary condition of the second kind are solved by the Jacobi elliptic functions, and next, (36) is replaced by $(\theta)_{x=1}=X(t)$, as $X(t)$ is the known function of t in this case, and therefore the problems of (33)~(36) are able to be treated as the problem of linear boundary condition of the first kind [5].

The non-linear boundary condition of the second kind is able to be reduced to the linear boundary condition of the first kind which is very simple, by the application of the Jacobi elliptic functions.

6. Summary

In the case to reduce the non-linear boundary condition of the second kind for combined thermal radiation and convection to the simple linear boundary condition of the first kind, the elliptic functions are very convenient.

The final calculation is reduced to the problem to find the elliptic functions in case k is given, and this is very easy.

The elliptic functions are able to calculate by the Theta functions with accuracy [2].

Murakawa, one of the authors, has thought of applying the elliptic functions to heat transfer problems for the first time.

Non-linear boundary condition, unsteady heat transfer and high accuracy in the numerical calculations are considered, which are necessary in the near future.

The usual finite-difference method, integral equations and variational problems are reduced to the simultaneous equations of many variables and high order, therefore the authors plan out not to apply and to omit the simultaneous equations, or to decrease the variables in the simultaneous equations by using the Jacobi elliptic functions.

References

1) B. Gay, Int. J. Heat and Mass Transfer, vol. **8**, no. 3, (1965), p. 507.
2) Harold T. Davis, Introduction to Non-linear Differential and Integral Equations, Dover Publication, (1960).
3) L. Collatz, The Numerical Treatment of Differential Equations, Springer-Verlag, (1966).
4) Y. S. Touloukian, Thermophysical Properties of High Temperature, Solid Metal, vol. **1**, **2**, **3**.
5) M. N. Özisik, Boundary Value Problems of Heat Conduction, International Textbook Company, (1968).