Basic Equations and Boundary Conditions of Dynamic Elastic Stability for Thin Walled Structural Members Subjected to Follower Loads

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Abstract

Basic equations and boundary conditions in which the effect of initial torsional moment M_z^0 is included for analysis of dynamic elastic stability of thin walled structural members are derived by applying the linearized finite displacement theory in continuum mechanics and using the principle of virtual work and D'Alembert principle.

1. Introduction

Many studies on stability problems of the structural members subjected to follower forces, i.e., nonconservative forces have been carried out since H. Ziegler [1] first presented and discussed his issue generally. The stability problems of thin walled members have also been studied by several researchers including R. S. Barsoum [2] among them.

Basic equations and boundary conditions for elastic stability of thin walled members subjected to unidirectional forces have been reported [3] to data by applying the linearized finite displacement theory and the finite displacement theory in which the infinitesimal terms of high order were considered. Basic equations and boundary conditions of thin walled members subjected to follower forces is also necessary for solving the nonconservative probelms of elastic stability. It is well known that the elastic stability analysis of nonconservative systems should be kinematically examined. In this paper, basic equations and boundary conditions for dynamic elastic stability of thin walled member subjected to general follower loads are presented by using the principle of virtual work, D'Alembert's principle and linearized finite displacement theory. It is assumed in deriving the equations that: the thickness of the plate composing the member is small compared with any characteristic dimensions of the cross section; the cross sectional dimensions are small compared with the length of the member; the shearing deformations of the middle surface, i.e., the surface which is lying midway through the plate composing the member vanish; and the cross section of the thin walled member is underformalble.

2. Stable Vibrating State with Infinitesmal Amplitude

Let's suppose a member which is vibrating with infinitesimal amplitude is sub-

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jected to distributed follower forces q_x, q_y and q_z, follower moments m_x, m_y and m_z, warping moment m_ω, end forces of member Q^0_{xi}, Q^0_{yi}, Q^0_{zi}, Q^0_{xj}, Q^0_{yj} and Q^0_{zj}, and end moments of member M^0_{xi}, M^0_{yi}, M^0_{zi}, $M^0_{\omega i}$, M^0_{xj}, M^0_{yj}, M^0_{zj} and $M^0_{\omega j}$.

In the cross section of thin walled member, let the shear center and the geometrical center be denoted by S and G respectively, let the origin of the right-handed rectangular coordinates (x, y, z) be at the geometrical center G of the end of member i as shown in Figure 1. The rectangular co-ordinates x and y coincided with the principal axes of cross section and z is the longitudinal axis through the geometrical center G.

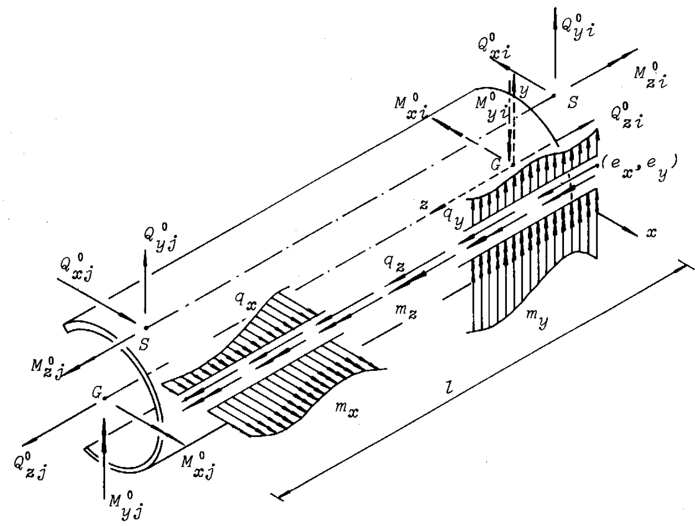

Figure 1 Structural member with thin walled open cross section in initial loading condition.

If displacements of the shear center in the direction of axes x, y and z are denoted by u^0_S, v^0_S and w^0_S respectively; displacements of the geometrical center by u^0_G, v^0_G and w^0_G respectively; and rotation about the axis z by θ°, as shown in Figure 2, then displacements of arbitrary point $P(x, y, z)$ in the cross section become

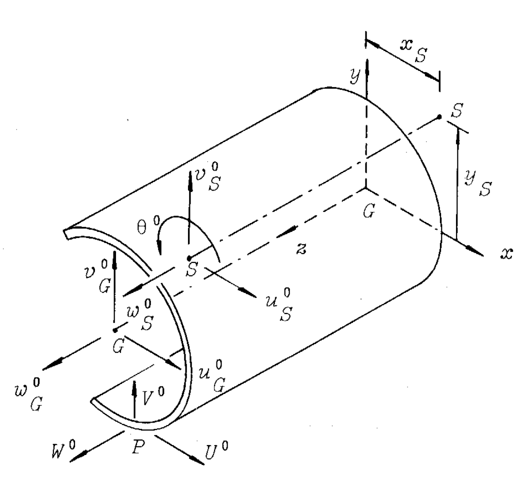

Figure 2 Displacements of member with thin walled open cross section in initial loading condition.

$$U^{\circ}(x,\ y,\ z,\ t)=u_S^0(z,\ t)-(y-y_S)\theta^{\circ}(z,\ t)$$

$$V^{\circ}(x,\ y,\ z,\ t)=v_S^0(z,\ t)+(x-x_S)\theta^{\circ}(z,\ t)$$

$$W^{\circ}(x,\ y,\ z,\ t)=w_G^0(z,\ t)-xu_S^{0\prime}-yv_S^{0\prime}+\omega_{nS}\theta^{0\prime}, \tag{2.1}$$

in which $\omega_{nS}=\omega_n-xy_S+x_Sy$ and $(\ ')=\partial/\partial z$. Here, ω_{nS} and ω_n are the normalized warping functions with respect to the shear center and the geometrical center, respectively. In this state, the strains of point P are

$$\varepsilon_x^0=\varepsilon_y^0=\gamma_{xy}^0=0,$$

$$\varepsilon_z^0=w_G^{0\prime}-xu_S^{0\prime\prime}-yv_S^{0\prime\prime}+\omega_{nS}\theta^{0\prime\prime},$$

$$\gamma_{yz}^0=\theta^{0\prime}\left\{\frac{\partial\omega_{nS}}{\partial y}+(x-x_S)\right\},$$

$$\gamma_{zx}^0=\theta^{0\prime}\left\{\frac{\partial\omega_{nS}}{\partial x}-(y-y_S)\right\}, \tag{2.2}$$

and the stresses are

$$\sigma_x^0=\sigma_y^0=\tau_{xy}^0=0,\quad \sigma_z^0=E\varepsilon_z^0,$$

$$\tau_{yz}^0=G\gamma_{yz}^0,\quad \tau_{zx}^0=G\gamma_{zx}^0. \tag{2.3}$$

3. Equation of Virtual Work in Unstable Vibrating Stable with Finite Amplitude

Let a unstable vibrating state with finite amplitude occur when the parameter with respect to the external forces change infinitesimally in the stable vibrating state with infinitesimal amplitude. In this state displacements of the sum of infinitesimal dis-

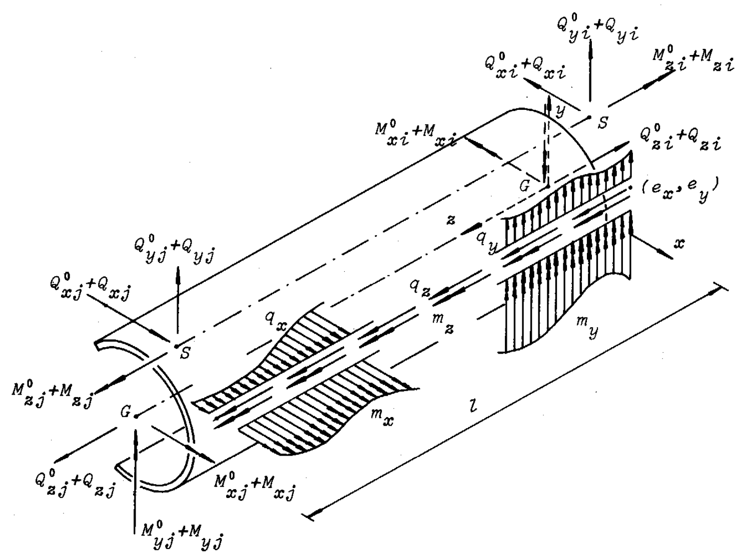

Figure 3 Structural member with thin walled open cross section in vibrating state with finite amplitude.

placements in stable vibration and finite displacements in unstable vibration are produced and the loading condition become as shown in Figure 3, in which the forces Q_{xi}, Q_{yi}, Q_{zi}, Q_{xj}, Q_{yj} and Q_{zj}, and the moments M_{xi}, M_{yi}, M_{zi}, $M_{\omega i}$, M_{xj}, M_{yj}, M_{zj} and $M_{\omega j}$ are the end forces and moments of the member caused by finite displacement, respectively.

If the additional finite displacements of the shear center are denoted by u_S, v_S and w_S and rotation by θ, diaplacements of arbitrary point $P(x, y, z)$ in the cross section become

$$\begin{aligned} U(x, y, z, t) &= u_S(z, t) - (y - y_S)\theta(z, t) \\ V(x, y, z, t) &= v_S(z, t) + (x - x_S)\theta(z, t) \\ W(x, y, z, t) &= w_G(z, t) - xu'_S - yv'_S + \omega_{nS}\theta'. \end{aligned} \tag{3.1}$$

Strains due to additional displacements (3.1) can be written as

$$\begin{aligned} \varepsilon_x &= \frac{\partial U}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right\}, \\ \varepsilon_y &= \frac{\partial V}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right\}, \\ \varepsilon_z &= \frac{\partial W}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right\}, \\ \gamma_{xy} &= \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) + \left(\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right), \\ \gamma_{yz} &= \left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) + \left(\frac{\partial U}{\partial y}\frac{\partial U}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\frac{\partial W}{\partial z}\right), \\ \gamma_{zx} &= \left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + \left(\frac{\partial U}{\partial z}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial z}\frac{\partial V}{\partial x} + \frac{\partial W}{\partial z}\frac{\partial W}{\partial x}\right), \end{aligned} \tag{3.2}$$

where $\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$ because cross section of the member is rigid. When linear strains are expressed by $\bar{\varepsilon}_z$, $\bar{\gamma}_{yz}$ and $\bar{\gamma}_{zx}$ and non-linear terms with respect to $\partial W/\partial z$ in equation (3.2) are neglected, the starins ε_z, γ_{yz} and γ_{zx} are:

$$\begin{aligned} \varepsilon_z &= \bar{\varepsilon}_z + \frac{1}{2}[\{u'_S - (y - y_S)\theta'\}^2 + \{v'_S + (x - x_S)\theta'\}^2], \\ \gamma_{yz} &= \bar{\gamma}_{yz} - \theta\{u'_S - (y - y_S)\theta'\}, \\ \gamma_{zx} &= \bar{\gamma}_{zx} + \theta\{v'_S + (x - x_S)\theta'\}, \end{aligned} \tag{3.3}$$

and the stresses are

$$\begin{aligned} &\sigma_x = \sigma_y = \tau_{xy} = 0, \\ &\sigma_z = E\varepsilon_z, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}. \end{aligned} \tag{3.4}$$

When virtual displacements δu_S, δv_S, δw_S, δu_G, δv_G, δw_G and $\delta\theta$, which are kinematically admissible variations, are introduced at a certain time in a state of vibration with finite amplitude, an equation of virtual work is obtained by using the principle of virtual work and D'Alembert's principle as

$$\int_T (\delta\Pi_{in}+\delta\Pi_{is}-\delta\Pi_a-\delta\Pi_a^E)dt=0, \tag{3.5}$$

in which $\delta\Pi_{in}$, $\delta\Pi_{is}$, $\delta\Pi_a$ and $\delta\Pi_a^E$ are the virtual works done by inertia forces, internal forces, external distributed loads and external end forces and moments, respectively, in unit time at a certain time t. These virtual works are evaluated as following.

(*Virtual work of inertia forces*)

Virtual work of inertia forces, $\delta\Pi_{in}$, becomes

$$\begin{aligned}\delta\Pi_{in} = \delta\Pi_{in}^0 - \int \{ & m(\ddot{u}_S+y_S\ddot{\theta})\delta u_S + m(\ddot{v}_S - x_S\ddot{\theta}) \\ & \times \delta v_S + m\ddot{w}_G\delta w_G + \mu I_{xx}\ddot{u}'_S\delta u'_S + \mu I_{yy}\ddot{v}'_S\delta v'_S \\ & + (\mu I_{pS}\ddot{\theta} + my_S\ddot{u}_S - mx_S\ddot{v}_S)\delta\theta + \mu I_\omega^S\ddot{\theta}'\delta\theta'\}dz,\end{aligned} \tag{3.6}$$

in which $\delta\Pi_{in}^0$ denotes the virtual work of inertia forces in the state of vibration with infinitesimal amplitude and can be written as

$$\begin{aligned}\delta\Pi_{in}^0 = -\int \{ & m(\ddot{u}_S^0+y_S\ddot{\theta}^\circ)\delta u_S + m(\ddot{v}_S^0 - x_S\ddot{\theta}^\circ)\delta v_S \\ & + m\ddot{w}_G^0\delta w_G + \mu I_{xx}\ddot{u}_S^{0\prime}\delta u'_S + \mu I_{yy}\ddot{v}_S^{0\prime}\delta v'_S + (\mu I_{pS}\ddot{\theta}^\circ \\ & + my_S\ddot{u}_S^0 - mx_S\ddot{v}_S^0)\delta\theta + \mu I_\omega^S\ddot{\theta}^{\circ\prime}\delta\theta'\}dz,\end{aligned} \tag{3.7}$$

$$\begin{aligned} & m=\mu\iint dxdy=\mu A, \quad I_{xx}=\iint_A x^2dxdy, \quad I_{yy}=\iint_A y^2dxdy, \\ & I_\omega^S=\iint_A \omega_{nS}^2dxdy, \quad I_{pS}=I_{xx}+I_{yy}+A(x_S^2+y_S^2).\end{aligned} \tag{3.8}$$

Here μ is the mass per unit volume and $(\dot{\ }) = \partial/\partial t$.

(*Virtual work of internal forces*)

Virtual work of internal forces, $\delta\Pi_{is}$, becomes

$$\begin{aligned}\delta\Pi_{is} = & \int_0^l\iint_A (\sigma_z^0\delta\varepsilon_z + \tau_{yz}^0\delta\gamma_{yz} + \tau_{zx}^0\delta\gamma_{zx})dxdydz \\ & + \int_0^l\iint_A (\sigma_z\delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx})dxdydz,\end{aligned} \tag{3.9}$$

where

$$\delta\varepsilon_z = \delta\bar{\varepsilon}_z + \frac{1}{2}\delta[\{u'_S-(y-y_S)\theta'\}^2 + \{v'_S+(x-x_S)\theta'\}^2],$$

$$\delta\gamma_{yz}=\delta\bar{\gamma}_{yz}-\delta[\theta\{u_S'-(y-y_S)\theta'\}],$$
$$\delta\gamma_{zx}=\delta\bar{\gamma}_{zx}+\delta[\theta\{v_S'+(x-x_S)\theta'\}]. \tag{3.10}$$

The first term on the right hand side of the equation (3.9) is rearranged as

$$\begin{aligned}\iiint(\sigma_z^0\delta\varepsilon_z+\tau_{yz}^0\delta\gamma_{yz}+\tau_{zx}^0\delta\gamma_{zx})dxdydz=\delta\Pi_{is}^0\\ +\int_0^l\Big[\frac{1}{2}Q_z(z,t)\delta\{(u_S'+y_S\theta')^2+(v_S'-x_s\theta')^2+r_0^2\theta'^2\}\\ -M_x^0(z,t)\delta(u_S'\theta'-\beta_y\theta'^2)-M_y^0(z,t)\delta(v_S'\theta'+\beta_x\theta'^2)\\ +2M_\omega^{S0}(z,t)\beta_\omega\theta'\delta\theta'\Big]dz+\int_0^l\Big[[Q_x^0(z,t)\delta(v_S'\theta+\beta_x\theta\theta')\\ +Q_y^0(z,t)\delta(-u_S'\theta+\beta_y\theta\theta')-M_z^{\omega 0}(z,t)\beta_\omega\delta(\theta\theta')\Big]dz,\end{aligned} \tag{3.11}$$

in which $\delta\Pi_{is}^0$ denotes the virtual work of internal forces in the state of vibration with infinitesimal amplitude and can be written as

$$\delta\Pi_{is}^0=\int_0^l\iint_A(\sigma_z^0\delta\bar{\varepsilon}_z+\tau_{yz}^0\delta\bar{\gamma}_{yz}+\tau_{zx}^0\delta\bar{\gamma}_{zx})dxdydz. \tag{3.12}$$

Here

$$\begin{aligned}&Q_x^0(z,t)=\iint_A\tau_{zx}^0dxdy,\quad Q_y^0(z,t)=\iint_A\tau_{yz}^0dxdy,\\ &Q_z^0(z,t)=EAw_G^{0\prime}=\iint_A\sigma_z^0dxdy,\\ &M_x^0(z,t)=-EI_{yy}v_S^{0\prime\prime}=\iint_A\sigma_z^0ydxdy,\\ &M_y^0(z,t)=EI_{xx}u_S^{0\prime\prime}=-\iint_A\sigma_z^0xdxdy,\\ &M_\omega^{S0}(z,t)=EI_\omega^S\theta^{\circ\prime\prime}=\iint_A\sigma_z^0\omega_{ns}dxdy,\quad M_z^{\omega 0}(z,t)=-\frac{\partial M_\omega^{S0}}{\partial z},\\ &r_0^2=\frac{1}{A}(I_{xx}+I_{yy}),\quad \beta_x=-x_S+\frac{I_{120}+I_{300}}{2I_{xx}},\\ &\beta_y=-y_S+\frac{I_{030}+I_{210}}{2I_{yy}},\quad \beta_\omega=\frac{I_{021}+I_{201}}{2I_\omega^S},\\ &I_{lmn}=\iint_A x^ly^m\omega_{ns}^n dxdy.\end{aligned} \tag{3.13}$$

The second term on the right hand side of the equation (3.9) is virtual work produced by nonlinear internal forces and virtual strains. In this paper, however, it is linearized and approximated as

$$\iiint(\sigma_z\delta\varepsilon_z+\tau_{yz}\delta\gamma_{yz}+\tau_{zx}\delta\gamma_{zx})dxdydz=\int_0^l(EAw'_G\delta w'_G$$
$$+EI_{xx}u''_S\delta u''_S+EI_{yy}v''_S\delta v''_S+EI^S_\omega\theta''\delta\theta''+GK\theta'\delta\theta')dz, \tag{3.14}$$

where

$$K=\left\{\frac{\partial\omega_{nS}}{\partial y}+(x-x_S)\right\}^2+\left\{\frac{\partial\omega_{nS}}{\partial x}-(y-y_S)\right\}^2. \tag{3.15}$$

The virtual work produced by initial torsional moment M^0_z being not included in the above virtual work $\delta\Pi_{is}$, the following virtual work [4] is added,

$$\int_0^l M^0_z(z,\ t)(\delta u''_S v'_S-u'_S\delta v''_S)dz. \tag{3.16}$$

Thus, substituting equations (3.11) and (3.14) into equation (3.9) and adding equation (3.16) to equation (3.9) yield the total virtual work of internal forces

$$\delta\Pi_{is}=\delta\Pi^0_{is}+\int_0^l\Big[Q^0_z(z,\ t)\{(u'_S+y_S\theta')(\delta u'_S+y_S\delta\theta')$$
$$+(v'_S-x_S\theta')(\delta v'_S-x_S\delta\theta')+r_0^2\theta'\delta\theta'\}-M^0_x(z,\ t)(\delta u'_S\theta'$$
$$+u'_S\delta\theta'-2\beta_y\theta'\delta\theta')-M^0_y(z,\ t)(\delta v'_S\theta'+v'_S\delta\theta'+2\beta_x\theta'\delta\theta'$$
$$+2M^{S0}_\omega(z,\ t)\beta\omega\theta'\delta\theta'+Q^0_x(z,\ t)(\beta_x\delta\theta\cdot\theta'+\beta_x\theta\delta\theta'+\delta v'_S\theta$$
$$+v'_S\delta\theta)+Q^0_y(z,\ t)(\beta_y\delta\theta\cdot\theta'+\beta_y\theta\cdot\delta\theta'-\delta u'_S\theta-u'_S\delta\theta)$$
$$-M^{\omega 0}_z(z,\ t)\beta_\omega(\delta\theta\cdot\theta'+\theta\delta\theta')+M^0_z(z,\ t)(\delta u''_S v'_S$$
$$-u'_S\delta v''_S)\Big]dz+\int_0^l(EAw'_G\delta w'_G+EI_{xx}u''_S\delta u''_S$$
$$+EI_{yy}v''_S\delta v''_S+EI^S_\omega\theta''\delta\theta''+GK\theta'\delta\theta')dz. \tag{3.17}$$

(Virtual work of distributed external loads)

Consider now the external forces q_x, q_y and q_z, and moments m_x, m_y and m_z, which act on the point $P_e(e_x,\ e_y,\ z)$ as shown in figure 1, and warping moment m_ω. It is assumed that the direction of vector of the external forces and moments except a warping moment m_ω change with rotational angle of the acting point, i.e., the external forces and moments are follower loads.

Displacements u_e, v_e and w_e, and rotational angles θ_{xe}, θ_{ye} and θ_{ze} at the point P_e are written as

$$u_e = u_S - (e_y - y_S)\theta, \quad v_e = v_S + (e_x - x_S)\theta,$$
$$w_e = w_G - e_x u'_S - e_y v'_S + \omega_{nSe}\theta', \quad \omega_{nSe} = w_n - e_x y_S + e_y x_S,$$
$$\theta_{xe} = -v'_e = -\{v'_S + (e_x - x_S)\theta'\},$$
$$\theta_{ye} = u'_e = \{u'_S - (e_y - y_S)\theta'\}, \quad \theta_{ze} = \theta. \tag{3.18}$$

When the point P_e is rotated at the angle θ_{xe}, θ_{ye} and θ_{ze}, initial external forces change as follows,

$$(q_x, 0, 0) \longrightarrow (q_x, q_x\theta, -q_x u'_e),$$
$$(0, q_y, 0) \longrightarrow (-q_y\theta, q_y, -q_y v'_e),$$
$$(0, 0, q_z) \longrightarrow (q_z u'_e\ q_z v'_e\ q_z),$$
$$(m_x, 0, 0) \longrightarrow (m_x, m_x\theta, -m_x u'_e),$$
$$(0, m_y, 0) \longrightarrow (-m_y\theta\ m_y\ -m_y v'_e),$$
$$(0, 0, m_z) \longrightarrow (m_z u'_e\ m_z v'_e\ m_z). \tag{3.19}$$

Thus, virtual work of the distributed loads, $\delta\Pi_a$, becomes

$$\delta\Pi_a = \delta\Pi_a^0 + \int_0^l \Big[\{-q_y\theta + q_z(u'_S - \bar{y}\theta')\}\delta u_s + [e_x\{q_x(u'_S - \bar{y}\theta')$$
$$+ q_y(v'_S + \bar{x}\theta')\} + \{m_x\theta + m_z(v'_S + \bar{x}\theta')\}]\delta u'_S$$
$$+ \{q_x\theta + q_z(v'_S + \bar{x}\theta')\}\delta v_s + [e_y\{q_x(u'_S - \bar{y}\theta') + q_y(v'_S + \bar{x}\theta')\}$$
$$+ \{m_y\theta - m_z(u'_S - \bar{y}\theta')\}]\delta v'_S - \{q_x(u'_S - \bar{y}\theta') + q_y(v'_S + \bar{x}\theta')\}\delta w_G$$
$$+ [\{q_y\theta - q_z(u'_S - \bar{y}\theta')\bar{y} + \{q_x\theta + q_z(v'_S + \bar{x}\theta')\}\bar{x} - \{m_x(u'_S - \bar{y}\theta')$$
$$+ m_y(v'_S + \bar{x}\theta')\}]\delta\theta + [-\omega_{nSe}\{q_x(u'_S - \bar{y}\theta') + q_y(v'_S + \bar{x}\theta')\}$$
$$+ \{m_y\theta - m_z(u'_S - \bar{y}\theta')\}\bar{x} - \{m_x\theta + m_z(v'_S + \bar{x}\theta')\}\bar{y}]\delta\theta' \Big] dz, \tag{3.20}$$

with $\bar{x} = e_x - x_S$ and $\bar{y} = e_y - y_S$, in which $\delta\Pi_a^0$ denotes virtual work done by the distributed loads in the state of vibration with infinitesimal amplitude and can be written as

$$\delta\Pi_a^0 = \int_0^l (q_x\delta u_e + q_y\delta v_e + q_z\delta w_e + m_x\delta\theta_{xe} + m_y\delta\theta_{ye} + m_z\delta\theta_{ze} + m_\omega\delta\theta'_{ze})dz. \tag{3.21}$$

(*Virtual work of end forces and moments of member*)

Virtual work produced by the end forces and moments shown in Figure 3, $\delta\Pi_a^E$, becomes

$$\delta\Pi_a^E = -(Q_{xi}^0 + Q_{xi})\delta u_S(0, t) + (Q_{xj}^0 + Q_{xj})\delta u_S(l, t)$$
$$-(Q_{yi}^0 + Q_{yi})\delta v_S(0, t) + (Q_{yj}^0 + Q_{yj})\delta v_S(l, t) + (M_{xi}^0$$

$$+M_{xi})\delta v'_G(0,\ t)-(M^0_{xj}+M_{xj})\delta v'_G(l,\ t)-(M^0_{yi}+M_{yi})$$
$$\times\delta u'_G(0,\ t)+(M^0_{yj}+M_{yj})\delta u'_G(l,\ t)-(Q^0_{zi}+Q_{zi})\delta w_G(0,\ t)$$
$$+(Q^0_{zj}+Q_{zj})\delta w_G(l,\ t)-(M^0_{zi}+M_{zi})\delta\theta(0,\ t)+(M^0_{zj}+M_{zj})$$
$$\times\delta\theta(l,\ t)-(M^0_{\omega i}+M_{\omega i})\delta\theta'(0,\ t)+(M^0_{\omega j}+M_{\omega j})\delta\theta'(l,\ t) \qquad (3.22)$$

with $\delta u_G(z,\ t)=\delta u_S(z,\ t)+y_S\delta\theta(z,\ t)$ and $\delta v_G(z,\ t)=\delta v_S(z,\ t)-x_S\delta\theta(z,\ t)$. The end forces and moments being expressed as

$$Q_{xi}=Q_x(0,\ t),\quad Q_{xj}=Q_x(l,\ t),\quad Q_{yi}=Q_y(0,\ t),\quad Q_{yj}=Q_y(l,\ t),$$
$$Q_{zi}=Q_z(0,\ t),\quad Q_{zj}=Q_z(l,\ t),\quad M_{xi}=M_x(0,\ t),\quad M_{xj}=M_x(l,\ t),$$
$$M_{yi}=M_y(0,\ t),\quad M_{yj}=M_y(l,\ t),\quad M_{zi}=M_z(0,\ t),\quad M_{zj}=M_z(l,\ t),$$
$$M_{\omega i}=M_\omega(0,\ t),\quad M_{\omega j}=M_\omega(l,\ t), \qquad (3.23)$$

$$\delta\Pi^E_a=\delta\Pi^{E0}_a+\{Q_x(z,\ t)\delta u_S(z,\ t)+Q_y(z,\ t)\delta v_S(z,\ t)-M_x(z,\ t)\delta v'_S(z,\ t)$$
$$+M_y(z,\ t)\delta u'_S(z,\ t)+Q_z(z,\ t)\delta w_G(z,\ t)+M_z(z,\ t)\delta\theta(z,\ t)$$
$$+M^S_\omega(z,\ t)\delta\theta'(z,\ t)\}|^l_0, \qquad (3.24)$$

in which $\delta\Pi^{E0}_a$ denotes virtual work done by end forces and moments in initial vibrating state and can be written as

$$\delta\Pi^{E0}_a=\{Q^0_x(z,\ t)\delta u_S(z,\ t)+Q^0_y(z,\ t)\delta v_S(z,\ t)-M^0_x(z,\ t)\delta v'_S(z,\ t)+M^0_y(z,\ t)$$
$$\times\delta u'_S(z,\ t)+Q^0_z(z,\ t)\delta w_G(z,\ t)+M^0_z(z,\ t)\delta\theta(z,\ t)+M^{S0}_\omega(z,\ t)\delta\theta'(z,\ t)\}|^l_0. \qquad (3.25)$$

Here $$M^{S0}_\omega=M^0_\omega+x_SM^0_x+y_SM^0_y,\ M^S_\omega=M_\omega+x_SM_x+y_SM_y \qquad (3.26)$$

and $|^l_0$ denotes the boundary value.

Since equation (3.5) must hold identically for any interval of time T, substitution of equations (3.6), (3.17), (3.20) and (3.24) in equation (3.5) and rearrangement of these equations in accordance with equation of virtual work in initial vibrating state

$$\int_T(\delta\Pi^0_{in}+\delta\Pi^0_{is}-\delta\Pi^0_a-\delta\Pi^{E0}_a)dt=0, \qquad (3.27)$$

results in

$$\int_0^l\{m(\ddot{u}_S+y_S\ddot{\theta})\delta u_S+m(\ddot{v}_S-x_S\ddot{\theta})\delta v_S+m\ddot{w}_G\delta w_G$$
$$+\mu I_{xx}\ddot{u}'_S\delta u'_S+\mu I_{yy}\ddot{v}'_S\delta v'_S+(\mu I_{pS}\ddot{\theta}+my_S\ddot{u}_S$$
$$-mx_S\ddot{v}_S)\delta\theta+\mu I^S_\omega\ddot{\theta}'\delta\theta'\}dz+\int_0^l(EAw'_G\delta w'_G$$
$$+EI_{xx}u''_S\delta u''_S+EI_{yy}v''_S\delta v''_S+EI^S_\omega\theta''\delta\theta''+GK\theta'\delta\theta')dz$$

$$+\int_0^l\Big[\Big\{Q_z^0(u_S'+y_S\theta')-M_x^0\theta'-Q_y^0\theta\Big\}\delta u_S'$$
$$+M_z^0v_S'\delta u_S''+\Big\{Q_z^0(v_S'-x_S\theta')-M_y^0\theta'+Q_x^0\theta\Big\}\delta v_S'$$
$$-M_z^0u_S'\delta v_S''+\{Q_x^0(v_S'+\beta_x\theta')-Q_y^0(u_S'-\beta_y\theta')$$
$$-M_z^{\omega0}\beta_\omega\theta'\}\delta\theta+[Q_z^0\{(u_S'+y_S\theta')y_S-(v_S'-x_S\theta')x_S+r_0^2\theta'\}$$
$$-M_x^0(u_S'-2\beta_y\theta')-M_y^0(v_S'+2\beta_x\theta')+2M_\omega^{S0}\beta_\omega\theta'+Q_x^0\beta_x\theta$$
$$+Q_y^0\beta_y\theta-M_z^{\omega0}\beta_\omega\theta]\delta\theta'\Big]dz-\int_0^l\Big[\{-q_y\theta+q_z(u_S'-\bar{y}\theta')\}\delta u_s$$
$$+[e_x\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}+\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}]\delta u_S'$$
$$+\{q_x\theta+q_z(v_S'+\bar{x}\theta')\}\delta v_S+[e_y\{q_x(u_S'-\bar{y}\theta')+q_y(u_S'+\bar{x}\theta')\}$$
$$+\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}]\delta v_S'-\{q_x(u_S'-\bar{y}\theta')$$
$$+q_y(v_S'+\bar{x}\theta')\}\delta w_G+[\{q_y\theta-q_z(u_S'-\bar{y}\theta')\}\bar{y}$$
$$+\{q_x\theta+q_z(v_S'+\bar{x}\theta')\}\bar{x}-\{m_x(u_S'-\bar{y}\theta')+m_y(v_S'+\bar{x}\theta')\}]\delta\theta$$
$$+[-\omega_{nSe}\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}$$
$$+\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}\bar{x}-\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}\bar{y}]\delta\theta'\Big]dz$$
$$-\{Q_x\delta u_S+Q_y\delta v_S-M_x\delta v_S'+M_y\delta u_S'+Q_z\delta w_G+M_z\delta\theta$$
$$+M_\omega^S\delta\theta'\}|_0^l=0, \qquad (3.28)$$

which is the equation of virtual work in the unstable vibrating state.

4. Basic Equations and Boundary Conditions for the Problems of the Dynamic Elastic Stability

The following basic equations and boundary conditions for the problems of the dynamic elastic stability are obtained because virtual displacements δu_S, δv_S, δw_G and $\delta\theta$ are arbitrary.

(*Basic equations*)

$$m(\ddot{u}_S+y_S\ddot{\theta})-\mu I_{xx}\ddot{u}_S''+EI_{xx}u_S''''-\{Q_z^0(u_S'+y_S\theta')\}'$$
$$+(M_x^0\theta)''+(M_z^0v_S')''+q_y\theta-q_z(u_S'-\bar{y}\theta')$$
$$+[e_x\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}+\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}]'$$
$$=0, \qquad (4.1)$$

$$m(\ddot{v}_S-x_S\ddot{\theta})-\mu I_{yy}\ddot{v}_S''+EI_{yy}v_S''''-\{Q_z^0(v_S'-x_S\theta')\}'$$

$$+(M_y^0\theta)''-(M_z^0u_S')''-q_x\theta-q_z(v_S'+\bar{x}\theta')$$
$$+[e_y\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}+\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}]'$$
$$=0, \tag{4.2}$$

$$m\ddot{w}_G-EAw_G''+q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')=0, \tag{4.3}$$

$$\mu I_{pS}\ddot{\theta}+m(y_S\ddot{u}_S-x_S\ddot{v}_S)-\mu I_\omega^S\ddot{\theta}''+EI_\omega^S\theta''''-GK\theta''$$
$$-\{Q_z^0(y_Su_S'-x_Sv_S')+Q_z^0r_S^2\theta'+2(M_x^0\beta_y-M_y^0\beta_x+M_\omega^{S0}\beta_\omega)\theta'\}'$$
$$+M_x^0u_S''+M_y^0v_S''-(Q_x^0\beta_x+Q_y^0\beta_y-M_z^{\omega0}\beta_\omega)'\theta$$
$$-\bar{y}\{q_y\theta-q_z(u_S'-\bar{y}\theta')\}-\bar{x}\{q_x\theta+q_z(v_S'+\bar{x}\theta')\}+m_x(u_S'-\bar{y}\theta')$$
$$+m_y(v_S'+\bar{x}\theta')+[-\omega_{nSe}\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}$$
$$+\bar{x}\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}-\bar{y}\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}]'=0, \tag{4.4}$$

(Boundary conditions)

$$[\mu I_{xx}\ddot{u}_S'-EI_{xx}u_S'''+Q_z^0(u_S'+y_S\theta')-(M_x^0\theta)'$$
$$-(M_z^0v_S')'-e_x\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}$$
$$-\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}-Q_x]\delta u_S|_0^l=0, \tag{4.5}$$

$$\left(EI_{xx}u_S''+M_z^0v_S'-M_y\right)\delta u_S'|_0^l=0, \tag{4.6}$$

$$[\mu I_{yy}\ddot{v}_S'-EI_{yy}v_S'''+Q_z^0(v_S'-x_s\theta')-(M_y^0\theta)'$$
$$+(M_z^0u_S')'-e_y\{q_x(u_S'-\bar{y}\theta')+q_y(v_S'+\bar{x}\theta')\}$$
$$-\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}-Q_y]\delta v_s|_0^l=0, \tag{4.7}$$

$$\left(EI_{yy}v_S''-M_z^0u_S'+M_x\right)\delta v_S'|_0^l=0, \tag{4.8}$$

$$(EAw_G'-Q_z)\delta w_G|_0^l=0, \tag{4.9}$$

$$[\mu I_\omega^S\ddot{\theta}'-EI_\omega^S\theta'''+GK\theta'+Q_z^0(y_Su_S'-x_sv_S'+r_S^2\theta')$$
$$-M_x^0u_S'-M_y^0v_S'+2(M_x^0\beta_y-M_y^0\beta_x+M_\omega^{s0}\beta_\omega)\theta'$$
$$+(Q_x^0\beta_x+Q_y^0\beta_y-M_z^{\omega0}\beta_\omega)\theta+\omega_{nSe}\{q_x(u_S'-\bar{y}\theta')$$
$$+q_y(v_S'+\bar{x}\theta')\}-\bar{x}\{m_y\theta-m_z(u_S'-\bar{y}\theta')\}+\bar{y}\{m_x\theta+m_z(v_S'+\bar{x}\theta')\}$$
$$-M_z]\delta\theta|_0^l=0, \tag{4.10}$$

$$(EI_\omega^S\theta''-M_\omega^S)\delta\theta'|_0^l=0, \tag{4.11}$$

in which $r_S^2=r_0^2+x_S^2+y_S^2$.

5. Conclusion

Basic equations and boundary conditions for analyzing the problems of dynamic elastic stability of thin walled structural members were presented by applying the linearized finite displacement theory in continuum mechanics and using the principle of virtual work and D'Alembert's principle. Investigation of the stability of structural members subjected to follower loads became possible with the application of equations and conditions.

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