On the Weak Formulation of the Stochastic Obstacle Problems

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Abstract

This paper is concerned with the weak formulation of stochastic obstacle problems. First, a mathematical model of a stochastic obstacle problem is given in a form of a variational inequality. The weak solution of the stochastic obstacle problem is defined. The relation between the weak solution and the strong one is stated. Secondly, under some conditions, it is proved that the weak solution exists by using the method of penalization. Finally, the existence of the maximum solution of the weak solution is shown. The mathematical formulation of stochastic obstacle problems by the strong solution is valid only for a one dimensional spatial variable, however, the weak solution proposed here enables to formulate the stochastic obstacle problems in a multi-dimensional spatial region.

1. Introduction

It is well known that many physical phenomena are modeled by ordinary and/or partial differential equations. Some nonlinear physical phenomena are modeled by differential inequalities rather than differential equalities. Such differential inequalities lead us to variational inequalities (Bensoussan[1], Panagiotopoulos[2], Baiocchi[3]). As typical examples of problems modeled by variational inequalities, dam problems, Stefan problems and obstacle problems, so-called, free boundary problems, are considered. In this paper, a weak formulation of the obstacle problem which is one of free boundary problems mentioned above is studied. Noting that the physical problems contain more or less random factors, our attention is focussed on a weak formulation of stochastic obstacle problems. First, the mathematical model of the stochastic obstacle problem is given in the form of the variational inequality. The definition of the weak solution of the stochastic variational inequality is stated. The relation between the strong solution (Haussmann & Pardoux[4]) and the weak solution proposed here is given. Secondly, by using the method of penalization, the existence of the weak solution is proved. Finally, it is shown that the weak solution has the maximum solution. The analytical method used in Haussmann & Pardoux[4] is valid only for a one dimensional spatial region and a strong regularity of data is required. On the other hand, the mathematical formulation by the weak solution is available for a multi

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-dimensional spatial region.

2. Definition of the Weak Solution

Let G be an open domain with a smooth boundary Γ in R^n. Consider the following stochastic differential inequality:

$$u(t,x) \leqq \psi(x), \qquad \text{in } T\times G \tag{2.1a}$$

$$du(t,x)+Au(t,x)dt+dw(t,x) \leqq f(t,x)dt, \qquad \text{in } T\times G \tag{2.1b}$$

$$(du(t,x)+Au(t,x)dt+dw(t,x)-f(t,x)dt)(u(t,x)-\psi(x))=0. \qquad \text{in } T\times G \tag{2.1c}$$

with the initial and boundary conditions

$$u(0,x)=u_0(x), \qquad \text{on } G \tag{2.2}$$

$$u(t,x)=0, \qquad \text{on } T\times\Gamma \tag{2.3}$$

where $A(\cdot)$ denotes an operator such that

$$A(\cdot)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial(\cdot)}{\partial x_j}), \tag{2.4}$$

w(t,x) is a Wiener process with a covariance operator Q(t), ψ and f are given functions respectively. The function ψ plays a role of an obstacle since the value of u(t,x) cannot exceed $\psi(x)$ from (2.1a). Physically, (2.1) is interpreted as follows: Let u(t,x) denote a temperature distribution in the region G at time t. The system (2.1) represents that the temperature u(t,x) is automatically controlled so as for the temperature u(t,x) not to exceed the prescribed temperature $\psi(x)$.

Define

$$I_C(\lambda)=\begin{cases}0, & \text{if } \lambda\leqq\psi\\ +\infty, & \text{if } \lambda>\psi.\end{cases}$$

Then, the subdifferential $\partial I_C(\lambda)$ is given by

$$\partial I_C(\lambda)=\begin{cases}0, & \text{if } \lambda<\psi\\ [0,\infty], & \text{if } \lambda=\psi\\ \text{empty}, & \text{if } \lambda>\psi.\end{cases} \tag{2.5}$$

By using the subdifferential (2.5), (2.1) can be rewritten by

$$u(t,x)\leqq\psi(x), \qquad \text{in } T\times G \tag{2.6}$$

$$du(t,x)+Au(t,x)dt+dw(t,x)+\partial I_C(u(t,x))dt \ni f(t,x)dt \qquad \text{in } T\times G \tag{2.7}$$

with (2.2) and (2.3).

In order to formulate (2.6) and (2.7), we introduce two Hilbert spaces such that

$$V=H_0^1(G)\subset H=L^2(G) \tag{2.8}$$

where $H_0^1(G)$ denotes a space $\{\phi \mid \phi\in H^1(G),\ \phi=0$ on Γ) and where $H^1(G)$ denotes a Sobolev space of the order 1 on G.

Identifying H wiht its dual, we have

$$V \subset H \subset V' = H^{-1}(G) \quad \text{(dual of } V). \tag{2.9}$$

Define

$$K = \{\phi \mid \phi \in V,\ \phi \leqq \psi \quad \text{a.e. in } T \times G\}. \tag{2.10}$$

Noting that the subdifferential $\partial I_k(u)$ is defined by

$$\partial I_k(u) = \{\chi(u) \in v' \mid \langle \chi(u),\ v-u \rangle \leqq 0,\ \text{for } {}^{\forall} v \in K\}. \tag{2.11}$$

(2.7) is rewritten by the stochastic variational inequality;

$$u(t) \in K \tag{2.12}$$

$$(u(t),\ \phi) + \int_0^t \langle Au(s),\ \phi \rangle ds + \int_0^t \langle \phi,\ dw(s) \rangle$$

$$+ \int_0^t \langle \chi(u),\ \phi \rangle ds = (u_0,\ \phi) + \int_0^t (f,\ \phi) ds,\ \text{for } {}^{\forall}\phi \in V \tag{2.13}$$

$$\int_0^t (\chi(u),\ v-u) ds \leqq 0, \qquad \text{for } {}^{\forall} v \in K \tag{2.14}$$

where $\langle \cdot, \cdot \rangle$ denotes a pairing between V' and V.

In Haussmann & Pardoux[4], under a strong restriction on data, the existence and uniqueness of the strong solution of the stochastic variational inequality (2.12) to (2.14) have been proved in the case where

$$I_k(\lambda) = \begin{cases} 0, & \text{if } \lambda \geqq 0 \\ +\infty & \text{if } \lambda < 0 \end{cases}$$

and $G =]0,1[$. In Rascanu[5], "weak solution" and "almost weak solution" are defined and the existence is proved, however, the "weak solution" and "almost weak solution" are somewhat unnatural to us. Then, we propose a new weak solution different from Rascanu and porve the existence of the weak solution proposed in this paper.

Definition 2.1: If u satisfies the followings, then u is called the weak solution of the stochastic variational inequality (2.12) to (2.14).

(i) $u \in L^2(\Omega \times T; V)$

(ii) $\int_0^{t_f} \langle \dot{v} + Au - f,\ v+m-u-w \rangle ds + \int_0^{t_f} (v+m-u-w, dm)$

$$+ \frac{1}{2} |m(t_f)|^2 + \frac{1}{2} |v(0) - u_0|^2 \geqq 0$$

for any v such that

$v \in H^1(T;V)$, $m \in M^2(T;H)$, $m - w \in L^2(\Omega \times T; V)$ and

$v + m - w \leqq \psi$ a.e. in $T \times G$

(iii) $u \leqq \psi$ a.e. in $T \times G$

where $M^2(T;H)$ denotes a space of square integrable H-valued martingales m with $m(0) = 0$.

3. Existence Theorem

In this section, we prove the existence of the weak solution of the stochastic variational inequality (2.12) to (2.14).

Theorem 3.1: With conditions

(C-1) : $u_0 \in L^2(\Omega;H)$ (Initial Value)

(C-2) : $Q \in L^1(H)$ (Covariance Operator of the System Noise)

(C-3) : $f \in L^2(T;V')$, (Input)

(C-4) : $^{\exists}\alpha > 0$ s.t. $\sum_{i,j}^{n} a_{ij}\xi_i\xi_j \geqq \alpha\sum_{i}^{n} |\xi_i|^2$ (Coercivity),

there exists a weak solution in the sense of the Definition 2.1 of the stochastic variational inequality (2.12) to (2.14).

In order to prove Theorem 3.1, consider the penalized equation associated with (2.12) to (2.14):

$$(u_\varepsilon(t), \phi) + \int_0^t \langle Au_\varepsilon, \phi\rangle ds + \frac{1}{\varepsilon}\int_0^t ((u_\varepsilon - \psi)^+, \phi) ds$$

$$+ \int_0^t (\phi, dw) = (u_0, \phi) + \int_0^t (f, \phi) ds, \quad \text{for } ^{\forall}\phi \in V \tag{3.1}$$

where $(\cdot)^+$ denotes a nonnegative part of $(\cdot)$.

Lemma 3.1: With the same conditions as in Theorem 3.1, there exists a unique solution u_ε of (3.1) such that

$$u_\varepsilon \in L^2(\Omega \times T;V) \cap L^2(\Omega;C(T;H)) \tag{3.2}$$

and the following estimate holds

$$\frac{1}{\varepsilon} E\{\int_0^{t_f} |(u_\varepsilon - \psi)^+|^2 ds\} \leqq \text{Const. (independent of } \varepsilon).$$

For the proof of Lemma 3.1, see Appendix A.

Proof of Theorm 3.1: It follows from (3.1) that

$$|v(t) - m(t) - u_\varepsilon(t) - w(t)|^2 - 2\int_0^t \langle Au_\varepsilon, v + m - u_\varepsilon - w\rangle ds$$

$$- \frac{2}{\varepsilon}\int_0^t ((u_\varepsilon - \psi)^+, v + m - u_\varepsilon - w) ds$$

$$= |v(0) - u_0|^2 - 2\int_0^t (f, v + m - u_\varepsilon - w) ds + |m(t)|^2$$

$$+ 2\int_0^t \langle \dot{v}, v + m - u_\varepsilon - w\rangle ds + 2\int_0^t (v + m - u_\varepsilon - w, dm) \tag{3.4}$$

for $^{\forall}v \in H^1(T;V)$ such that $v + m - w \leqq \psi$ a.e. in $T \times G$ and $m - w \in L^2(\Omega \times T; V)$.

Noting that

$$((u_\varepsilon - \psi)^+, v + m - u_\varepsilon - w) = ((u_\varepsilon - \psi)^+, v + m - w - \psi)$$

$$-|(u_\varepsilon-\psi)^+|^2\leqq 0, \tag{3.5}$$

(3.4) yields

$$\int_0^t\langle\dot{v}+Au_\varepsilon-f,\ v+m-u_\varepsilon-w\rangle ds+\int_0^t(v+m-u_\varepsilon-w,\ dm)$$
$$+\frac{1}{2}|m(t)|^2+\frac{1}{2}|v(0)-u_0|^2\geqq 0. \tag{3.6}$$

From Lemma 3.1, we can extract a subsequence $u_{\varepsilon'}$ of u_ε such that

$$u_{\varepsilon'}\rightarrow u \text{ weakly in } L^2(\Omega\times T;V). \tag{3.7}$$

It follows frm (3.7) that

$$\lim_{\varepsilon'}\inf E\{\int_0^{t_f}\langle Au_{\varepsilon'},\ u_{\varepsilon'}\rangle ds\}\geqq E\{\int_0^{t_f}\langle Au,\ u\rangle ds\}. \tag{3.8}$$

From (3.6) to (3.8), we have

$$\int_0^{t_f}\langle\dot{v}+Au-f,\ v+m-u-w\rangle ds+\int_0^{t_f}(v+m-u-w,\ dm)$$
$$+\frac{1}{2}|m(t_f)|^2+\frac{1}{2}|v(0)-u_0|^2\geqq 0. \tag{3.9}$$

Furthermore, from (3.7), we have

$$E\{\int_0^{t_f}|(u-\psi)^+|^2 ds\}\leqq\lim_{\varepsilon}\inf E\{\int_0^{t_f}|(u_\varepsilon-\psi)^+|^2 ds\}$$
$$\text{(from (3.3))}$$
$$=0. \tag{3.10}$$

The relation (3.10) implies that

$$u\leqq\psi \quad \text{a.e. in } T\times G \tag{3.11}$$

The proof has thus been completed.

Theorem 3.2: With (C-1) to (C-4), the following relation holds for any weak solution u:

$$u(t)\leqq u^*(t) \quad \text{a.e.} \quad \text{in } T\times G \tag{3.12}$$

where $u^*(t)$ is the weak solution obtained as the limit of the solution of the penalized equation (3.1).

Lemma 3.2: With (C-1) to (C-4), the following estimates holds

$$\int_0^{t_f}\langle\dot{\theta},\ u-u_\varepsilon\rangle ds+\frac{1}{\varepsilon}\int_0^{t_f}((u_\varepsilon-\psi)^+,\ \theta)ds$$
$$\geqq\int_0^{t_f}\langle A(u-u_\varepsilon),\ \theta\rangle ds \tag{3.13}$$

for any $\theta\in L^2(\Omega;H^1(T;V))$, $\theta(t_f)=0$ and θ is nonnegative if u is a weak solution.

For the proof of Lemma 3.2, see Appendix B.

Proof of Theorem 3.2; Let θ_λ be the solution of

$$-\lambda\, \dot{\theta}_\lambda + \theta_\lambda = (u - u_\varepsilon)^+ \quad \text{in } T\times G \tag{3.14}$$

$$\theta_\lambda(t_f) = 0. \tag{3.15}$$

From (3.14) and (3.15), we have

$$\theta_\lambda(t) = \frac{1}{\lambda}\int_0^{t_f} \exp\{(t-s)/\lambda\}\ (u-u_\varepsilon)^+ ds. \tag{3.16}$$

Equations (3.14) and (3.16) yield

$$\int_0^{t_f} \langle \dot{\theta}_\lambda,\ u-u_\varepsilon\rangle ds \geqq 0. \tag{3.17}$$

In (3.13), setting as $\theta=\theta_\lambda$ and using (3.17), we have

$$\frac{1}{\varepsilon}\int_0^{t_f} ((u_\varepsilon-\psi)^+,\ \theta_\lambda)ds \geqq \int_0^{t_f} \langle A(u-u_\varepsilon),\ \theta_\lambda\rangle ds. \tag{3.18}$$

Noting that $\theta_\lambda \to (u-u_\varepsilon)^+$ in $L^2(\Omega\times T;H)$, it follows from (3.10) that

$$E\{\int_0^{t_f} \| (u-u_\varepsilon)^+ \|^2 ds\} \leqq 0.$$

Therefore,

$$u \leqq u_\varepsilon \quad \text{a.e.} \quad \text{in } T\times G, \text{ w.p.1.} \tag{3.19}$$

Lemma 3.1 and (3.19) yield

$$u \leqq u^* \quad \text{a.e.} \quad \text{in } T\times G, \text{ w.p.1.} \tag{3.20}$$

The proof has thus been completed.

References

1) Bensoussan A. "Application of Variational Inequalities in Stochastic Control" (North-Holland) (1982)

2) Baiocchi C. "Variational and Quasivariational Inequalities" (John Wiley) (1984)

3) Panagiotopoulos P.D. "Inequality Problems in Mechanics and Applications" (Birkhäuser) (1985)

4) Haussmann U.G. & Pardoux E. "Stochastic Variational Inequalities of Parabolic Type", Appl. Math. Optim., 20, 163-192 (1989)

5) Rascanu A. "Existence for a Class of Stochastic Parabolic Variational Inequalities", Stochastics, 5, 201-239 (1981)

Appendices

Appendix A. (Proof of Lemma 3.1): Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of H which made up with elements of V. Consider the finite dimensional stochastic equation

$$(u_\varepsilon^n(t),\ e_i) + \int_0^t \langle Au_\varepsilon^n,\ e_i\rangle ds + \frac{1}{\varepsilon}\int_0^t ((u_\varepsilon^n-\psi)^+,\ e_i)ds$$

$$+\int_0^t (e_i,\ dw^n) = (u_0^n,\ e_i) + \int_0^t (f,\ e_i)ds, \text{ for } 1\leqq i\leqq n \tag{A.1}$$

For the function v such that

$v \in \mathrm{span}[e_1, e_2, \cdots\cdots, e_n]$, $v \leqq \psi^n$ a.e. in $T \times G$,

applying Ito's lemma to $|u_\varepsilon^n(t) - v|^2$, we have

$$|u_\varepsilon^n(t) - v|^2 + 2\int_0^t \langle Au_\varepsilon^n, u_\varepsilon^n - v\rangle ds + \frac{2}{\varepsilon}\int_0^t ((u_\varepsilon^n - \psi^n)^+, u_\varepsilon^n - v)ds$$

$$+2\int_0^t (u_\varepsilon^n - v, dw^n) = |u_0^n - v|^2 + 2\int_0^t (f, u_\varepsilon^n - v)ds$$

$$+\sum_{i=1}^n \int_0^t (Qe_i, e_i)ds \tag{A.2}$$

Noting that

$$((u_\varepsilon^n - \psi^n)^+, u_\varepsilon^n - v) = |(u_\varepsilon^n - \psi^n)^+|^2 + ((u_\varepsilon^n - \psi^n)^+, \psi^n - v)$$
$$\geqq |(u_\varepsilon^n - \psi^n)^+|^2. \tag{A.3}$$

(A.2) yields

$$|u_\varepsilon^n(t) - v|^2 + 2\alpha\int_0^t \| u_\varepsilon^n - v \|^2 ds + \frac{2}{\varepsilon}\int_0^t |u_\varepsilon^n - \psi^n)^+|^2 ds$$

$$+2\int_0^t (u_\varepsilon^n - v, dw^n) \leqq C_1 + \frac{1}{\delta}\int_0^t \| f \|_{V'}^2 ds + \delta\int_0^t \| u_\varepsilon^n - v \|^2 ds \tag{A.4}$$

where

$$C_1 \geqq 2[\| u_0 \|^2 + |v|^2] + \int_0^t \mathrm{tr}[Q]ds.$$

Choosing δ as $2\alpha - \delta > C \geqq 0$ in (A.4) and using Gronwall's inequality, we have

$$E\{\int_0^{t_f} \| u_\varepsilon^n - v \|^2 ds\} + \frac{1}{\varepsilon}E\{\int_0^{t_f} |(u_\varepsilon^n - \psi^n)|^2 ds\} \leqq C_2 \tag{A.5}$$

where C_2 is a constant independent of n and ε.
From (A.5), we can extract a subsequence $u_\varepsilon^{n'}$ of u_ε^n for a fixed ε such that

$$u_\varepsilon^{n'} \to u_\varepsilon \text{ weakly in } L^2(\Omega \times T; V). \tag{A.6}$$

From the monotone property of the operator $(\cdot)^+$, we have

$$\frac{1}{\varepsilon}E\{\int_0^{t_f} |(u_\varepsilon - \psi)^+|^2 ds\} \leqq C_3 \text{(independent of } \varepsilon).$$

Appendix B (Proof of Lemma 3.2): Consider

$$(\dot{z}_\varepsilon(t), \phi) + \langle Au_\varepsilon, \phi\rangle + \frac{1}{\varepsilon}((u_\varepsilon - \psi)^+, \phi) = (f, \phi) \text{ for } {}^\forall\phi \in V \tag{B.1}$$

$$z_\varepsilon(0) = u_0, \tag{B.2}$$

Multiplying (B.1) by $v+m-w$, where $v \in H^1(T;V)$, $m \in M^2(T;H)$, $m-w \in L^2(\Omega \times T;H)$ and $v+m-w \leqq \psi$ a.e. in $T \times G$, we have

$$(\dot{z}_\varepsilon, v+m-u) + \langle Au_\varepsilon, v+m-u-w\rangle$$
$$+\frac{1}{\varepsilon}((u_\varepsilon - \psi)^+, v+m-u-w) = \langle f, v+m-u-w\rangle \tag{B.3}$$

From the Definition 2.1, we have

$$\int_0^{t_f}\langle f,\ v+m-u-w\rangle ds \leqq \int_0^{t_f}\langle \dot{v}+Au,\ v+m-u-w\rangle ds$$
$$+\int_0^{t_f}(v+m-u-w,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2+\frac{1}{2}\,|\,v(0)-u_o\,|^2 \tag{B.4}$$

Using (B.4) in (B.3), we obtain

$$\int_0^{t_f}\langle \dot{v}+Au-Au_\varepsilon-\dot{z}_\varepsilon-\frac{1}{\varepsilon}(u_\varepsilon-\psi)^+,\ v+m-u-w\rangle ds$$
$$+\int_0^{t_f}(v+m-u-w,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2+\frac{1}{2}\,|\,v(0)-u_o\,|^2\geqq 0 \tag{B.5}$$

Let $r\in L^2(\Omega;H^1(T;V))$ such that $r(t_f)=0$ and r is nonnegative if u is a weak solution. Substituting $v-\frac{1}{\lambda}r$ $(\lambda>0)$ for v in (B.5), we have

$$\int_0^{t_f}\langle \dot{v}-\frac{1}{\lambda}\dot{r}+Au-Au_\varepsilon-\dot{z}_\varepsilon-\frac{1}{\varepsilon}(u_\varepsilon-\psi)^+,\ v-\frac{1}{\lambda}r+m-u-w\rangle ds$$
$$+\int_0^{t_f}(v-\frac{1}{\lambda}r+m-u-w,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2+\frac{1}{2}\,|\,v(0)-u_o\,|^2\geqq 0 \tag{B.6}$$

Noting that

$$\frac{1}{2}\,|\,v(t_f)-\frac{1}{\lambda}r(t_f)+m(t_f)-z_\varepsilon(t_f)\,|^2$$
$$=\frac{1}{2}\,|\,v(0)-\frac{1}{\lambda}r(0)-u_o\,|^2+\int_0^{t_f}\langle \dot{v}-\frac{1}{\lambda}\dot{r}-\dot{z}_\varepsilon,\ v-\frac{1}{\lambda}r+m-z_\varepsilon\rangle ds$$
$$+\int_0^{t_f}(v-\frac{1}{\lambda}r+m-z_\varepsilon,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2 \tag{B.7}$$

In (B.7), taking $r=0$, we have

$$\frac{1}{2}\,|\,v(t_f)+m(t_f)-z_\varepsilon(t_f)\,|^2=\frac{1}{2}\,|\,v(0)-u_o\,|^2$$
$$+\int_0^{t_f}\langle \dot{v}-\dot{z}_\varepsilon,\ v+m-z_\varepsilon\rangle ds+\int_0^{t_f}(v+m-z_\varepsilon,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2 \tag{B.8}$$

By subtracting (B.8) from (B.7) and noting $r(t_f)=0$, we obtain

$$(v(0)-u_o,\ -\frac{1}{\lambda}r(0))+\frac{1}{2}\,|\,\frac{1}{\lambda}r(0)\,|^2+\int_0^{t_f}\langle \dot{v}-\dot{z}_\varepsilon,\ -\frac{1}{\lambda}r\rangle ds$$
$$+\int_0^{t_f}\langle -\frac{1}{\lambda}\dot{r},\ v-\frac{1}{\lambda}r+m-z_\varepsilon\rangle ds+\int_0^{t_f}(-\frac{1}{\lambda}r,\ dm)=0. \tag{B.9}$$

It follows from (B.6) and (B.9) that

$$\int_0^{t_f}\langle \dot{v}-\dot{z}_\varepsilon,\ v+m-u-w\rangle ds+\int_0^{t_f}\langle -\frac{1}{\lambda}\dot{r},\ z_\varepsilon-u-w\rangle ds$$
$$+\int_0^{t_f}\langle Au-Au_\varepsilon-\frac{1}{\varepsilon}(u_\varepsilon-\psi)^+,\ v-\frac{1}{\lambda}r+m-u-w\rangle ds$$
$$+\int_0^{t_f}(v+m-u-w,\ dm)+\frac{1}{2}\,|\,m(t_f)\,|^2+\frac{1}{2}\,|\,v(0)-u_o\,|^2\geqq 0. \tag{B.10}$$

Multiplying (B.10) by λ and tends to $\lambda\to 0$, from Lemma 3.1, we have

$$E\{\int_0^{t_f}\langle \dot{r},\ z_\varepsilon - u - w\rangle ds\} - E\{\int_0^{t_f}\langle A(u-u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+,\ r\rangle ds\} \geqq 0 \tag{B.11}$$

Since $z_\varepsilon = u_\varepsilon - w$, (B.11) yields

$$E\{\int_0^{t_f}\langle \dot{r},\ u - u_\varepsilon\rangle ds\} + E\{\int_0^{t_f}(\frac{1}{\varepsilon}(u_\varepsilon - \psi)^+,\ r)ds\}$$

$$\geqq E\{\int_0^{t_f}\langle A(u-u_\varepsilon),\ r\rangle ds\}. \tag{B.12}$$

(B.12) implies (3.15).