Analysis of Dynamic Stability of Framed Structures

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Abstract

An analytical method of dynamic instability of three-dimensional framed structures to which it is difficult to apply the theory of continuum mechanics is proposed in the two cases of regarding and disregarding the effect of damping, in the cases where the effect of forced vibration is neglected. The equations of motion of Mathieu-Hill type are set up using the linearized finite displacement method, and the equations of boundary frequency in the regions of dynamic instability of columns and arches are derived applying the proposed method. The results show good agreement with the boundary values of a column simply supported at the both ends.

Introduction

It is generally known that the critical loads of the columns or the framed structures as shown in Fig. 1 are considerably affected by the frequencies and amplitudes of the periodic loads. The problem for estimating the critical loads of a column subjected to the horizontal periodic loads as shown in Fig. 1 (a) is regarded to be the same as that for obtaining the natural frequencies affected by an axial force and this is comparatively readily solved. On the other side, parametrically excited vibration occurs in the structures if the periodic and non-periodic loads act in the same direction as shown in Fig. 1 (b) and (d). Many studies [1, 3, 4] have been carried out in this connection using the method of continuum mechanics.

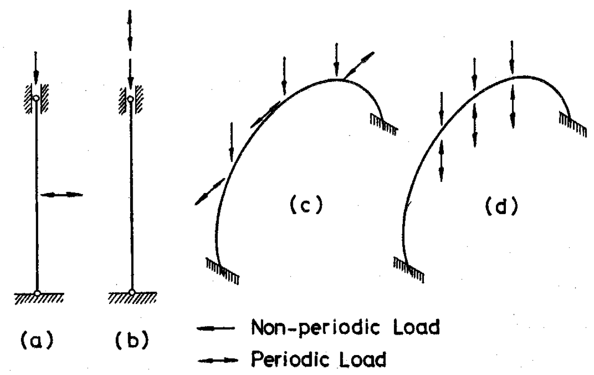

Fig. 1 Structures Subjected to the Periodic and Non-periodic Loads.

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In this study, assuming that the shape of framed structures exhibits regular geometry, that is, shallow arches, flat shell, etc., are excluded, framed structures to which it is difficult to apply the continuum mechanics are taken up and the dynamic stability of them, in the case where parametrically excited vibration occurs, is investigated.

In parametrically excited unstable vibration, the effect of damping should be considered since it has comparatively large influence. It is noted that, if damping is taken into account, the regions of instability become narrow. Moreover, it is also known that there occurs the resonance by forced vibration in the case where the frequencies of the periodic loads are near to the natural frequencies of the structures and the forced vibration exerts influence on the dynamic stability. However, since the forced vibration has no influence upon the dynamic stability if the frequencies are far enough from a natural frequency with same modes as ones due to periodic loads, the effect of forced vibration may be neglected.

In this paper, dynamic stability is investigated, regarding or disregarding the damping, in the case where the effect of forced vibration is neglected. Although there are several cases in which the instability due to the combination resonance in addition to the simple parametric resonance is important for some kinds of structures with certain boundary conditions under certain loads, only the instability due to the simple parametric resonance is here dealt with.

In the analysis, the equations of motion with small amplitudes in the stable vibration are first derived. Expressing the equations in the unstable vibration by the displacements in the stable vibration and perturbed ones (i.e., additional displacements), the equations for boundaries of dynamic instability of the structure are established and the boundary values of the regions of dynamic stability are founded. Following assumptions are made:

(1) all the external loads act on the nodes of the structure,
(2) all the periodic loads have a constant period,
(3) the amplitudes of all the periodic loads and magnitudes of all the non-periodic loads vary at a constant ratio respectively and are much smaller than critical loads for prebuckling mode,
(4) the external loads have no masses,
(5) the stresses in the members of the structure do not exceed the proportional limit,
(6) local buckling and instability phenomenon do not happen in changing the magnitudes of the periodic and non-periodic loads and
(7) the masses of the structure are concentrated on the nodes, i.e., the lumped mass system is considered as the substitute.

In the calculation, it is presumed that the periodic solutions on the boundary of the regions of instability are close to harmonic vibration since it is verified in the study of Y. Sugiyama [2] that, although this assumption is not valid on the upper boundary of the principal region, they are appropriate in the steady state motion.

In the last chapter, columns and parabolic arches are analyzed by the proposed method and the results are compared with those of the simply supported column already analyzed by the theory of continuum mechanics.

Equations of Motion

When the magnitudes of non-periodic loads and amplitudes and frequencies of periodic loads are kept within the characteristic ranges of the structure, i.e., they are within the regions of dynamic stability, a very small change in displacements is produced for their infinitesimal change of them and the amplitudes of the displacements do not unboundedly increase with time. But, when some of the values mentioned above exceed the characteristic ranges, that is, they enter the regions of dynamic instability, the pattern of vibration is quite different from the mode in the stable vibration and the unstable phenomenon in which the amplitudes of the displacements unboundedly increase with time takes place. In the former state, the equations of motion with infinitesimal amplitudes hold approximately, but in the latter state, they no longer hold and the equations with finite amplitudes are required.

In this paper, the equations of motion with infinitesimal amplitudes are set up supposing that the magnitudes of the non-periodic loads and the amplitudes and frequencies of the periodic loads take certain values adjacent to the boundary of the regions of stability and the structure is kept in the state of stable vibration. Next, supposing the state in which the values mentioned above undergo slight changes and the structure vibrates unstably, the equations of motion with finite amplitudes are obtained from the dynamic equilibrium of nodes of the deformed structure. In this case, it is assumed that the internal forces in each member are linearly proportional to the displacements of nodes, that is, Hooke's law is satisfied and the linearized finite displacement method is applicable. The equations of motion and the solutions of general three-dimensional rigid framed structure with m nodes in total as shown in Fig. 2 will be presented in the following.

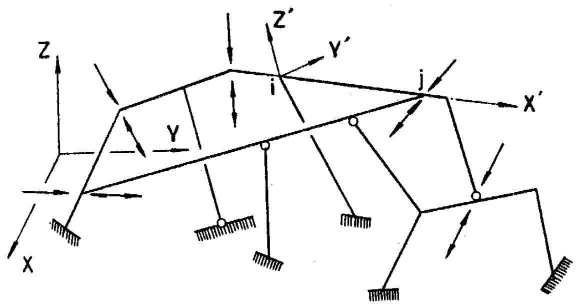

Fig. 2 Framed Structure Subjected to the Periodic and Non-periodic Loads.

Equations of motion with infinitesimal amplitudes- Let us consider the case where non-periodic load $\mathbf{P}_{0i}$ and periodic load $\bar{\mathbf{P}}_i \sin \omega t$ act on the *i-th* node of the structure, where ω is a circular frequency. We represent the displacements of the *i-th* node by a vector $\mathbf{d}_i$, especially, the displacements due to non-periodic and periodic loads by $\mathbf{d}_{0i}$ and $\bar{\mathbf{d}}_i$ respectively.

$$\mathbf{d}_i=\{u_i,\ v_i,\ w_i,\ \theta_{xi},\ \theta_{yi},\ \theta_{zi}\}^T \qquad (i=1,\ 2,\ldots,\ m) \tag{1}$$

Furthermore, we represent the coordinates of the *i-th* node by a vector $\mathbf{f}_i$ and the rotation angle of the principal axes of the cross section at the end i of the member ij by α_{ij}.

$$\mathbf{f}_i=\{x_i,\ y_i,\ z_i\}^T \qquad (i=1,\ 2,\ldots,\ m) \tag{2}$$

Now, let the local rectangular coordinates $(x',\ y',\ z')$, be established as shown in Fig. 2, taking the member axis ij as x'-axis and the principal axes of the cross section at the end i as y'- and z'-axes respectively. The above mentioned angle is a rotation angle required to make y'-axis parallel to the $x-y$ plane by revolving y'- and z'-axes about x'-axis.

Representing the end forces at the end i of the member ij referred to the local coordinates by a vector $\mathbf{N}'_{ij}$ and the transformation matrix by $\mathbf{T}_{ij}$, by which $\mathbf{N}'_{ij}$ is transformed into $\mathbf{N}_{ij}$ in reference to the gloval coordinates $(x,\ y,\ z)$, the equations of motion of the *i-th* node are represented in the following form since the transformation matrix $\mathbf{T}_{ij}$ is a function of $\mathbf{f}_i$, $\mathbf{f}_j$ and α_{ij}, and the end force vector $\mathbf{N}'_{ij}$ is a function of $\mathbf{f}_i$, $\mathbf{f}_j$, α_{ij}, $\mathbf{d}_{0i}+\bar{\mathbf{d}}_i$ and $\mathbf{d}_{0j}+\bar{\mathbf{d}}_j$:

$$\begin{aligned}\mathbf{W}_i\frac{d^2\bar{\mathbf{d}}_i}{dt^2}=&-\sum_{j=1}^{m}\mathbf{T}_{ij}(\mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij})\mathbf{N}'_{ij}(\mathbf{d}_{0i}+\bar{\mathbf{d}}_i,\ \mathbf{d}_{0j}+\bar{\mathbf{d}}_j,\ \mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij})\\&-\mathbf{P}_{0i}-\bar{\mathbf{P}}_i\sin\omega t\end{aligned} \tag{3}$$

where the matrix $\mathbf{W}_i$ is a mass matrix of the *i-th* node.

A set of equations of motion is obtained by applying Eq. (3) to each node of the framed structure, which is rewritten in the following matrix form:

$$\mathbf{W}\frac{d^2\bar{\mathbf{d}}}{dt^2}+\mathbf{K}_1(\mathbf{d}_0+\bar{\mathbf{d}})=-\mathbf{P}_0-\bar{\mathbf{P}}\sin\omega t \tag{4}$$

$$\mathbf{W}=\begin{Bmatrix}\mathbf{W}_1 & & & 0\\ & \mathbf{W}_2 & & \\ 0 & & \ddots & \\ & & & \mathbf{W}_m\end{Bmatrix},\quad \mathbf{d}_0=\begin{Bmatrix}\mathbf{d}_{01}\\ \mathbf{d}_{02}\\ \vdots\\ \mathbf{d}_{0m}\end{Bmatrix},\quad \bar{\mathbf{d}}=\begin{Bmatrix}\bar{\mathbf{d}}_1\\ \bar{\mathbf{d}}_2\\ \vdots\\ \bar{\mathbf{d}}_m\end{Bmatrix},$$

$$\mathbf{P}_0=\begin{Bmatrix}\mathbf{P}_{01}\\ \mathbf{P}_{02}\\ \vdots\\ \mathbf{P}_{0m}\end{Bmatrix},\quad \bar{\mathbf{P}}=\begin{Bmatrix}\bar{\mathbf{P}}_1\\ \bar{\mathbf{P}}_2\\ \vdots\\ \bar{\mathbf{P}}_m\end{Bmatrix}. \tag{5}$$

where $\mathbf{K}_1$ is the stiffness matrix of the structure.

Equations of motion with finite amplitudes- Let it be supposed that the magnitudes of the non-periodic loads and the amplitudes and frequencies of the periodic loads slightly change and take certain values adjacent to the boundary of the regions of dynamic instability. In this case, certain finite perturbed additional displacements will be produced. Denote these additional displacements of the *i-th* node by $\Delta\bar{\mathbf{d}}_i$. Then a equation of motion of the *i-th* node is expressed approximately as follows:

$$\mathbf{W}_i \frac{d^2}{dt^2}(\bar{\mathbf{d}}_i+\Delta\bar{\mathbf{d}}_i) = -\sum_{j=1}^{m}\mathbf{T}_{ij}(\mathbf{f}_i+\Delta\mathbf{f}_i,\ \mathbf{f}_j+\Delta\mathbf{f}_j,\ \alpha_{ij}+\Delta\alpha_{ij})\mathbf{N}'_{ij}(\mathbf{d}_{0i}+\bar{\mathbf{d}}_i$$

$$+\Delta\bar{\mathbf{d}}_i, \mathbf{d}_{0j}+\bar{\mathbf{d}}_j+\Delta\bar{\mathbf{d}}_j,\ \mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij}) - \mathbf{P}_{0i} - \bar{\mathbf{P}}_i \sin\omega t \tag{6}$$

Expand $\mathbf{T}_{ij}$ and $\mathbf{N}'_{ij}$ in series and neglect the terms higher than the second order of $\Delta\bar{\mathbf{d}}$, and put again $\mathbf{d}$ for $\Delta\bar{\mathbf{d}}$, then Eq. (6), using Eq. (3), becomes as follows:

$$\mathbf{W}_i \frac{d^2\mathbf{d}_i}{dt^2} = -\sum_{j=1}^{m}\mathbf{T}_{ij}(\mathbf{f}_i,\mathbf{f}_j,\ \alpha_{ij})\mathbf{N}'_{ij}(\mathbf{d}_i,\ \mathbf{d}_j,\ \mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij}) - \sum_{j=1}^{m}\{\mathbf{H}_{iij}(\mathbf{f}_i,\ \mathbf{f}_j,$$

$$\alpha_{ij},\ \mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij})\mathbf{d}_i + \mathbf{H}_{jij}(\mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij},\ \mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij})\mathbf{d}_j\} \tag{7}$$

where

$$\mathbf{H}_{iij}(\mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij},\ \mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}) = \left\{\frac{\partial\mathbf{T}_{ij}}{\partial x_i}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\ \frac{\partial\mathbf{T}_{ij}}{\partial y_i}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\right.$$

$$\frac{\partial\mathbf{T}_{ij}}{\partial z_i}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\ \lambda_{ij}\frac{\partial\mathbf{T}_{ij}}{\partial\alpha_{ij}}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\ \mu_{ij}\frac{\partial\mathbf{T}_{ij}}{\partial\alpha_{ij}}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),$$

$$\left.\nu_{ij}\frac{\partial\mathbf{T}_{ij}}{\partial\alpha_{ij}}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij})\right\} \tag{8}$$

$$\mathbf{H}_{jij}(\mathbf{f}_i,\ \mathbf{f}_j,\ \alpha_{ij},\ \mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}) = \left\{\frac{\partial\mathbf{T}_{ij}}{\partial x_j}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\ \frac{\partial\mathbf{T}_{ij}}{\partial y_j}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}),\right.$$

$$\left.\frac{\partial\mathbf{T}_{ij}}{\partial z_j}(\mathbf{N}'_{0ij}+\bar{\mathbf{N}}'_{ij}), \mathbf{O},\ \mathbf{O},\ \mathbf{O}\right\} \tag{9}$$

in which λ_{ij}, μ_{ij} and ν_{ij} are direction cosines of x'-axis of member ij and $\mathbf{N}'_{0ij}$ and $\bar{\mathbf{N}}'_{ij}$ are respectively the end force vectors at end i of the member ij due to non-periodic load $\mathbf{P}_0$ and periodic load $\bar{\mathbf{P}}$ $\sin\omega t$.

Let the loads acting on the *i-th* node in non-periodic and periodic fundamental loadings be denoted by $\tilde{\mathbf{P}}_{0i}$ and $\tilde{\mathbf{P}}_i \sin\omega t$ respectively, then $\mathbf{P}_{0i}$ and $\bar{\mathbf{P}}_i$ satisfy the following relations respectively:

$$\mathbf{P}_{0i} = P_0\tilde{\mathbf{P}}_{0i}, \quad \bar{\mathbf{P}}_i = \bar{P}\tilde{\mathbf{P}}_i \tag{10}$$

where P_0 and $\bar{P}$ denote the ratios of actual and fundamental loads for non-periodic and periodic loads respectively, that is, the magnitudes of actual loads $\mathbf{P}_{0i}$ and $\bar{\mathbf{P}}_i$ are respectively P_0 and $\bar{P}$ times as large as those of the respective fundamental loads.

Therefore, when the end force vector of end i of the member ij under non-periodic fundamental loading is denoted by $\tilde{\mathbf{N}}'_{0ij}$ in reference to the local coordinates, $\mathbf{N}'_{0ij}$ is expresses in the following form:

$$\mathbf{N}'_{0ij}=P_0\tilde{\mathbf{N}}'_{0ij} \tag{11}$$

Similarly, when the amplitude of the end force vector of the same member end under periodic fundamental loading is denoted by $\tilde{\mathbf{N}}'_{ij}$, $\bar{\mathbf{N}}'_{ij}$ is expressed by $\tilde{\mathbf{N}}'_{ij}$:

$$\bar{\mathbf{N}}'_{ij}=\bar{P}\tilde{\mathbf{N}}'_{ij}\sin\omega t \tag{12}$$

Since $\tilde{\mathbf{N}}'_{ii}$ is the amplitude of end force vector due to fundamental loads $\tilde{\mathbf{P}}_i\sin\omega t$ ($i=1, 2,\ldots, m$), it is a function of ω. But $\tilde{\mathbf{N}}'_{ij}$ may be considered to be the same as the end force vector due to nonperiodic loads $\tilde{\mathbf{P}}_i$ ($i=1, 2,\ldots, m$) if ω is sufficiently apart form the natural frequencies mentioned above. Hence, Eq. (7) becomes,

$$\mathbf{W}_i\frac{d^2\mathbf{d}_i}{dt^2}=-\sum_{j=1}^{m}\mathbf{T}_{ij}(\mathbf{f}_i, \mathbf{f}_j, \alpha_{ij})\mathbf{N}'_{ij}(\mathbf{d}_i, \mathbf{d}_j, \mathbf{f}_i, \mathbf{f}_j, \alpha_{ij})-P_0\sum_{j=1}^{m}\{\mathbf{H}_{iij}(\mathbf{f}_i, \mathbf{f}_j, \alpha_{ij}, \tilde{\mathbf{N}}'_{0ij})\mathbf{d}_i+\mathbf{H}_{jij}(\mathbf{f}_i, \mathbf{f}_j, \alpha_{ij}, \tilde{\mathbf{N}}'_{0ij})\mathbf{d}_j\}-\bar{P}\sin\omega t\sum_{j=1}^{m}\{\mathbf{H}_{iij}(\mathbf{f}_i, \mathbf{f}_j, \alpha_{ij}, \tilde{\mathbf{N}}'_{ij})\mathbf{d}_i+\mathbf{H}_{jij}(\mathbf{f}_i, \mathbf{f}_j, \alpha_{ij}, \tilde{\mathbf{N}}'_{ij})\mathbf{d}_j\} \tag{13}$$

This is the equation of motion with finite amplitudes of the *i-th* node.

A set of equations is obtained by applying Eq. (13) to each node of the frame, which is indicated in the matrix form, thus:

$$\mathbf{W}\frac{d^2\mathbf{d}}{dt^2}+P_0\mathbf{K}_3\mathbf{d}+\bar{P}\sin\omega t\,\mathbf{K}_2\mathbf{d}+\mathbf{K}_1\mathbf{d}=\mathbf{O} \tag{14}$$

where $\mathbf{W}$ is the mass matrix which is the same as Eq. (5), $\mathbf{K}_1$ is the stiffness matrix of the structure, $\mathbf{K}_2$ and $\mathbf{K}_3$ are initial stress matrices due to non-periodic loads $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}_0$ respectively, and

$$\mathbf{d}=\begin{Bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ \vdots\\ \mathbf{d}_m\end{Bmatrix} \tag{15}$$

Boundary Frequencies and Critical Loads

Eq. (14) is the Mathieu-Hill equation. It is well-known that the periodic solutions of this equation with the period equal to or two times that of the external load bound the regions of stability and instability; more exactly, two solutions with identical periods bound the regions of instability and the two solutions with different periods bound the regions of stability [1].

Express the latter periodic solution in the following form [1]:

$$\mathbf{d}=\boldsymbol{\alpha}\sin\tfrac{1}{2}\omega t+\boldsymbol{\beta}\cos\tfrac{1}{2}\omega t \tag{16}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are independent of time, then the boundary frequencies of the principal region of dynamic instability are approximately calculated by the following equation.

$$|\mathbf{K}_1 \pm \tfrac{1}{2}\bar{P}\mathbf{K}_2 + P_0\mathbf{K}_3 - \tfrac{1}{4}\omega^2\mathbf{W}| = 0 \tag{17}$$

Similarly, express the former periodic solution in the following form [1]:

$$\mathbf{d} = \tfrac{1}{2}\boldsymbol{\beta}_0 + \boldsymbol{\alpha}\sin\omega t + \boldsymbol{\beta}\cos\omega t \tag{18}$$

where $\boldsymbol{\beta}_0$ is independent of time, and we have the approximate boundary frequencies for the second region of dynamic instability:

$$|\mathbf{K}_1 + P_0\mathbf{K}_3 - \omega^2\mathbf{W}| = 0 \tag{19}$$

$$\begin{vmatrix} \mathbf{K}_1 + P_0\mathbf{K}_3 & \bar{P}\mathbf{K}_2 \\ \tfrac{1}{2}\bar{P}\mathbf{K}_2 & \mathbf{K}_1 + P_0\mathbf{K}_3 - \omega^2\mathbf{W} \end{vmatrix} = 0 \tag{20}$$

Boundary Frequencies and Critical Loads when the Effect of Damping is Considered

Consider the damping force proportional to the derivative of the displacement with respect to time in the vibration of structure, then Eq. (14) is transformed into the form below:

$$\mathbf{W}\frac{d^2\mathbf{d}}{dt^2} + \mathbf{D}\frac{d\mathbf{d}}{dt} + P_0\mathbf{K}_3\mathbf{d} + \bar{P}\sin\omega t\mathbf{K}_2\mathbf{d} + \mathbf{K}_1\mathbf{d} = 0 \tag{21}$$

Since the damping matrix **D** cannot be easily obtained in general, let it be assumed that the damping matrix **D** of the structure during the vibration with infinitesimal amplitudes can be applied to **D** and that it can be diagonalized by modal matrix.

When the damping constants for each mode of free vibration are known, the damping matrix can be evaluated by the modal analysis as follows:

Denote the first, second,..., and *m-th* natural circular frequencies of the system, which satisfy the equation of motion of undamped free vibration, by ω_1, ω_2,... and ω_m and the normalized modal vectors corresponding to the respective natural circular frequencies by Φ_1, Φ_2,... and Φ_m, then the damping matrix **D** satisfies the following relation according to the previous assumption:

$$[\Phi]^T\mathbf{D}[\Phi] = \begin{pmatrix} 2h_1\omega_1 & & 0 \\ & 2h_2\omega_2 & \\ & & \ddots \\ 0 & & 2h_m\omega_m \end{pmatrix} = [2h\omega] \tag{22}$$

where $[\Phi] = [\Phi_1, \Phi_2, \ldots, \Phi_m]$ and h_1, h_2,... and h_m are the damping constants corresponding to the respective modes of free vibration. Therefore, **D** is expressed utilizing the damping constants, thus

$$\mathbf{D} = \mathbf{W}[\Phi][2h\omega][\Phi]^T\mathbf{W} \tag{23}$$

Let us now express the periodic solution of Eq. (21) using the relation of Eq. (16) to calculate the boundary frequencies of the principal region of dynamic instability approximately by the following equation of boundary frequency:

$$\begin{vmatrix} \mathbf{K}_1+P_0\mathbf{K}_3-\frac{1}{4}\omega^2\mathbf{W} & \frac{1}{2}\bar{P}\mathbf{K}_2-\frac{1}{2}\omega\mathbf{D} \\ \frac{1}{2}\bar{P}\mathbf{K}_2+\frac{1}{2}\omega\mathbf{D} & \mathbf{K}_1+P_0\mathbf{K}_3-\frac{1}{4}\omega^2\mathbf{W} \end{vmatrix}=0 \qquad (24)$$

Similarly, expressing the periodic solution of Eq. (21) by Eq. (18), we have the approximate boundary frequencies of the second region of dynamic instability:

$$\begin{vmatrix} \mathbf{K}_1+P_0\mathbf{K}_3 & \bar{P}\mathbf{K}_2 & \mathbf{0} \\ \frac{1}{2}\bar{P}\mathbf{K}_2 & \mathbf{K}_1+P_0\mathbf{K}_3-\omega^2\mathbf{W} & -\omega\mathbf{D} \\ \mathbf{0} & \omega\mathbf{D} & \mathbf{K}_1+P_0\mathbf{K}_3-\omega^2\mathbf{W} \end{vmatrix}=0 \qquad (25)$$

Applications

Columns- The regions of dynamic instability about the principal axis y of the cross section for columns made of aluminium having the cross section shown in Fig. 3 (a) are estimated using the proposed method. Three cases, shown in Fig. 4, are investigated. Especially, when the effect of damping is considered, the damping constant h for each mode of free vibration in unloading condition is 0.05 and the regions of dynamic instability is estimated.

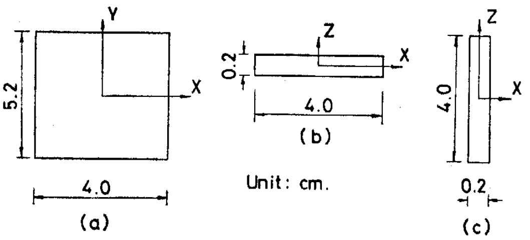

Fig. 3 Shapes of Cross Sections.

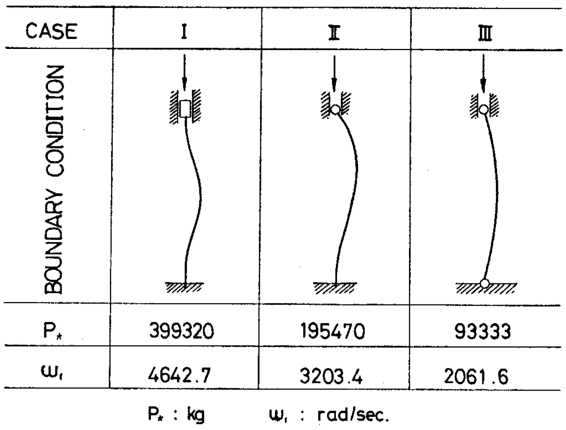

Fig. 4 Columns Subjected to the Periodic and Non-periodic Loads.

In the analysis these columns are substituted by the lumped mass systems, as shown in Fig. 5 (a), consisting of seven nodes. And it is assumed that the loading condition of a downward unit load acting on the top is regarded as the non-periodic fundamental loading, and the loading condition of a unit periodic load $1\sin\omega t$, acting in the same manner, as the periodic fundamental loading. By P_*, specify the value of P_0 at the static critical state, and by ω_1 the first natural circular frequency. These are shown in Fig. 4.

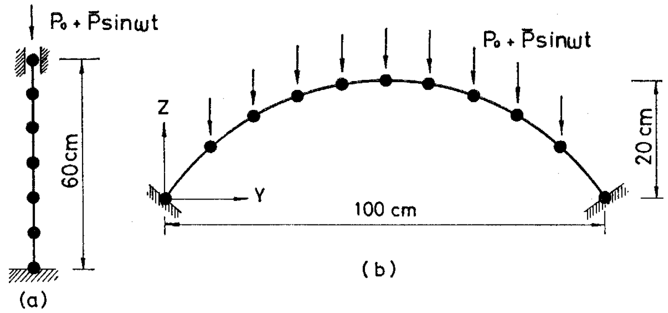

Fig. 5 Lumped Mass Systems.

The boundary values of the regions of dynamic instability are founded by the following procedure: When the effect of damping is neglected, a value of P_0 (the magnitude of critical load) is evaluated after $\bar{P}$ and ω are given. When the effect of damping is considered, the damping matrix **D** is first calculated at the unloading state and a value of P_0 is evaluated against the given values of $\bar{P}$ and ω. Subsequently, those results are represented by the values of $\omega/2\Omega$, together with the excitation parameter μ, where Ω is the first natural circular frequency of the structure subjected to the non-periodic loads and $\mu=\bar{P}/2(P_*-P_0)$.

The calculated results are partially shown in Tables 1, 2 and 3. The values in

Table 1. Boundary Values of the Regions of Dynamic Instability of a Column for Case I.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	0.725	1.334 (1.313)	0.364	0.810 (0.798)
	0.474	1.233 (1.214)	0.296	0.852 (0.839)
	0.340	1.175 (1.158)	0.203	0.907 (0.893)
	0.259	1.138 (1.122)	0.171	0.924 (0.911)
	0.117	1.064 (1.057)	0.095	0.964 (0.952)
Second	0.322	0.508 (0.500)	0.539	0.329 (0.323)
	0.142	0.507 (0.500)	0.191	0.489 (0.482)
	0.080	0.506 (0.500)	0.102	0.501 (0.495)
	0.042	0.504 (0.500)	0.051	0.503 (0.499)
	0.026	0.501 (0.500)	0.026	0.500 (0.500)
Case of considering the effect of damping				
Principal	1.164	1.114 (1.092)	0.686	0.649 (0.641)
	0.802	1.203 (1.180)	0.628	0.710 (0.701)
	0.586	1.114 (1.092)	0.525	0.811 (0.801)
	—	—	0.482	0.855 (0.845)
	—	—	0.445	0.896 (0.888)
Second	2.076	0.484 (0.475)	0.699	0.094 (0.092)
	1.056	0.461 (0.452)	0.676	0.184 (0.182)
	—	—	0.643	0.269 (0.267)

Table 2. Boundary Values of the Regions of Dynamic Instability of a Column for Case II.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	3.897	2.241 (2.213)	0.443	0.756 (0.746)
	0.682	1.312 (1.297)	0.288	0.853 (0.844)
	0.384	1.163 (1.151)	0.197	0.906 (0.896)
	0.159	1.085 (1.076)	0.140	0.935 (0.927)
	0.112	1.061 (1.054)	0.091	0.959 (0.953)
Second	0.310	0.506 (0.500)	0.266	0.468 (0.463)
	0.198	0.505 (0.500)	0.184	0.487 (0.483)
	0.100	0.503 (0.500)	0.098	0.498 (0.495)
	0.040	0.499 (0.500)	0.040	0.499 (0.499)
	0.024	0.496 (0.500)	0.024	0.496 (0.500)
Case of considering the effect of damping				
Principal	0.975	1.344 (1.328)	0.473	0.781 (0.772)
	0.563	1.192 (1.178)	0.385	0.845 (0.836)
	0.375	1.112 (1.099)	0.317	0.894 (0.885)
	0.268	1.057 (1.046)	0.264	0.933 (0.924)
	—	—	0.225	0.969 (0.960)
Second	1.135	0.484 (0.477)	0.666	0.185 (0.181)
	—	—	0.568	0.341 (0.338)

Table 3. Boundary Values of the Regions of Dynamic Instability of a Column for Case III.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	0.869	1.363 (1.367)	0.401	0.772 (0.774)
	0.519	1.229 (1.232)	0.317	0.824 (0.826)
	0.356	1.160 (1.164)	0.207	0.888 (0.890)
	0.201	1.093 (1.096)	0.143	0.923 (0.926)
	0.132	1.061 (1.064)	0.104	0.944 (0.946)
Second	0.465	0.499 (0.500)	0.650	0.196 (0.197)
	0.262	0.499 (0.500)	0.507	0.347 (0.348)
	0.167	0.499 (0.500)	0.351	0.433 (0.434)
	0.116	0.499 (0.500)	0.159	0.486 (0.487)
	0.065	0.499 (0.500)	0.084	0.495 (0.496)
Case of considering the effect of damping				
Principal	0.789	1.300 (1.303)	0.662	0.595 (0.597)
	0.478	1.180 (1.183)	0.414	0.784 (0.787)
	0.329	1.118 (1.121)	0.330	0.840 (0.842)
	0.187	1.054 (1.057)	0.217	0.909 (0.912)
	0.121	1.018 (1.021)	0.112	0.978 (0.981)
Second	0.994	0.486 (0.487)	0.654	0.197 (0.198)
	—	—	0.525	0.353 (0.355)

parentheses in the column of $\omega/2\Omega$ are the boundary frequencies of a simply supported column, obtained by Eqs. (26), (27), (28) and (29), approximately derived in accordance with the theory of continuum mechanics [1].

Case of neglecting the effect of damping:

Principal region of dynamic instability-

$$\omega/2\Omega=\sqrt{(1\pm\mu)} \tag{26}$$

Second region of dynamic instability-

$$\omega/2\Omega=\tfrac{1}{2},\ \omega/2\Omega=\tfrac{1}{2}\sqrt{(1-2\mu^2)} \tag{27}$$

Case of considering the effect of damping:

Principal region of dynamic instability-

$$\omega/2\pi=\sqrt{\{1-\tfrac{1}{2}\Delta^2\pm\sqrt{(\mu^2-\Delta^2+\tfrac{1}{4}\Delta^4)}\}} \tag{28}$$

Second region of dynamic instability-

$$\omega/2\Omega=\tfrac{1}{2}\sqrt{\{1-\mu^2\pm\sqrt{\mu^4-\Delta^2(1-\mu^2)}\}} \tag{29}$$

where

$$\Delta=2\varepsilon/\{\omega_1\sqrt{(1-P_0/P_*)}\},\qquad \varepsilon=h\omega_1.$$

From the tables, it is recognized that results calculated by the proposed method agree well with the boundary frequencies analyzed by the theory of continuum mechanics [5].

Arches- The regions of dynamic instability of the parabolic arches made of aluminium are analyzed. The arches have the same axis as shown in Fig. 5 (b) and their cross sections are shown in Figs. 3 (b) and (c). Similarly, in the case of columns, when the effect of damping is considered, the damping constant is assumed to be 0.05.

In the analysis the arches are divided into ten segments and their masses are concentrated on each node. And it is assumed that the condition of unit loads acting downwards on each node is the non-periodic fundamental loading, and condition of unit periodic loads with the magnitude of $1\sin\omega t$, acting in the sane manner, is the periodic fundamental loading.

(*Arches deforming in their planes*) The arches with cross section shown in Fig. 3 (b) are investigated for the three kinds of boundary conditions as shown in Fig. 6.

The values of P_* at the static critical state and the first natural circular frequencies ω_1 of the arches deforming in their planes are shown in Fig. 6.

The procedure of calculation for boundary values of dynamic instability is similar to that of columns. The results are partially shown in Tables 4, 5 and 6. The values in parentheses in the column of $\omega/2\Omega$ are the boundary frequencies of simply supported column, evaluated by the approximate Eqs. (26), (27), (28) and (29).

It will be clear from the tables that the boundary frequencies of the regions of dynamic instability of the arches illustrated in Fig. 5 (b) and 6 can be evaluated utilizing the approximate Eqs. (26), (27), (28) and (29) for simply supported column, if we permit the difference of about 1%.

(*An arch deforming out of its plane*) An arch with the cross section shown in Fig. 3 (c), which is clamped at both the ends, is observed. A value of P_* and the first natural circular frequency ω_1 of an arch deforming out of its plane are

CASE	I	II	III
BOUNDARY CONDITION			
$\bar{P}$	1.0	1.0	0.2
P_*	.2.0999	1.3406	0.9049
ω_1	135.53	107.36	84.13

$\bar{P}$ and P_* : kg ω_1 : rad/sec.

Fig. 6 Arches Subjected to the Periodic and Non-periodic Loads.

Table 4. Boundary Values of the Regions of Dynamic Instability of an Arch Deforming in its Plane for Case I.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	0.632	1.284 (1.278)	0.484	0.727 (0.719)
	0.526	1.240 (1.235)	0.365	0.803 (0.797)
	0.447	1.206 (1.203)	0.305	0.839 (0.834)
	0.337	1.157 (1.156)	0.236	0.877 (0.874)
	0.264	1.123 (1.124)	0.172	0.909 (0.909)
Second	0.573	0.503 (0.500)	0.580	0.289 (0.286)
	0.438	0.502 (0.500)	0.457	0.385 (0.381)
	0.345	0.502 (0.500)	0.338	0.442 (0.439)
	0.278	0.501 (0.500)	0.245	0.470 (0.469)
	0.191	0.499 (0.500)	0.179	0.482 (0.484)
Case of considering the effect of damping				
Principal	0.610	1.262 (1.255)	0.588	0.656 (0.649)
	0.509	1.219 (1.215)	0.488	0.730 (0.723)
	0.433	1.187 (1.184)	0.370	0.808 (0.802)
	0.327	1.140 (1.140)	0.261	0.873 (0.869)
	0.257	1.107 (1.109)	0.176	0.917 (0.918)
Second	1.111	0.501 (0.495)	0.638	0.226 (0.219)
	0.762	0.498 (0.493)	0.585	0.291 (0.286)
	0.412	0.487 (0.485)	0.357	0.454 (0.450)

Table 5. Boundary Values of the Regions of Dynamic Instability of an Arch Deforming in its Plane for Case II.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\mu/2\Omega$
Case of neglecting the effect of damping				
Principal	0.745	1.323 (1.321)	0.525	0.694 (0.689)
	0.606	1.267 (1.268)	0.478	0.729 (0.723)
	0.505	1.226 (1.227)	0.394	0.782 (0.783)
	0.430	1.193 (1.196)	0.299	0.839 (0.837)
	0.324	1.145 (1.150)	0.197	0.892 (0.896)
Second	0.426	0.501 (0.500)	0.631	0.228 (0.225)
	0.335	0.499 (0.500)	0.517	0.344 (0.342)
	0.270	0.498 (0.500)	0.453	0.387 (0.384)
	0.222	0.497 (0.500)	0.333	0.442 (0.441)
	0.186	0.496 (0.500)	0.204	0.476 (0.479)
Case of considering the effect of damping				
Principal	1.789	1.666 (1.659)	0.480	0.730 (0.726)
	1.228	1.487 (1.483)	0.437	0.760 (0.755)
	0.921	1.379 (1.377)	0.397	0.785 (0.782)
	0.725	1.306 (1.305)	0.362	0.807 (0.804)
	—	—	0.302	0.842 (0.841)
Second	1.091	0.501 (0.497)	0.581	0.292 (0.289)
	0.545	0.495 (0.494)	0.458	0.389 (0.387)
	0.315	0.484 (0.486)	0.299	0.472 (0.469)

Table 6. Boundary Values of the Regions of Dynamic Instability of an Arch Deforming in its Plane for Case III.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	0.528	1.239 (1.236)	0.332	0.819 (0.817)
	0.415	1.191 (1.190)	0.257	0.864 (0.862)
	0.339	1.159 (1.157)	0.202	0.894 (0.893)
	0.240	1.115 (1.114)	0.162	0.916 (0.915)
	0.159	1.077 (1.077)	0.133	0.932 (0.931)
Second	0.496	0.500 (0.500)	0.632	0.226 (0.223)
	0.345	0.501 (0.500)	0.456	0.384 (0.382)
	0.253	0.500 (0.500)	0.289	0.458 (0.457)
	0.194	0.500 (0.500)	0.277	0.474 (0.473)
	0.124	0.500 (0.500)	0.147	0.490 (0.489)
Case of considering the effect of damping				
Principal	0.675	1.284 (1.282)	0.509	0.710 (0.707)
	0.513	1.221 (1.219)	0.384	0.793 (0.792)
	0.404	1.174 (1.174)	0.295	0.849 (0.848)
	0.331	1.144 (1.143)	0.205	0.901 (0.900)
	0.202	1.087 (1.086)	0.149	0.932 (0.932)
Second	1.374	0.500 (0.497)	0.690	0.118 (0.110)
	0.759	0.495 (0.494)	0.635	0.227 (0.223)
	0.476	0.490 (0.490)	0.556	0.318 (0.315)

$$P_* = 0.8196\ kg, \quad \omega_1 = 50.265\ rad/sec.$$

The procedure of calculation of the boundary values is similar to that of the column, and the results are shown in Table 7. The values in parentheses in the column of $\omega/2\Omega$ indicate the boundary frequencies of a simply supported column, calculated form the approximate Eqs. (26), (27), (28) and (29).

Table 7. Boundary Values of the Regions of Dynamic Instability of an Arch Deforming out of its plane.

Region	Upper bound		Lower bound	
	μ	$\omega/2\Omega$	μ	$\omega/2\Omega$
Case of neglecting the effect of damping				
Principal	0.460	1.214 (1.208)	0.469	0.735 (0.729)
	0.279	1.133 (1.131)	0.331	0.823 (0.818)
	0.189	1.090 (1.091)	0.240	0.876 (0.872)
	0.138	1.063 (1.067)	0.137	0.929 (0.929)
	0.083	1.031 (1.041)	0.071	0.955 (0.964)
Second	1.992	0.505 (0.500)	0.593	0.276 (0.273)
	0.495	0.503 (0.500)	0.364	0.432 (0.429)
	0.218	0.501 (0.500)	0.200	0.481 (0.479)
	0.059	0.491 (0.500)	0.121	0.498 (0.493)
	0.076	0.493 (0.500)	0.077	0.495 (0.497)
Case of considering the effect of damping				
Principal	0.433	1.177 (1.172)	0.479	0.743 (0.737)
	0.263	1.101 (1.099)	0.341	0.835 (0.830)
	0.181	1.065 (1.062)	0.248	0.891 (0.888)
	0.130	1.031 (1.036)	0.144	0.949 (0.948)
	0.098	1.008 (1.018)	0.093	0.980 (0.982)
Second	1.949	0.500 (0.495)	0.600	0.277 (0.274)
	0.436	0.473 (0.469)	0.408	0.457 (0.449)

Thus, it can again be said that we are able to evaluate the boundary frequencies of a fixed arch with loading condition illustrated in Fig. 5 (b) using the approximate equations.

Conclusion

The equations of motion with finite amplitudes are obtained using the linearized finite displacement method and the analysis of dynamic elastic stability of general three-dimensional framed structures subjected to the periodic loads is presented.

In the two cases of regarding and disregarding the effect of damping, the boundary values calculated by the proposed method show good agreement with those obtained by the theory of continuum mechanics for the dynamic instability of a simply supported column. Therefore, it can be said that the proposed method is useful to estimate the dynamic instability of the frames. In addition, when the regions of dynamic instability of columns except a simply supported column and arches chosen in applications are expressed with respect to $\omega/2\Omega-\mu$ plane, With this in the case where the non-periodic fundamental loading is equal to the periodic ones, it may be said that the boundary frequencies of the regions of instability of columns and arches chosen in applications can be obtained from the approximate Eqs. (26), (27), (28) and (29) irrespective of column and arch and of difference in boundary conditions. However, if these loadings are not equal, these approximate equations are not available.

Although the applications for columns and arches only are shown, the proposed method can equally be applied to various kinds of framed structures. Accordingly, their dynamic elastic instability accompanying the simple parametric resonance can be investigated in the same way.

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Numerical calculations were carried out by FACOM 230–60 and 75 installed in the Computer Center of the Kyushu University.

Appendix I. References

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Appendix II. Notations

The following symbols are used in this paper:

$\mathbf{D}$ =damping matrix;

$\mathbf{d}$ =displacement vector;

$\bar{\mathbf{d}}, \mathbf{d}_0$ =displacement vectors due to the periodic and non-periodic loads, respectively;

$\mathbf{d}_i$	=displacement vector of *i-th* node;
$\bar{\mathbf{d}}_i$, $\mathbf{d}_{0i}$	=displacement vectors of *i-th* node due to periodic and non-periodic loads, respectively;
$\mathbf{f}_i$	=coordinates of *i-th* node which are expressed in vector form;
h_k	=damping constant for *k-th* natural free vibration;
i	=number specifying the node considered;
j	=number specifying the node adjacent to the node cosidered;
$\mathbf{K}_1$	=stiffness matrix of framed structure;
$\mathbf{K}_2$, $\mathbf{K}_3$	=intial stress matrices under non-periodic fundamental loads $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}_0$, respectively;
m	=total number of nodes of the structure;
$\mathbf{N}_{ij}$, $\mathbf{N}'_{ij}$	=end force vectors of end i of member ij refered to the global and local coordinate axes, respectively;
$\bar{\mathbf{N}}'_{ij}$, $\mathbf{N}'_{0ij}$	=end force vectors of end i of member ij under the periodic and non-periodic loads, respectively, referring to the local coordinate axes;
$\tilde{\mathbf{N}}'_{ij}$, $\tilde{\mathbf{N}}'_{0ij}$	=end force vectors of end i of member ij under the periodic and non-periodic fundamental loads, respectively, referring to the local coordinate axes;
$\bar{\mathbf{P}}$	=amplitude of vector of periodic load acting on the framed structure;
$\bar{\mathbf{P}}_i$, $\tilde{\mathbf{P}}_i$	=amplitudes of vectors of the periodic actual and fundamental loads acting on *i-th* node, respectively;
$\mathbf{P}_0$	=vector of the non-periodic load;
$\mathbf{P}_{0i}$, $\tilde{\mathbf{P}}_{0i}$	=vectors of the non-periodic actual and fundamental loads acting on *i-th* node, respectively;
$\bar{P}$, P_0	=ratios of the actual load to the fundamental load under the periodic and non-periodic loads, respectively;
P_*	=P_0 value of static critical state;
$\mathbf{T}_{ij}$	=transformation matrix;
u_i, v_i, w_i	=displacements of *i-th* node in the directions of x-, y- and z-axes, respectively;
$\mathbf{W}$	=mass matrix of framed structure;
$\mathbf{W}_i$	=mass matrix of *i-th* node;
x, y, z	=global coordinate axes;
x_i, y_i, z_i	=global coordinates of *i-th* node;
x', y', z'	=local coordinate axes of member ij;
α_{ij}	=rotation angle of principal axes of cross section of end i about member axis ij;
$\Delta\bar{\mathbf{d}}_i$	=increment of displacement vector of *i-th* node due to periodic load, i. e., additional displacement vector;
$\Delta\mathbf{f}_i$	=increment of coordinates of *i-th* node by deformation;
$\Delta\alpha_{ij}$	=increment of rotation angle α_{ij} by deformation of framed structure;
θ_{xi}, θ_{yi}, θ_{zi}	=rotation angles of *i-th* node about the axes parallel to the global coordinate axes, respectively;
λ_{ij}, μ_{ij}, ν_{ij}	=direction cosines of member ij;
μ	=excitation parameter;
Φ_k	=*k-th* normalized modal vector;
$[\Phi]$	=modal matrix;
ω	=circular frequency of the periodic load;
ω_k	=*k-th* natural circular frequency; and
Ω	=first natural circular frequency of structure subjected to the non-periodic load.