Observing the State of the Discrete Linear Time-Varying Systems

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Abstract

In this paper, the concepts of Luenberger observer, which estimate the state of a continuous linear plant on the basis of measurements of the outputs, were generalized for discrete linear time-varying systems. The results obtained indicate that the minimal-order observer-estimator is a special kind of the observer-estimator introduced in this paper.

1. Introduction

The problem of estimating the entire state vector of the system to be controlled, is of fundamental importance, when designing feedback control systems. Thus, for continuous linear time-invariant systems governed by

$$\mathring{\mathbf{X}} = A\mathbf{X} + B\mathbf{U} \tag{1}$$

where, $\mathbf{X}$ is an $(n, 1)$ state vector, $\mathbf{U}$ is an $(r, 1)$ input vector, A is an (n, n) system matrix, and B is an (n, r) distribution matrix. One might design a feedback law of the form $\mathbf{U}(t) = \mathbf{U}(\mathbf{X}(t), t)$ which could be implemented if $\mathbf{X}(t)$ were available. If the entire state vector cannot be measured, the control law deduced in the form $\mathbf{U}(\mathbf{X}(t), t)$ cannot be implemented. An approach which implements the control law in this situation is the design of system that produces an approximation to the state vector. This system is called an observer or Luenberger observer[1]. This observer reconstructes the entire state vector through input measurement, output measurement and system matrices A, H. In this paper, the concepts of observer are applied to the discrete linear time varying systems. The results obtained unify the these extended by several researchers[2].

The structure of the paper is as follows. In section 2, we define some basic notions. In section 3, we discuss the basic elements of observers for the discrete linear time-varying systems. In section 4, we indicate that the minimal-order observer-estimator is a special kind of the observer-estimator introduced in this paper. In section 5, we indicate that this observer-estimator can predict the estimate of next stage.

2. Definitions

We shall consider the discrete linear time-varying systems described by the following state and output equations, respectively,

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$$\mathbf{X}(k+1)=A(k)\mathbf{X}(k)+B(k)\mathbf{U}(k) \tag{2}$$

$$\mathbf{Y}(k)\quad=H(k)\mathbf{X}(k) \tag{3}$$

where, $k=0, 1, 2, 3,\ldots$, $\mathbf{X}(k)\in R^n$, $\mathbf{U}(k)\in R^r$, $\mathbf{Y}(k)\in R^m$, and $H(k)$ is of rank M. The notion of observer-estimator introduced earlier is now given a precise meaning in the following definition.

definition; A q-dimensional discrete linear time-varying system

$$\mathbf{Z}(k+1)=F(k)\mathbf{Z}(k)+C(k)\mathbf{Y}(k)+E(k)\mathbf{U}(k) \tag{4}$$

is called as an (p, q) state observer-estimator for the system of (2) and (3) if and only if there exist an $(n, (p+m))$ matrix $W(k+1)$ and an (p, q) matrix $T(k+1)$ satisfying

$$\lim_{k\to\infty}\left[\mathbf{X}(k+1)-W(k+1)\begin{bmatrix}T(k+1)\mathbf{Z}(k+1)\\ \mathbf{Y}(k+1)\end{bmatrix}\right]=\mathbf{O}_n \tag{5}$$

where, rank $F(k)=q$, $C(k)$ is an (q, m) matrix and $E(k)=an(q, r)$ matrix. $(p=n-m)$

3. Observer-estimator for deterministic systems.

Initially, consider the problem of observing the linear function of $\mathbf{X}(k+1)$, $S(k+1)\mathbf{X}(k+1)$.

lemma 1; The linear function of the state $\mathbf{Z}(k+1)$ of (4), $T(k+1)\mathbf{Z}(k+1)$ is an estimate of $S(k+1)\mathbf{X}(k+1)$ if the following conditions hold (where rank $S(k+1)=p$)

condition a	$[[G(k)]]<1$ where $[[G]]$: norm of matrix G	(6)
condition b	$S(k+1)A(k)-G(k)S(k)=T(k+1)C(k)H(k)$	(7)
condition c	$G(k)T(k)=T(k+1)F(k)$	(8)
condition d	$S(k+1)B(k)=T(k+1)E(k)$	(9)

proof. Sufficiency is proved by noting that the error vector

$\mathbf{e}(k+1)=T(k+1)\mathbf{Z}(k+1)-S(k+1)\mathbf{X}(k+1)$ is governed by

$$\begin{aligned}
&T(k+1)\mathbf{Z}(k+1)-S(k+1)\mathbf{X}(k+1)\\
&=T(k+1)F(k)\mathbf{Z}(k)+T(k+1)C(k)\mathbf{Y}(k)+T(k+1)E(k)\mathbf{U}(k)\\
&\quad-S(k+1)A(k)\mathbf{X}(k)-S(k+1)B(k)\mathbf{U}(k)\\
&=(T(k+1)C(k)H(k)-S(k+1)A(k))\mathbf{X}(k)+T(k+1)F(k)\mathbf{Z}(k)\\
&\quad+(T(k+1)E(k)-S(k+1)B(k))\mathbf{U}(k)\\
&=G(k)T(k)\mathbf{Z}(k)-G(k)S(k)\mathbf{X}(k)\\
&=G(k)(T(k)\mathbf{Z}(k)-S(k)\mathbf{X}(k))
\end{aligned} \tag{10}$$

from (10), finally

$$\lim_{k\to\infty} T(k+1)\mathbf{Z}(k+1)=S(k+1)\mathbf{X}(k+1) \qquad \text{if} \quad [[G(k)]]<1$$

Q. E. D.

We note that the result of lemma 1 can be easily extended to observing the state $\mathbf{X}(k+1)$ of (2).

lemma 2; $W(k+1)\begin{bmatrix} T(k+1)\mathbf{Z}(k+1) \\ \mathbf{Y}(k+1) \end{bmatrix}$ is an estimate of the state $\mathbf{X}(k+1)$, if there exist $P(k+1)$ and $V(k+1)$ such that $T(k+1)\mathbf{Z}(k+1)$ estimates $S(k+1)\mathbf{X}(k+1)$ and

$$W(k+1)\begin{bmatrix} S(k+1) \\ H(k+1) \end{bmatrix}=P(k+1)S(k+1)+V(k+1)H(k+1)=I_n \tag{11}$$

where, $W(k+1)$ is the $(n, (p+m))$ matrix $[P(k+1),\ V(k+1)]$.
proof.

$$\begin{bmatrix} S(k+1) \\ H(k+1) \end{bmatrix}\mathbf{X}(k+1)=\begin{bmatrix} T(k+1)\mathbf{Z}(k+1) \\ \mathbf{Y}(k+1) \end{bmatrix}-\begin{bmatrix} G(k)\mathbf{e}(k) \\ \mathbf{O}_m \end{bmatrix} \tag{12}$$

Applying $W(k+1)$ to both sides, we obtain

$$W(k+1)\begin{bmatrix} S(k+1) \\ H(k+1) \end{bmatrix}\mathbf{X}(k+1)=W(k+1)\begin{bmatrix} T(k+1)\mathbf{Z}(k+1) \\ \mathbf{Y}(k+1) \end{bmatrix} - W(k+1)\begin{bmatrix} G(k)\mathbf{e}(k) \\ \mathbf{O}_m \end{bmatrix} \tag{13}$$

$$\mathbf{X}(k+1)=W(k+1)\begin{bmatrix} T(k+1)\mathbf{Z}(k+1) \\ \mathbf{Y}(k+1) \end{bmatrix}-W(k+1)\begin{bmatrix} G(k)\mathbf{e}(k) \\ \mathbf{O}_m \end{bmatrix} \tag{14}$$

$$\mathbf{X}(k+1)-P(k+1)T(k+1)\mathbf{Z}(k+1)-V(k+1)\mathbf{Y}(k+1) = -W(k+1)\begin{bmatrix} G(k)\mathbf{e}(k) \\ \mathbf{O}_m \end{bmatrix} \tag{15}$$

$$\lim_{k\to\infty}(\mathbf{X}(k+1)-P(k+1)T(k+1)\mathbf{Z}(k+1)-V(k+1)\mathbf{Y}(k+1)) = -\lim_{k\to\infty} W(k+1)\begin{bmatrix} G(k)\mathbf{e}(k) \\ \mathbf{O}_m \end{bmatrix} = \mathbf{O}_n(\lim_{k\to\infty}\mathbf{e}(k)=\mathbf{O}_P) \tag{16}$$

Q. E. D.

4. Minimal-order observer-estimator

An minimal-order observer-estimator is one in which the transformation T relating the state of the (p, q) observer introduced in this paper to the state of the original system is the identify transformation.
From the result of lemma 1,

condition a $\quad S(k+1)A(k)-G(k)S(k)=C(k)H(k) \quad (17)$

condition b $\quad F(k)=G(k) \quad (18)$

condition c $\quad S(k+1)B(k)=E(k) \quad (19)$

from equation (17) and (18), we obtain:

$$S(k+1)A(k)-F(k)S(k)=C(k)H(k) \tag{20}$$

and

$$E(k)=S(k+1)B(k) \tag{21}$$

Where, if we choose $F(k)=S(k+1)A(k)P(k)$,

$C(k)=S(k+1)A(k)V(k)$, we obtain:

$$S(k+1)A(k)-S(k+1)A(k)P(k)S(k)=S(k+1)A(k)V(k)H(k) \tag{22}$$

$$S(k+1)A(k)(P(k)S(k)+V(k)H(k)-I_n)=O_{(p,n)} \tag{23}$$

Here, from the equation (23), we obtain

$$P(k)S(k)+V(k)H(k)=I_n \tag{24}$$

The equation (24) is fundamental for an minimal-order observer-estimator considered by Tea and Athans. Then, from definition and lemma 1, we obtain ($k=0, 1, 2\ldots$)

$$\begin{aligned}\mathbf{Z}(k+1)-S(k+1)\mathbf{X}(k+1)&=F(k)(\mathbf{Z}(k)-S(k)\mathbf{X}(k))\\&=S(k+1)A(k)P(k)(\mathbf{Z}(k)-S(k)\mathbf{X}(k))\end{aligned} \tag{25}$$

and

$$\hat{\mathbf{X}}(k)=P(k)\mathbf{Z}(k)+V(k)\mathbf{Y}(k) \tag{26}$$

Where, $P(k)$ and $V(k)$ are determined partly by the equation (24) and the condition $[[S(k+1)A(k)P(k)]]<1$. We can reconstruct $\mathbf{X}(k)$, by the equation (26).

5. Prediction of the estimate of next stage by an observer-estimator

An observer-estimator, which predict the estimate of next stage, is one in which the transformation $S(k+1)$ relating the state of (p, q) observer to the state of original system is the identity transformation, $(S(k+1)=I_n)$.

condition a $\quad G(k)=A(k)-T(k+1)C(k)H(k) \quad (27)$

condition b $\quad G(k)T(k)=T(k+1)F(k) \quad (28)$

condition c $\quad B(k)=T(k+1)E(k) \quad (29)$

Where, if we choose $F(k)=P(k+1)G(k)T(k)$, from the equation (28), we obtain:

$$T(k+1)P(k+1)=I_n \tag{30}$$

and from lemma 1

$$\begin{aligned}\mathbf{X}(k+1)-T(k+1)\mathbf{Z}(k+1)&=G(k)(\mathbf{X}(k)-T(k)\mathbf{Z}(k))\\&=(A(k)-T(k+1)C(k)H(k))(\mathbf{X}(k)\\&\quad-T(k)\mathbf{Z}(k))\end{aligned} \tag{31}$$

and

$$\hat{\mathbf{X}}(k+1)=T(k+1)\mathbf{Z}(k+1) \tag{32}$$

Where, $P(k+1)$, $T(k+1)$ and $C(k)$ are determined partly by the equation (30) and the condition $[[A(k)-T(k+1)C(k)H(k)]]<1$. We can predict the estimate of next stage.

6. Conclusion

An observer-estimator is constructed to estimate the state of a linear discrete-time plant on the basis of deterministic measurements. It is shown that the minimal-order observer-estimator is a special kind of the observer-estimator in this paper.

7. Referencecs

1) D. G. Luenberger, "An Introduction To Observers." I. E. E. E., Trans. Automatic Control, Vol. AC–16, No. 6, pp. 596–602, 1971
2) E. Tse and M. Athans, "Optimal Minimal-Order Observer-Estimators for Discrete Linear Time-Varying System." I. E. E. E., Trans. Automatic Control, Vol. AC–15, No. 4, pp. 416–426, 1970