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Determining Image Flow from Multiple Frames Based on the Continuity Equation

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Abstract

An integration-based temporal-optimization is proposed for the determination of optical flow. The method is based on the continuity equation of fluids. Dynamic scenes are regarded as a kind of hydrodynamic flow (called image flow tentatively). A generalized expression of the continuity equation for the image flow is shown explicitly. The continuity image sequences under assumptions of incompressible fluids without sources or sinks and of temporal constancy of the flow field. Improvements in accuracy and reliability of the method is confirmed by simulation experiments.

1. Introduction

Determination of the motion parameters from image sequence is attracted increasing attention from three-dimensional object recognition and shape perception[(1)~(3)]. Analysis of the relative motion between a camera and textured objects results in an apparent two-dimensional (2-D) velocity field which is denoted optical flow[(4),(5)]. Evaluation of such a 2-D optical flow field offers useful 3-D information to recover the structure of the scene.

The gradient-based methods which utilize a relationship between motion of a textured surface and the partial derivatives of image brightness function are known as a representative method to determine the flow field[(4),(6)]. The gradient constraint equation does not by itself provide a means for calculating optical flow, since only one independedt measurement is available from the image sequence at a "point", while the flow velocity has two components[(4)]. We need a second constraint. Two different approaches have been investigated in the gradient-based methods. Global optimization minimizes an error function based upon the gradient constraint and an assumption of local smoothness of optical flow variations over the entire image. Iterative implementations to optimize the flow field have been tested in the approach[(4),(7)]. Local optimization assumes that optical flow is constant in a local space and apply the gradient constraint equation to two (or more) points to obtain a full solution[(6)]. These traditional approaches are founded on the same assumption that the observed brightness of any object point is constant over time. Observer's visual point is fixed on a feature point of the moving object, and he tries to trace the movement of the point. This description corresponds to the Lagrange representation of continuity equation in

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the transport theory.

In the previous letter[8], we proposed a different approach based on a field theory. We paid our attention not to the moving object itself but to the background field. We regarded the moving objects as only visualizing matters like as dusts on the hydrodynamic flow. The velocity field was assumed to be constant in a short period. In this report, we will develop our approach and propose several methods such as integration based temporal optimization and non-stationary velocity field approach to obtain reliable optical flow field from multiple image frames. Constrast to the traditional methods, our approach correspond to the Euler representation of the continuity equation.

2. Traditional Approaches

The assumptioin of the gradient-based methods are expressed as:

$$g(x, y, t)=g(x+\delta x, y+\delta y, t+\delta t), \tag{1}$$

where $g(x, y, t)$ and $g(x+\delta x, y+\delta y, t+\delta t)$ is brightness function (or image function) of the image at points $p_i(x, y, t)$ and $p_j(x+\delta x, y+\delta y, t+\delta t)$, respectively. Expanding the image fuction in a Taylor's series around the point p_1 we obtain

$$g(x+\delta x, y+\delta y, t+\delta t) \\ =g(x, y, t)+g_x\delta x+g_y\delta y+g_t\delta t+h.o.t. \tag{2}$$

where $g_x(=\partial g/\partial x)$, $g_y(=\partial g/\partial y)$, $g_t(=\partial g/\partial t)$ are the partial derivatives of the image function. In the limit as $\delta t \to 0$ the eq.(2) becomes the gradient constraint equation[4]

$$g_x u+g_y v+g_t=0 \tag{3}$$

where $u(=\lim_{\delta t \to 0} \frac{\delta x}{\delta t})$ and $v(=\lim_{\delta t \to 0} \frac{\delta y}{\delta t})$. Since we neglected the higher order term (h.o.t) in eq.(2), the above equation shows an approximate relationship for a finite time interval $\delta t(\neq 0)$. According to this restriction $(\delta t \to 0)$, we do not have a reasonable background to deal with multiple image frames simultaneously to evaluate the optical flow. Therefore, only two or three frames have been utilized to obtain the instantaneous optical flow in the traditional approaches[6]. This guarantees good time resolutioin of the opticla flow, however, the reliability of the methods was not enough. And to obtain the full solutions, we need the second constraint such as the smoothness of optical flow variations over the entire image (global optimization[4]) or constant velocity assumption within a local area (local optimization[6]). These constraints degrade the spatial resolution of the flow field and may bring erratic and unreliable solution at object boundaries.

3. Transport Theory Approach

3.1. *Basic Formulation*

Recently, there have been proposed several new trials to explain optical flow based on the continuity equation of the hydrodynamic flow[7,9]. As pointed out by Schunck[9] and Nagel[10], the flow of features out of an area enclosed by a loop δ C may be equal in magnitude and opposite in sign to the instantaneous rate of change of the "features density g" integrated over this area:

$$\int_{\delta C} g\ \vec{V}\cdot\vec{n}\ dC = -\frac{\partial}{\partial t}\int_{\delta S} g\ dS \tag{4}$$

where $\vec{n}$ is a unit length normal vector pointing to the outside of the loop δC and $\vec{V}$ is the velocity of the flow. In hydrodynamics, the above equation is recognized as the continuity equation for fluids. The density of fluids and the current density of the flow may correspond to g (feature density or image function) and g $\vec{V}$ (feature current density or image flow density), respectively. The fundamental idea of eq. (4) is based on the transport theory. Observer's visual point is fixed on a local area δS, and just observing the rate of changes of the feature density within the area. There is no need to trace object movement. (This approach corrsponds to the Euler representation in the hydrodynamic equation.) The difference of situations between the transport theory approach and the traditional one is illustrated in Fig.1. In this paper, we tentatively call the flow of features as "image flow" $\vec{v}$=(p,q) which obeys the continuity equation of fluids instead of optical flow $\vec{V}$=(u,v). In the limitδS →0, we obtain a differential expression for eq. (4)

$$g_t + \mathrm{div}(g\ \vec{V}) = g_t + \vec{V}\cdot\mathrm{grad}(g) + g\ \mathrm{div}(\vec{V}) = 0 \tag{5}$$

The same euqation is introduced by Fitzpartrick[11] for the study of energy density images such as those from CAT scans, and by Schunck[9] for token density images. Schunck[9] pointed out that the third term, g·div($\vec{V}$), can only be neglected if the motion of the scene is parallel to the image plane.

For the reflective images, however, we believe that the following assumptions are reasonable:

1) If the motion of a rigid scene is parallel to the image plane, the image flow is regarded as an incompressible fluid which has non-uniform density without diffusion.
2) Whenever surface elements in the scene move towards or away from the image plane, there exist sources or sinks in the image flow field.
3) If the occluding motion is included in a dynamic scene, we can also imagine the existence of sources or sinks in the flow field.

Considering the above situations, we have to start from a generalized expression[7] for the continuity equation instead of eq. (5):

$$g_t + \vec{V}\cdot\mathrm{grad}(g) + g\ \mathrm{div}(\vec{V}) - S = 0 \tag{6}$$

where S means a rate of brighness formation from the sources or sinks in the field. The eq. (6) represents a very generalized relation which can be applied to any fluids including image flow. Of course, we can imagine special image sequences such as diffusion phenomena (div($\vec{V}$) ≠0, S=0) and temporal variation of illumination (div ($\vec{V}$) =0, S≠0). These special images may be regarded as compressible fluids. For normal problem of the 3D scene analysis, we can assume that the image flow is an

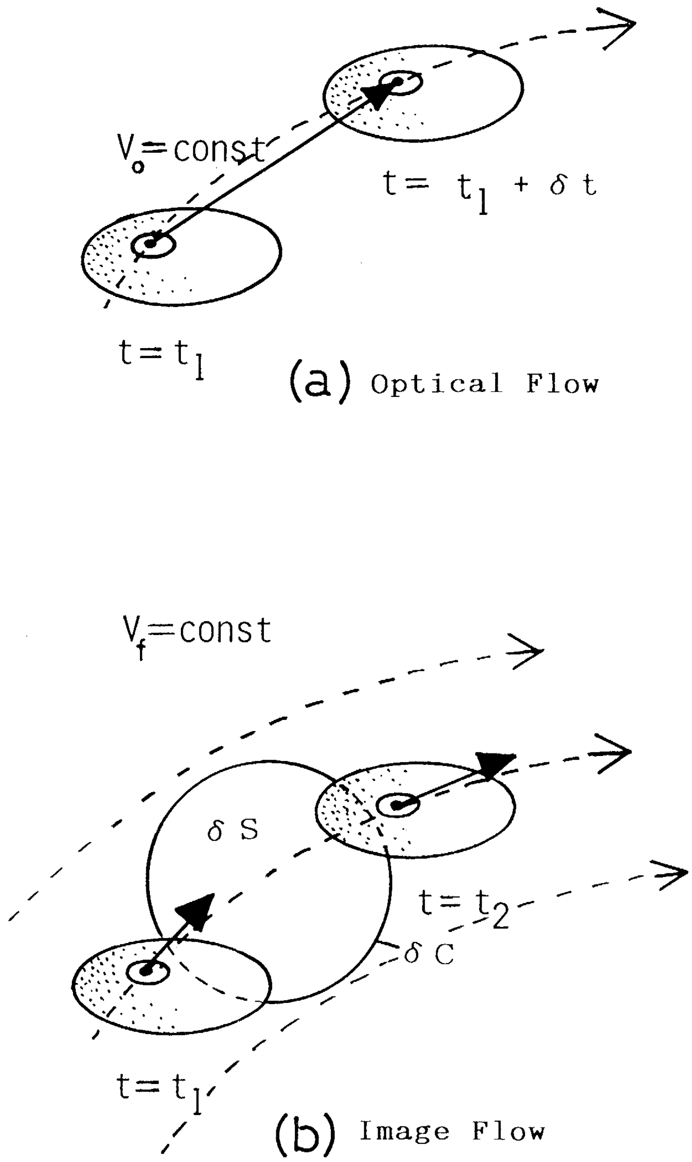


Fig.1 Difference of viewpoint for the gradient constraint equation. (a)Traditional approaches trace the object motion under assumption of constant object-velocity (V_0=constant) within a short interval δt. (b)The transport theory approach counts the flow rate under the assumption of constant velocity-field(V_1=constant) within a fixed small area δS and within a time interval t_2-t_1.

incompressible fluid. In this paper, we restricted our discussion to the incompressible image flows. The assumption of incompressible fluid is expressed by the Lagrange operator (D/Dt)

$$Dg/Dt = g_t + \vec{V} \cdot \mathrm{grad}(g) = 0 \qquad (7)$$

Combining the equations (6) and (7), we obtain

$$S = g\ \mathrm{div}(\vec{V}) \tag{8}$$

The most critical discrepancy between Schunck's equation (eq. (5)) and our proposal (eq. (6)) is an interpretation of the divergence of the flow velocity. As pointed by Schunck[9], the divergence term can only be neglected if the motion of the scene is parallel to the image plane. We dare say that the term is equal to the rate of brightness formation from the sources or sinks for the incompressible image flow. As mentioned before, the occluding motion and the movement toward or away from the image plane are the origins of sources and sinks. Because of a singularity of the flow field, it seems difficult to analyze these cases by the continuity equation. In the following, we will concentrate our discussion to the restricted situation ($S=0$ and $\mathrm{div}(\vec{V})=0$). In this case, we can utilize the continuity equation of the incompressible fluid (eq. (4) or (7)) to analyze the image flow field. Apparently, eq. (7) coincides with eq. (3) in 2-D representation. This indicates that the incompressible image flow is equivalent to the optical flow.

3.2. *Temporal optimization for stationary field*

3.2.1 Integration formula

For incompressible fluids without sources or sinks, the equation (4) represents a precise relationship. Contrast to the case in eq. (3), we are not obliged to take the limit $\delta t \to 0$ to obtain eq. (7) If the velocity field $\vec{V}$ is supposed to be stationary within a time interval $\delta t (= t_2 - t_1)$, we can write down a pair of equations determining image flow at a local area δS around a fixed point p(x,y) as

$$\int_{\delta C} g(x,y,t_1)\ \vec{V}\cdot\vec{n}\ dC = -\frac{\partial}{\partial t}\int_{\delta S} g(x,y,t_1)\,dS \tag{9,1}$$

$$\int_{\delta C} g(x,y,t_2)\ \vec{V}\cdot\vec{n}\ dC = -\frac{\partial}{\partial t}\int_{\delta S} g(x,y,t_2)\,dS \tag{9,2}$$

Contrast to the traditional approach (the simple local optimization or spatial local-optimization (SLO)[6], constraint equations are gathered not from a neighboring small area (spatial) but from a succeeding short interval (temporal) of image sequence. Assumption of the stationary field is based on the fact that the flow field $\vec{V}(x,y,t) = (p,q)$ is slowly varying compared to gray value $g(x,y,t)$. Tentatively, we call this approach as an "integration based temporal local-optimization (IB-TLO)". Or actually, this is not already the "gradient-based method", but might be regarded as " integration-based method" or "flow-based" method. When we apply the eq.(9) to a rectangular area as shown in Fig.2, we obtain

$$p(A_1 - C_1) + q(B_1 - D_1) = -\frac{\partial}{\partial t} G(t_1) \tag{10,1}$$

$$p(A_2 - C_2) + q(B_2 - D_2) = -\frac{\partial}{\partial t} G(t_2) \tag{10,2}$$

where

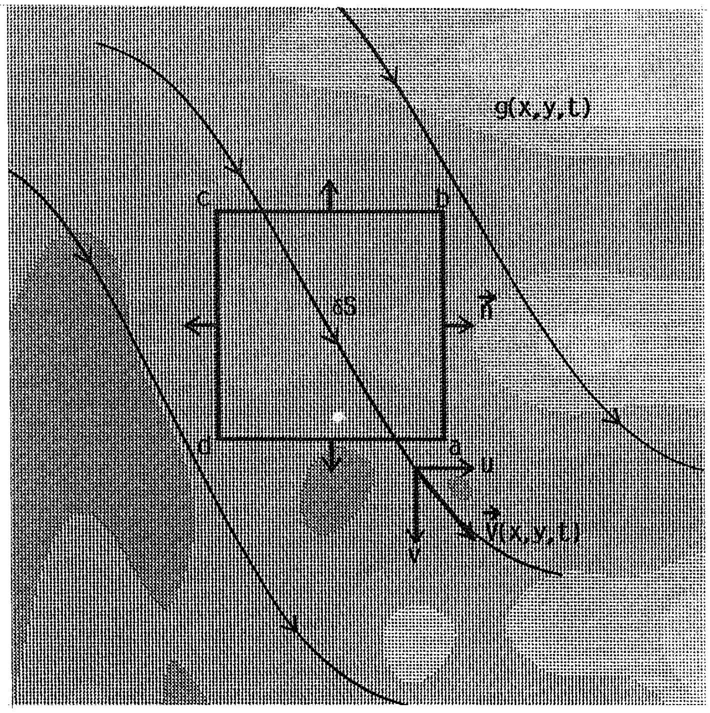


Fig.2 Application of IB-TLO (Integration based temporal local-optimization) method to a small rectangular area.

$$A=\int_{a}^{b} g\ dy,\ B=\int_{b}^{c} g\ dx,\ C=\int_{c}^{d} g\ dy,\ D=\int_{d}^{a} g\ dx, \tag{11}$$

and

$$G=\int_{\delta S} g\ dx\ dy. \tag{12}$$

3.2.2 Differential formulation

As it has been proposed in the previous report[8], we can write down a differential formula of eq.(7) for two image frames at time t_1 and t_2 as

$$g_x(x,y,t_1)p+g_y(x,y,t_1)q=-g_t(x,y,t_1) \tag{13,1}$$

$$g_x(x,y,t_2)p+g_y(x,y,t_2)q=-g_t(x,y,t_2) \tag{13,2}$$

When we have a sequence of images (more than 3 frames), we can introduce the simple optimization technique. Constraint equations from a group of sequential frame points are gathered to produce an overdetermined system of linear equations:

$$\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \cdot & \\ \cdot & \\ \frac{\partial g_n}{\partial x} & \frac{\partial g_n}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = -\frac{\partial}{\partial t} \begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ g_n \end{pmatrix} \quad (14)$$

where $g_i = g(x,y,t_i)$. The least squares minimization is utilized to solve the system. In the previous report, we call this approach as a "temporal local-optimization (TLO)" method. Similar optimization technique is also applicable to the integrative formulation in eq.(9).

3.3 *Deterministic approach for non-stationary field*

In the preceding section, we assume the stationariness (temporal constancy) of the flow field. Here, we try to formulate a deterministic approach for the analysis of non-stationary field. Now, we assume a temporal smoothness (continuity) of the velocity field. Expanding the velocity field $\vec{V}(x,y,t)$ at $t=t_0+\delta t$ in a Taylor's series around $t=t_0$ we obtain

$$\vec{V}(x,y,t) = \vec{V}(x,y,t_0) + (\partial \vec{V}/\partial t)\,\delta t + (1/2)\,(\partial^2\vec{V}/\partial t^2)\,(\delta t)^2 + h.o.t \quad (15)$$

When we consider up to the first order term, unknown variables are $p(=V_x)$, $q(=V_y)$, $\partial p/\partial t$, and $\partial q/\partial t$. We need four independent equations to obtain full solution. According to our temporal optimization approach, determinintic equations for four succeeding time points are given as

$$g_x(t)\ p \quad + g_y(t)\ q \quad + g_t(t) \quad = 0 \quad (16\text{-}1)$$

$$g_x(t+\delta t)\,(p+\frac{\partial p}{\partial t}\,\delta t) \quad + g_y(t+\delta t)\,(q+\frac{\partial q}{\partial t}\,\delta t) \quad + g_t(t+\delta t) = 0 \quad (16\text{-}2)$$

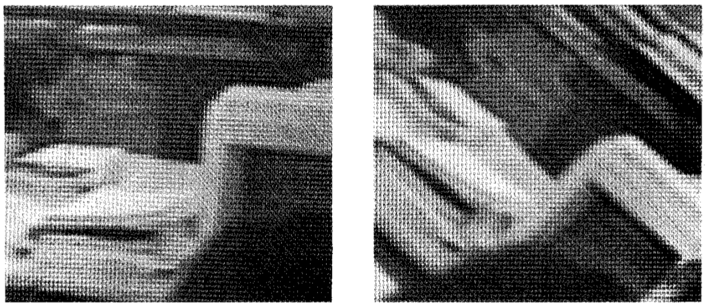
$$g_x(t+2\delta t)\,(p+\frac{\partial p}{\partial t}\,2\delta t) + g_y(t+2\delta t)\,(q+\frac{\partial q}{\partial t}\,2\delta t) + g_t(t+2\delta t) = 0 \quad (16\text{-}3)$$

$$g_x(t+3\delta t)\,(p+\frac{\partial p}{\partial t}\,3\delta t) + g_y(t+3\delta t)\,(q+\frac{\partial q}{\partial t}\,3\delta t) + g_t(t+3\delta t) = 0 \quad (16\text{-}4)$$

where $\vec{V}(t) = \vec{V}(x,y,t)$ and $\vec{V}(t+\delta t) = \vec{V}(x,y,t+\delta t)$. If we take into account up to the second order term in eq.(15), we need six independent equations to obtain six unknown variables ($p, q, \partial p/\partial t$, $\partial^2 p/\partial t^2$, and $\partial^2 q/\partial t^2$). The similar approach can be applied to the integration formula in eq.(9).

4. Simulation Experiments

Effectiveness of the proposed methods is investigated with artificial image sequences. Here, we demonstrate a typical example of the results. A static scene of real world (an office scene) is rotated clockwise (1.0 degree/frame) artificially around



(a) (b)

Fig.3 Snapshots of analyzed image sequence obtained by artificial rotation of a real picture (an office scene).

the center of picture and transformed to an image sequence. Snapshots of the scene are shown in Fig.3. The analyzed velocity fields are summarized in Fig.4(a−c). Trying to make an impartial evaluation, the same number of linear equations (n=25) are gathered to produce the overdetermined system (see details in Fig.4). Consequently, the calculation cost is kept almost constant for three methods. Each figures correspond to the result obtained by SLO method (a), by TLO method (b) and by IB-TLO method (c), respectively. These representative results demonstrate clearly the difference of accuracy and reliability of the respective methods. Apparently, the proposed integration formula (IB-TLO method) is superior than the other methods. The new approach used more reliable information than the traditional approaches (SLO method and TLO method).

5. Concluding Remarks

In this paper, we proposed new approaches based on the transport theory. In the traditional approach, the well known gradient constraint equation (see eq.(3)) is valid only for $t \rightarrow 0$ limit. Constant velocity motion of the moving object during a short time-interval is assumed in the analysis. Consequently, considering the general characteristics of dynamic scenes, simultaneous manipulation of multiple image frames has not been accepted in the traditional optical flow analysis. In our approach, we abandon to trace the object motion, but employ to estimate the flow rate of the "image flow" passing through a local small area. The observer's view point is fixed on the local area (Euler representation). We can deal with multiple frames to evaluate image flow field under the assumption that the field is in stationary state (or changing gradually with time). This is exactly the same point of view to the spatio-temporal correlation approach[12,13], which we have been proposed recently. As pointed out by Subbarao (1989)[14], most of researches until now have concentrated on measuring only the instantaneous optical flow. He suggests that general methods for measuring optical flow in the "spatio-temporal" domain needs to be investigated. Recent investi-

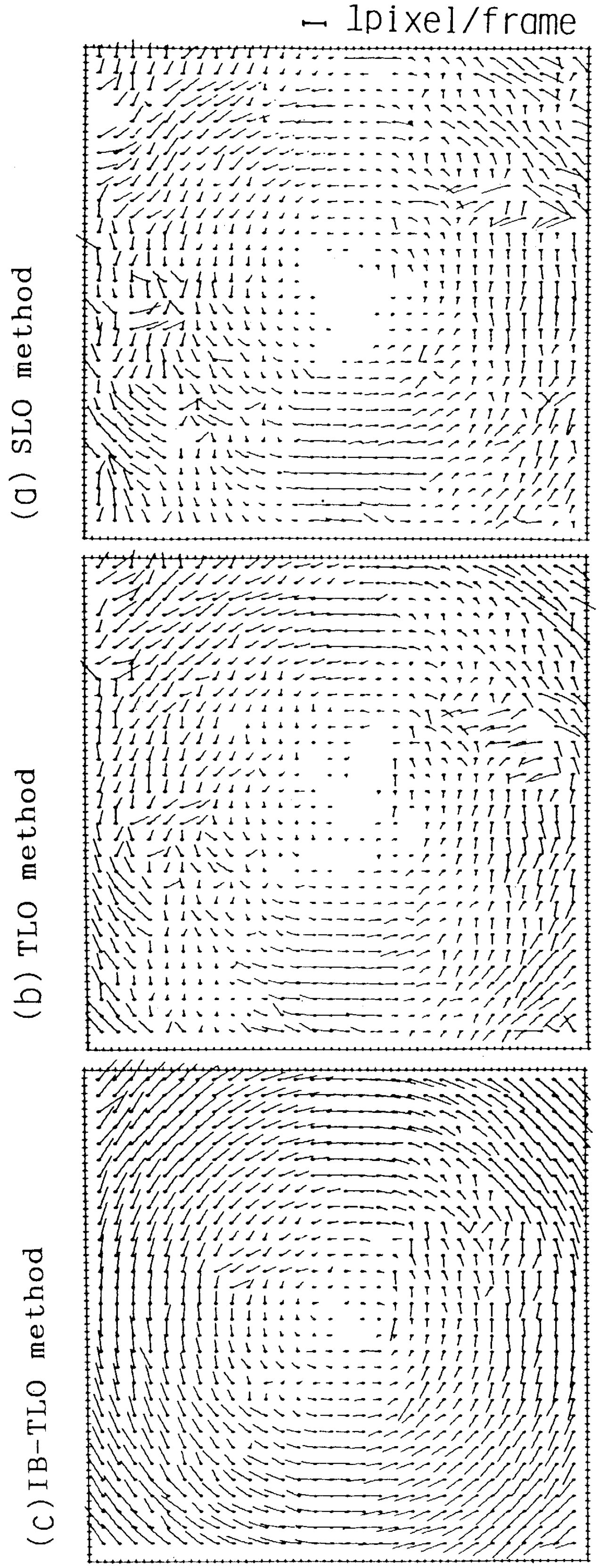


Fig.4 Obtained optical flow fields by (a)SLO method, by (b)TLO method, by (c)IB-TLO method. An overdetermined set of 25 equations is gathered from a rectangular 5×5 window in SLO method. 25 frames of sequential images are utilized for the temporal optimization in TLO and IB-TLO methods. A rectangular 3×3 window is used as the fixed area S in IB-TLO method.

gation of Blanck and Anandan[15] and our approach may be good candidates for measuring optical flow in the "spatio-temporal" domain.

Tne new approaches proposed here have advantages of 1) good spatial resolution (by principle in TLO method) and 2) high accuracy (especially in IB-TLO method) compared with the traditional approach (SLO method) under the condition that the same number on overdetermined equations are gathered. Detailed examinations including the investigation of the nonstationary approach, analysis of real image sequences and comparison with our spatio-temporal correlation method are under way. For future development of the research, it is essential to investigate the image sequences including occluding motion and motion toward the depth direction. The proposed continuity equation for the image flow may give a clue to attack these difficult problems.

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