The Drop-size Distributions in Well-developed Convective Rainclouds: Part 3

On the normal distributions of liquid water content of raindrops with size

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Abstract

The new equation of raindrop-size distribution will be proposed in this paper. It may be represented by the normal distribution of liquid water content per unit volume of air with size. Almost of the conventional distribution (e.g. Marshall — Palmer, and Best equation) and the observed and computed stationary distribution (reported in Part 1 and Part 2 of this paper) are to obey to this rule. Some physical considerations on such a rule will be made and the characteristics of the derived rain parameters will be described for radar meteorology.

1. Introduction

The history of raincloud physics has two representative equations for the drop-size distribution. One was contributed by Marshall and Palmer[4] who dealt directly with the number density of drops in a size interval. The other was given by Best[2] who modified the unsuitability of those exponential type disrributions, noting some characteristics of distribution of liquid water comprised by drops in some size range.

Independently the writer had some interests in the distribution of liquid water content with drop size in the works of Part 1[5] and Part 2[6] of this report. As having been described in Fig. 6 of Part 1, the observed stationary distribution at Histoyoshi is well characterized by the shift of the concentrated region with size in the distribution of liquid water content from the *M-P* distributions; consequently the values of D_0 (median volume diameter) increasing. The ideal lines in Fig. 6 of Part 1 were introduced under the assumption that a given total liquid water content to a unit volume of air would be allotted equally to each group of drops in a equally divided size interval. Here again the family of the ideal distributionis illustrated in Fig. 1. That is to say, the ideal distribution could be expressed as

$$N_D = AD^{-3} \ (A\text{: constant}) \qquad \cdots\cdots\cdots(1)$$

because the derivative of liquid water content to drop size, derived from Eq. (1), is constant, as following equations

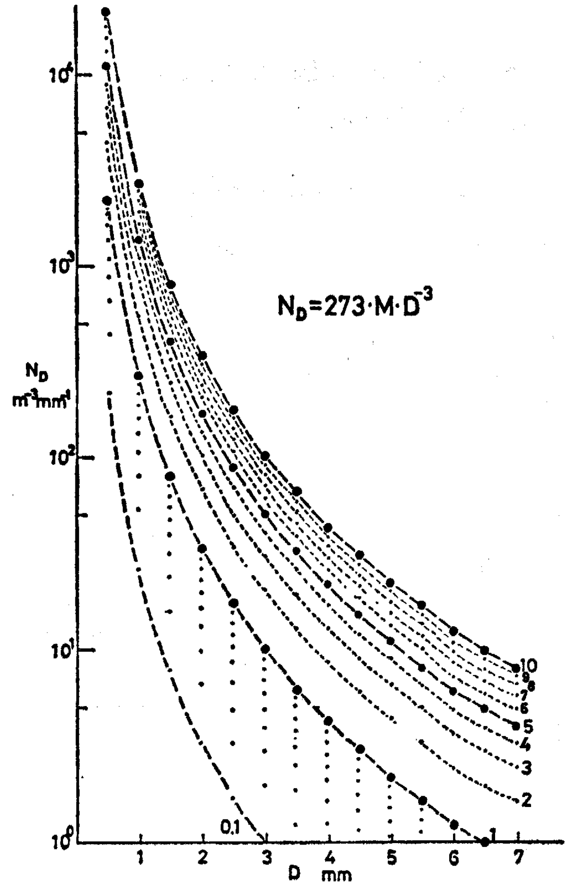

Fig. 1 The family of the ideal distribution ($N_D=AD^{-3}$) with liquid water content M (g/m³).

$$\frac{\partial}{\partial D}(L.W.C.)=\frac{\partial}{\partial D}\int_0^\infty \frac{\pi}{6}\cdot\rho\cdot N_D\cdot D^3\,dD$$

$$=\frac{\partial}{\partial D}\int_0^\infty A\cdot D^{-3}\cdot\frac{\pi}{6}\cdot\rho\cdot D^3\,dD=\frac{\partial}{\partial D}(K\cdot D_{max})=0 \quad \text{.........(2)}$$

where $K=\frac{\pi}{6}\cdot\rho\cdot A=\text{const}$, ρ: density of water, D max: allowed constant maximum diameter in Nature for ∞. Proportional coefficient A in Eq. (1) will be determined by giving the value of liquid water content and the maximum diameter.

$$L.W.C.=A\cdot\frac{\pi}{6}\cdot\rho\cdot D_{max}$$

$$\therefore A=\frac{L.W.C.}{D_{max}}\cdot\frac{6}{\rho\cdot\pi} \quad \text{.........(3)}$$

If for the maximum diameter is adopted the observed value of 7 mm at Hitoyoshi, and units of $L.W.C.$ and D max are given by g/m³ and mm, respectively,

$$A=2.73\times10^2\times\text{L.W.C.} \qquad (\text{mm}^{-1}) \quad \text{.........(4)}$$

Therefore the actual distribution of which total liquid water content is M can be evaluated by seeing the distribution of the ratio $m(D)$ of the actual number density N_D to the ideal, as

$$m(D)=(N_D)\text{ actual}/(2.73\times10^2\times M\cdot D^{-3}) \qquad \cdots\cdots\cdots(5)$$

Thus we can see the distribution of the liquid water with size not directly but relatively to the ideal distribution. From Eq. (5), $m(D)dD$ represents the so-called probability density of liquid water of drops in size interval between D and $D+dD$. And then the total liquid water content and the mean value of diameter can be calculated by the following equations.

$$M=\int_0^{D\max} M\cdot m(D)\,dD \qquad \cdots\cdots\cdots(6)$$

$$\bar{D}=\int_0^{D\max} m(D)\cdot D\,dD \qquad \cdots\cdots\cdots(7)$$

The purpose of this paper is to show that $m(D)$ might be obeyed to distribute in the form of the normal distribution for almost of the various types of drop-size distribution, and then the calculated rain parameters are in good accorcance with the observed. Therefore, $\bar{D}$, the calculated mean value of diameter by Eq. (7), approximates well to the median volume diameter, D_0. Derived various relationship of rain parameters will be discussed, and a convenient parameter diagram will be introduced for radar meteorology.

2. Examples of normalization of the distribution of liquid water content with drop-size

In calculating Eqs. (6) and (7), $\Delta D=0.5$ mm and D max$=7$ mm were adopted, and $m(D)$ was determined by dividing Eq. (5) $\frac{N_D}{A\cdot M\cdot D^{-3}}$ into $\frac{N_D}{AD^{-3}}$ and $\frac{1}{M}$. Where $\frac{N_D}{AD^{-3}}=M'(D)$, which is the ratio of N_D to the ideal distribution for $M=1$. By use of Fig.1, $M'(D)$ may be simply and directly obtained by reading the value of the point in the families of $A\cdot M\cdot D^{-3}$ curve, corresponding to the value of N_D. And M may be given by calculating $\sum_{D=0.25}^{D=6.75} M'(D)\Delta D/\sum_{D=0.25}^{D=6.75}\Delta D$. $\frac{M'(D)\Delta D}{\Sigma\Delta D}$ is the distributed liquid water content for drop of D in each same size interval $\Delta D=0.5$ mm. The representative value of class interval for N_D was simply given by adopting the median value of each class, as at 0.25, 0.75, ……, 6.75 mm diameter. So that the total number of class is 14.

Fig.2 (a) and (b) show the results in the case of the observed stationary distributions at Hitoyoshi as described in Part 1. First, the observed distribution of number density, which is shown in the figure (a) by the hitogram, was transformed into the liquid water distribution as shown in the figure (b), using Fig.1. Second, from the observed distribution of liquid water, the mean value

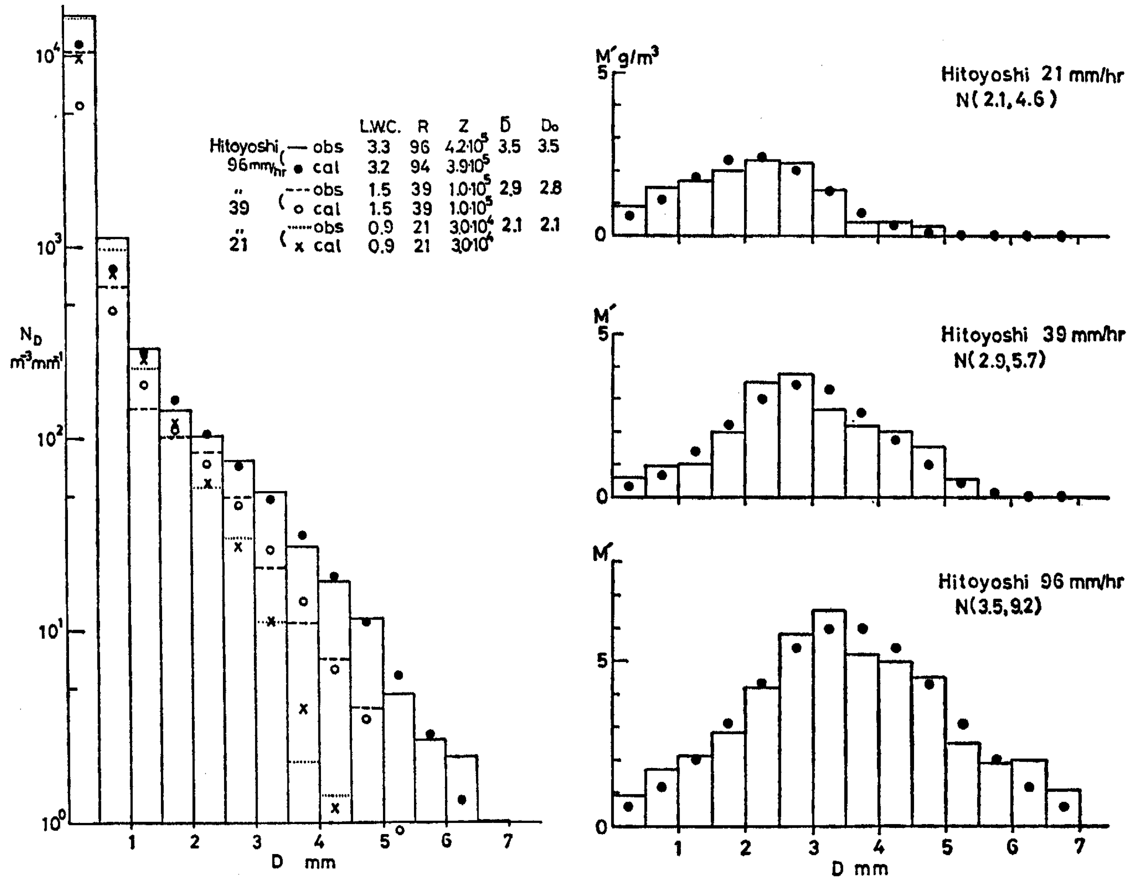

Fig. 2–a Fig. 2–b

Fig. 2 Examples of the normalized drop-size distribution for the observed spectra at Hitoyoshi. The histogram and the plot of some symbols indicate the observed (obs.) and the normalized (cal.) distribution, respestively. The distribution of the number concentration of drops is shown in the figure (a), and the distribution of the liquid water content with size, M', is shown in the figure (b).

of D:$\bar{D}$, and the variance of D:σ^2, were calculated, and then the normal distribution $N(\bar{D},\sigma^2)$with those parameters would be obtained as shown in the figure (b). Of course, the obtained normal distribution is the truncated one between $D=0$ and 7 mm. In the figure (a) the obtained $\bar{D}$ and D_0 are also indicated. Third, from the determined normal distribution, the distribution of number density, $<N_D>$, was calcualted by

$$<N_D> = A\cdot D^{-3}\cdot\frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(D-\bar{D})^2}{2\sigma^2}} \qquad \cdots\cdots\cdots(8)$$

The obtained normalized distribution, $<N_D>$, is plotted by the circles in the figure (a), in which the rain parameters, from both of the observed and the calculated from Eq.(8), are also shown. Thus, the truncated normal distribution of $N(\bar{D}, \sigma^2)$, and the derived parameters well represent and approximate to the observed.

Other examples of normalizing by the similar procedures are shown in Fig. 3 and 4. The former figure represents the results of application for the *M-P* distribution and the latter includes the results for the computed distributions obtained in Fig. 1 and Fig. 5 in Part 2, and Best's distribution for $D_0=3.5$ mm. In application for Best's distribution, the equation $1-F=\exp\{-(D/a)^n\}$ was transformed for convenience, as follows. As the variable F in the equation (a, n: the constants determined by rate of precipitation) represents the fraction of liquid water in the air comprised by drops with diameter less than D, we can easily obtain the relation $D_0=(0.69)^{\frac{1}{n}}\cdot a$ by giving $F=\frac{1}{2}$. Therefore, $a=D_0\cdot(0.69)^{-\frac{1}{n}}$, and the liquid water amount of the size interval dD might be given by

$$\frac{dF}{dD}=n\cdot 0.69\cdot D_0^{-n}\cdot D^{n-1}\cdot\exp\left\{-0.69\left(\frac{D}{D_0}\right)^n\right\} \qquad \cdots\cdots\cdots(9)$$

Then we can obtain $\frac{dF}{dD}$ from Eq.(9), by giving the mean value of $n=2.25$ by Best, and the appropriate value of D_0.

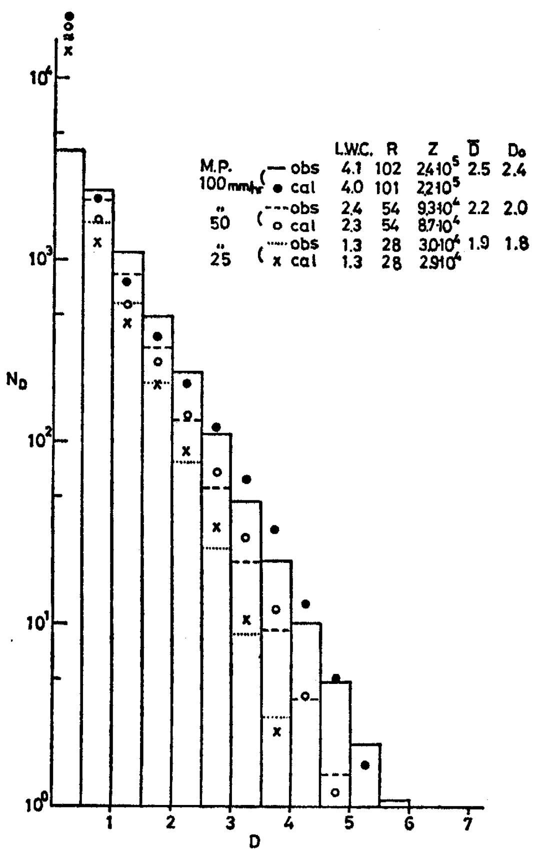

Fig. 3–a

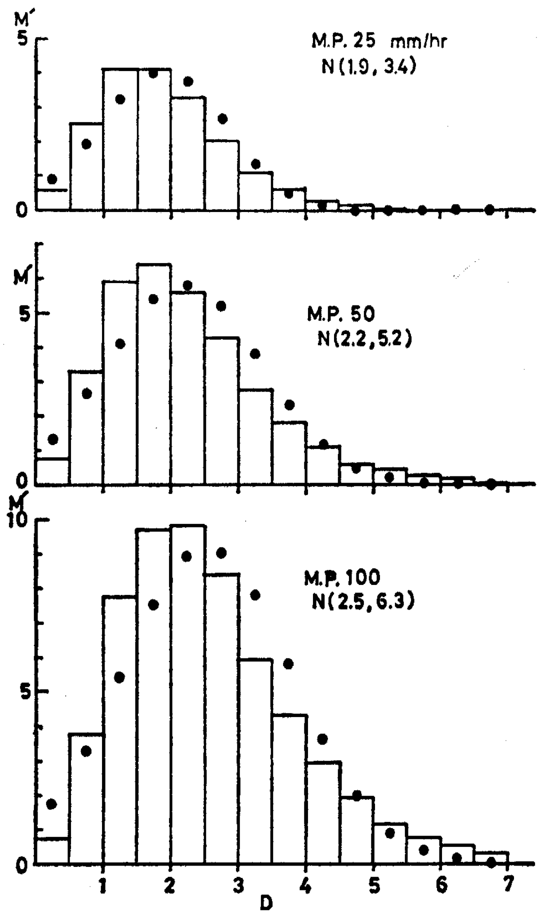

Fig. 3–b

Fig. 3 The cases of the M–P distribution, samely as Fig. 2.

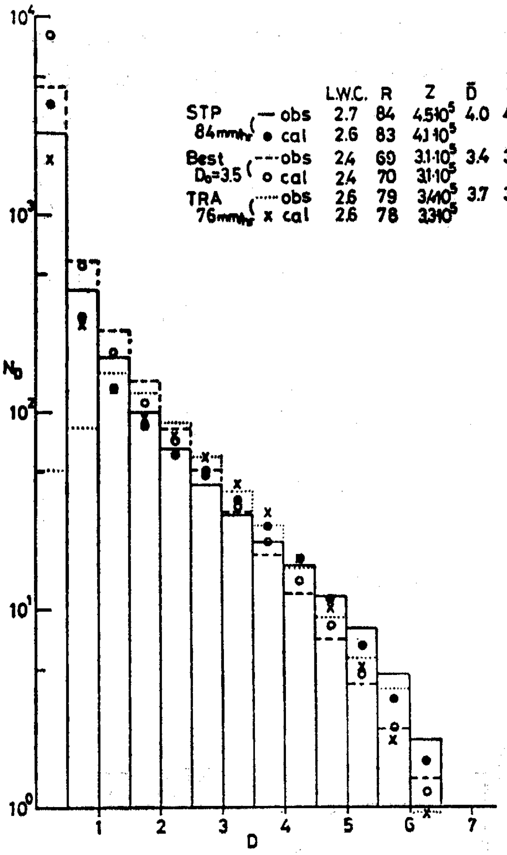

Fig. 4–a

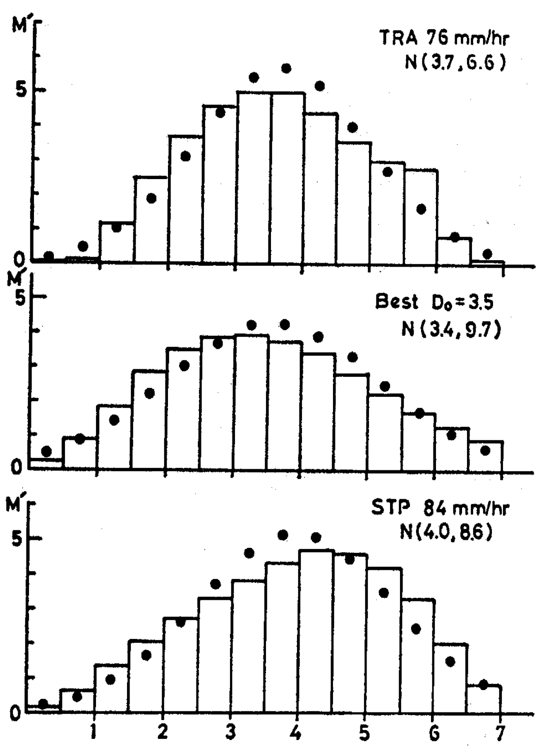

Fig. 4–b

Fig. 4 Another case, samely as Fig. 2. Where, Best $D_0=3.5$ is of the case of $D_0=3.5$ mm of the Best distribution, and STP 87 and TRA 79 are of the cases of the computed distributions in the well mixed cloud and by the transport equation, respectively, under the stochastic processes as described in Part 2 of this paper.

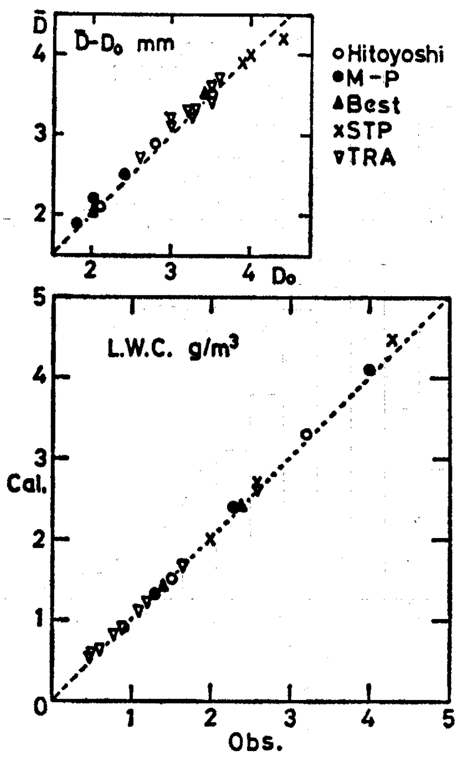

Fig. 5 Upper figure: comparison of the mean diameter $\bar{D}$ and the median volume diameter D_0 in the original distribution. Lower figure: comparison of the liquid water content calculated from the truncated normal distribution with one of the original distribution.

As shown in Fig. 3 and 4, it may be found that normalization is applicable for various types of distribution. In fact, most of normalized distributions for another cases of Fig. 1 and Fig. 5 in Part 2, and also for each stage of the computed distribution of raindrops under the process of coalescence and breakup, are well representative to the original distribution. Fig. 5 shows the correlation of the rain parameters (D_0, and $L.W.C.$(obs.)) of the original distribution to the derived values ($\bar{D}$, and $L.W.C.$(cal.)) from the corresponding normalized distribution. It is fairly seen that all of the derived values are within $\pm 10\%$ of the original. Then we can say that almost all of the raindrop size distribution might be controlled to obey the rule of normal distribution with size of liquid water content in a unit volume of air. Physical consideration on this rule will be referred in §4.

3. Rain parameter diagram

Normal distribution of liquid water amount with drop-size is determined by only the combination of the mean diameter $\bar{D}$ and the variance σ^2(σ: standard deviation of D). As shown in the preceding section; combination of $\bar{D}$ and σ would be dependent on the type of rain as such, in the weak continuous rain where the M-P distribution is suitable, both of $\bar{D}$ and σ are small, and in the developed convective rain where the stationary distribution as Hitoyoshi's is fittable, they are large. We are frequently coming across the necessity to know the drop-size distribution in some echo on the radar scope, for weather forcasting, rainfall measurement by radar, and so on. For those purposes, the rain parameter diagram was made and shown in Fig. 6, under the assumption of the drop size range of $0\sim 7$ mm diameter as described in the introduction. And the values of Z and R are given in case that the liquid water content is 1 g/m³.

From the figures (a) and (b) of Fig. 6, we can know the value of Z and R when knowing $L.W.C.$, σ and $\bar{D}$, and reversely the value of σ and $\bar{D}$ by knowing $L.W.C.$, R and Z. Ordinarily we can know the values of R and Z from the rain gauge network on the surface and by radar, respectively. If the estimate of liquid water content by some means, for example, the upper sounding within a raincloud or the direct measurement by air-plane, is possible, the drop-size distribution can be determined by the observed Z, R and $L.W.C.$ Thus, the figures (a) and (b) can answer the residual two parameters with knowing three of five parameters as described above.

On the other hand, in the figure (c), we may determine Z/R independent of $L.W.C.$ by knowing σ and $\bar{D}$. If the correlation between σ and $\bar{D}$ is established, we can obtain either of both of σ and $\bar{D}$ with knowing Z/R. An example for $\sigma-\bar{D}$ relation is shown in Fig. 7. In the figure, the plots are representative for the characteristic values for three examples of the $M-P$ distribution, three of observed at Hitoyoshi, and two of Best's distribution. Generally it will be

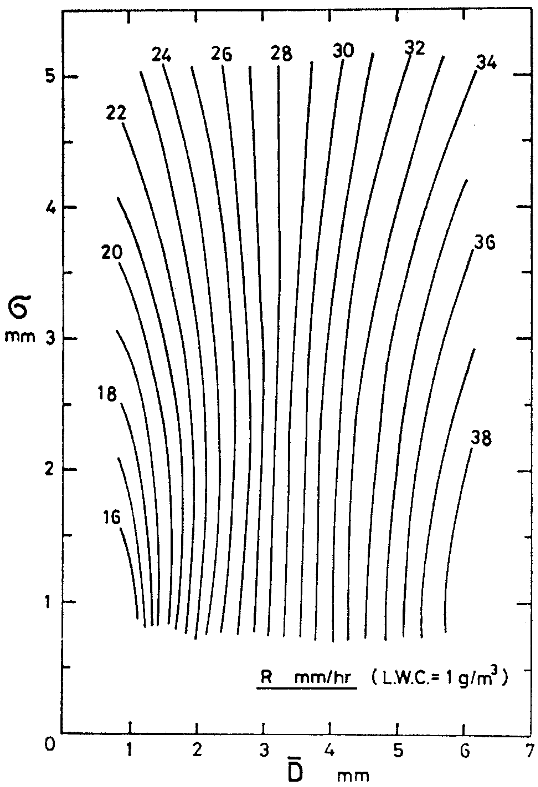

Fig. 6-a

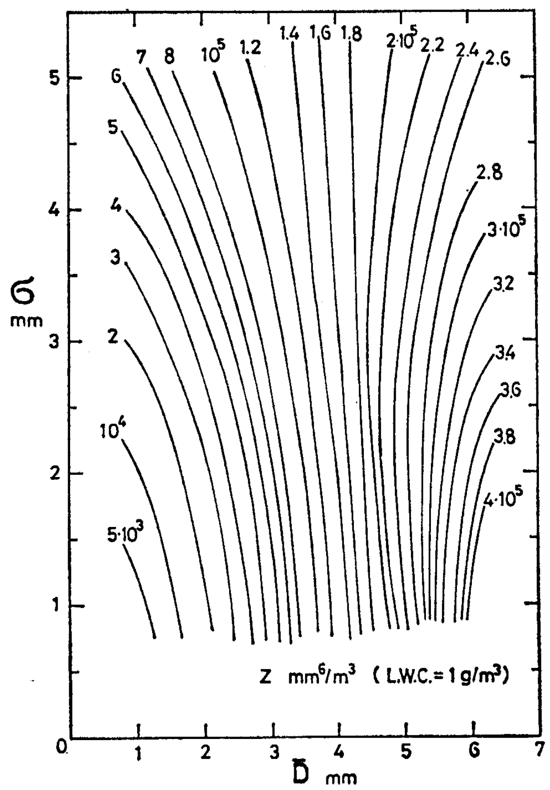

Fig. 6-b

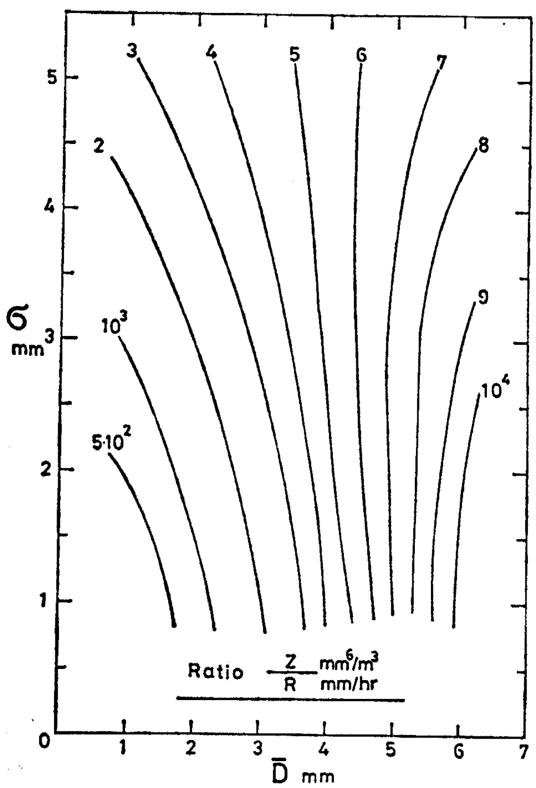

Fig. 6-c

Fig. 6 Rain parameter diagram.

a) Relation between R, σ, and $\bar{D}$ ($\sim D_0$) for a truncated normal distribution, $N(\bar{D}, \sigma^2)$, with liquid water content L. W. C. = 1 g/m^3.

b) Relation between Z, σ, and $\bar{D}$, samely as a).

c) Relation between the relation Z/R, σ, and $\bar{D}$. (independent of L. W. C.)

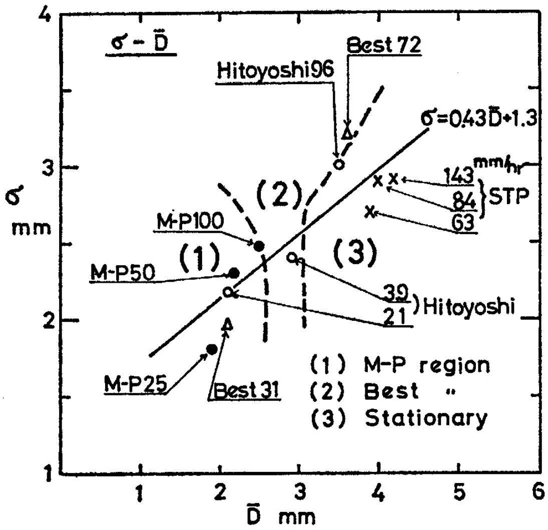

Fig. 7 Correlation between σ and $\bar{D}$ for various types of rain. Three regions are indicated in correspondence to the rain type.

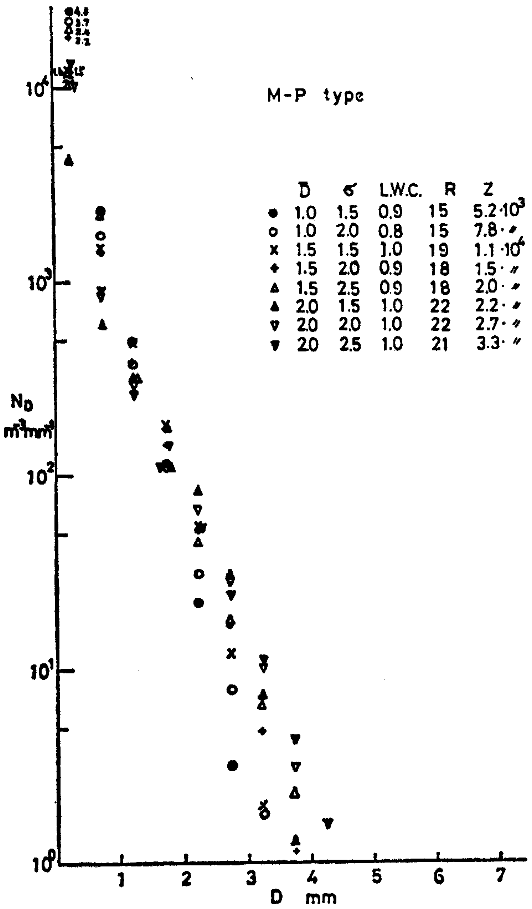

Fig. 8–a

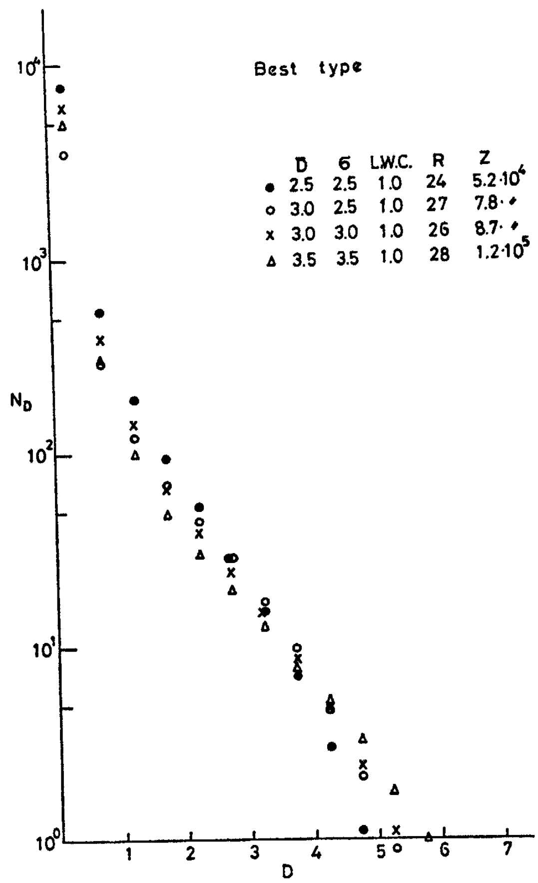

Fig. 8–b

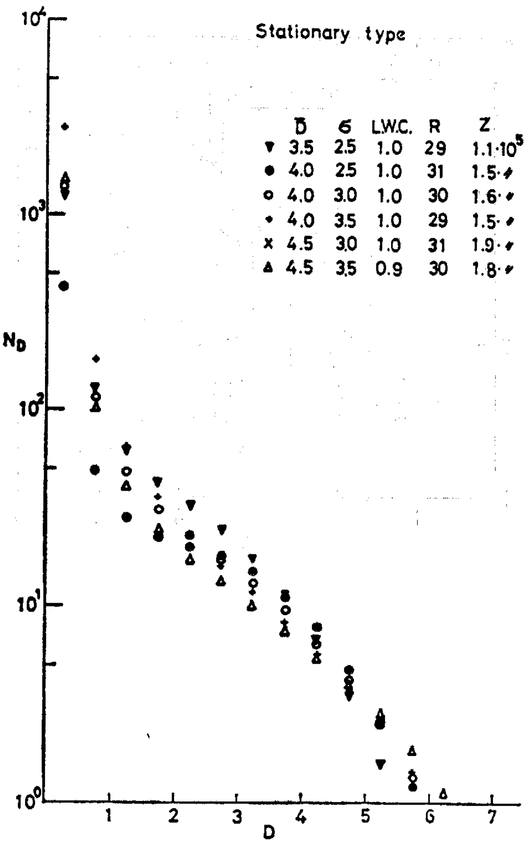

Fig. 8–c

Fig. 8 Families of drop-size distribution determined by the truncated normal distribution $N(\bar{D},\sigma^2)$.

a) σ of 1.5~2.5 mm and $\bar{D}$ of 1.0~2.5 mm for the M–P type rain.

b) σ of 2.5~3.0 and $\bar{D}$ of 2.0~3.0 for the Best type rain.

c) σ of 2.0~3.5 and $\bar{D}$ of 3.0~4.5 for the well-developed rain.

found that σ is increased in proportion to $\bar{D}$, and their regression line is calculated to be $\sigma=0.43\ \bar{D}+1.3$, although the data are quite scarse. And various kinds of the combination of σ and $\bar{D}$ would be expected to converge around this line. Furthermore, the expected distributions near this line would be divided in three types of distribution as shown in Fig. 8(a), (b) and (c). (a) is for the combinations of σ(1.5~2.5 mm) and $\bar{D}$(1.0~2.5 mm), and (b) is for those of σ(2.5~3.0) and $\bar{D}$(2.0~3.0), and (c) is for those of σ(2.0~3.5) and $\bar{D}$(3.0~4.5). The distribution families of the figure (a) are quite similar in form to the $M-P$ distribution, and then their σ and $\bar{D}$ may be available for weak continuous rain. In (b) and (c), they intend to array apart from the exponential type like the $M-P$ at the vicinity of the point of $D=1$ mm, and come to appear flat in medium drop size range. The families in (c) are for the stationary distribution as observed in the heavy rain at Hitoyoshi and obtained in the numerical experiments under the stochastic model. And then the families in (b) show the transitional distribution which might be experienced at rain shower, and be for the Best's distribution. The regions of those main three rain type are indicated in Fig. 7.

Thus, if Z by radar, R by rain gauge, and rain type by some information as radar are known, we can derive easily the combination of σ and $\bar{D}$ by the in-

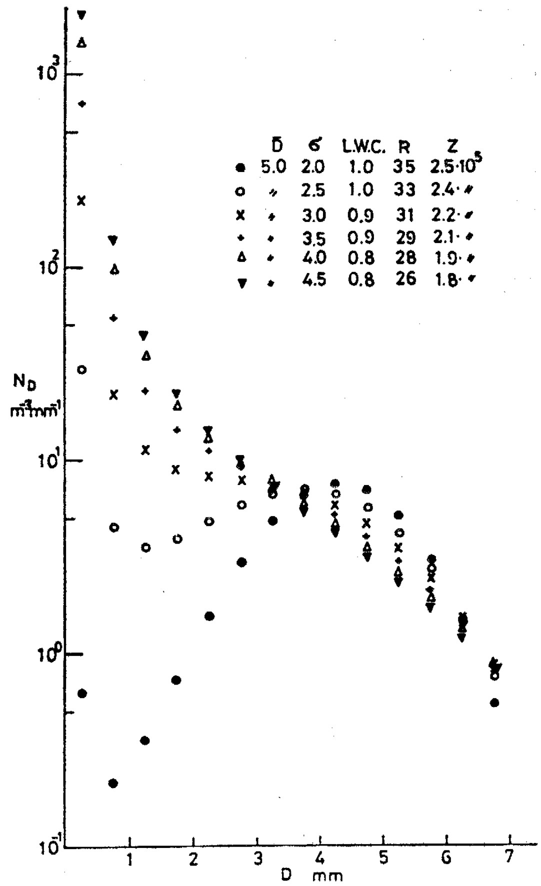

Fig. 9 Family of drop-size distribution in the case of $\bar{D} = 5$ mm in $N(\bar{D},\sigma^2)$.

tersecting point of the $\sigma-\bar{D}$ line and the Z/R curve on the graph of Fig. 6 (c). And from the obtained σ and $\bar{D}$, we can know also *L. W. C.* by either of both of graphs of Fig. 6 (a) and (b), such as $L.W.C.=\frac{R(\text{observed})}{R(L.W.C.=1\text{g/m}^3)}$ or $\frac{Z(\text{observed})}{Z(L.W.C.=1\text{g/m}^3)}$
If the values of *L.W.C.* from separately (a) and (b) were quite different to each other, we can know that the size distribution might never belong to the above described type of rain, any longer. For example, the complete flat distribution, as like frequently observed at the beginning stage of convective rain as described in Part 1 may be in the region of large $\bar{D}$. Fig. 9 shows the distribution families for the combinations of $\bar{D}=5.0$ mm and σ of $2\sim4.5$ mm, in which the distributions at σ of $2\sim3$ mm are in similar form to the complete flat distribution.

But, as concerned with the average distribution in a rainfall of which life time is more than ten minutes, the $\sigma-\bar{D}$ relation described above would be available. And then we may deal with the quite different value to the expected by the $\sigma-\bar{D}$ relationship, as from the quite singular rainclouds.

4. Considerations

As described in the previous sections, almost all of the drop-size distributions in the various types of rain can be represented by the normal distribution of liquid water content. But what is the physical explanation for that rule? Two expected reasons will be considered in this section. One may be a effect of a difference of falling velocity of drops to each other. From the results by Atlas[1] and also by this report, the ratio ($\bar{v}/v_0$) of the fall velocity of the mean diameter of drops, $\bar{v}$, to one of the median volume diameter, v_0, seemed to be unity for the size distribution of any type of rain. This may mean that the liquid water content in the falling rain parcel at $\bar{v}$ is equally divided into two parts of size range at $\bar{D}(=D_0)$. The falling parcel has been called "bubble" or "cell" in the series of this report. When we see the size distribution falling with the same speed as the bubble, the number density or the fraction of liquid water content of drops with the different speed compared to us will become less and less.

The relative decrement of number density or liquid water content of other drops besides $\bar{D}$ drop may be shown by the velocity difference between them, that, the ratio (γ) of space number density or liquid water content after Δt time from the beginning of fall is

$$\gamma=1-\frac{|\Delta v_D|\Delta t}{\bar{v}\Delta t}=1-\frac{|\Delta v_D|}{\bar{v}} \qquad \cdots\cdots\cdots(10)$$

, where Δv_D=fall velocity difference between v_D for drop of diameter D and $\bar{v}$ for the mean diameter $\bar{D}$. In Eq. (10), using the fall velocity function $v=kD^n$ (k, n: constant) by Gunn and Kinzer[3],

$$\gamma=\begin{cases} 2-\sqrt{\dfrac{D}{\bar{D}}} & \text{when } 7\text{mm}\geqq D\geqq \bar{D} \\ \sqrt{\dfrac{D}{\bar{D}}} & \text{when } \bar{D}\geqq D\geqq 0 \end{cases} \qquad \cdots\cdots\cdots(11)$$

is obtained. Fig. 10 shows the γ distribution with size for each case of $\bar{D}$= 1, 2, 3.5 and 5 mm. From the figure we can find nearly symmetrical distribution of γ which may be very similar to the liquid water amount distribution $m(D)$ for various types of rain. Thus, it might be seen from the γ distribution that the distribution of liquid water must be destined nearly to the normal distribution even if initially the same liquid water amount was allotted to each size interval. Fig. 11 was obtained for the case of $\bar{D}=2.0$ and $\bar{D}=3.5$ in Fig. 10 under the same procedures as described in §2. For present two cases, the initial distribution of liquid water content was given homogeneously to drops of each size interval by 5 g/m³ as shown in Fig. 1. Then, the median volume diameter and the total liquid water content for both two cases in Fig. 10 are D_0=2.8 mm, $L.W.C.$=2.8 g/m³, and D_0=3.7, $L.W.C.$=3.7, respectively. The adaptability to the normal distribution in Fig. 11 is not so enough as the examples in the cases in §2. This is because of adopting the assumption of the homogeneous allotting for the initial liquid water amount distribution with size. Actually in nature an existence of such a ideal distribution might be expected to be very rare due to the change of population of drops in some

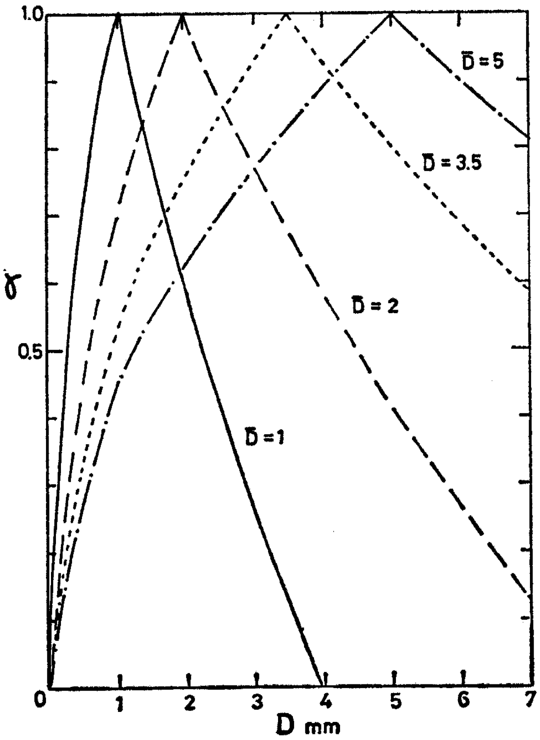

Fig. 10 Relative change of size spectra caused by the difference of falling velocity of drops when noting a drop of mean diameter. The curve was obtained from Eq. (11).

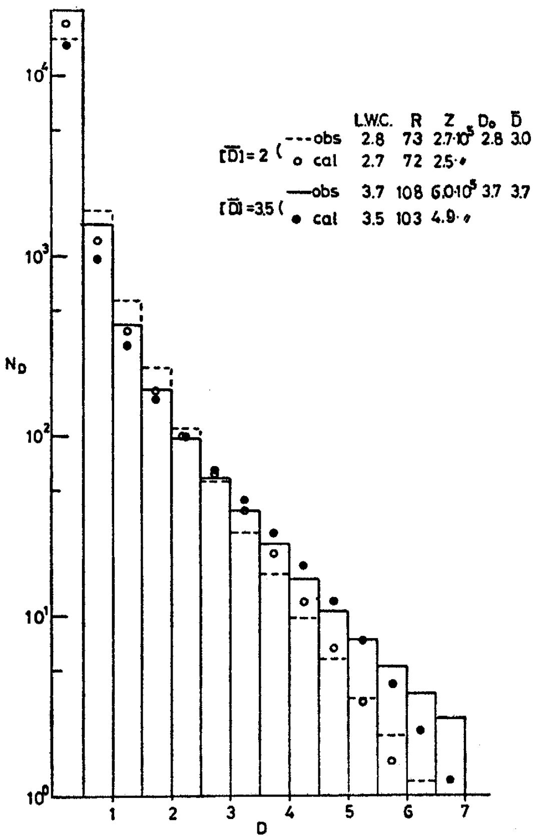

Fig. 11–a

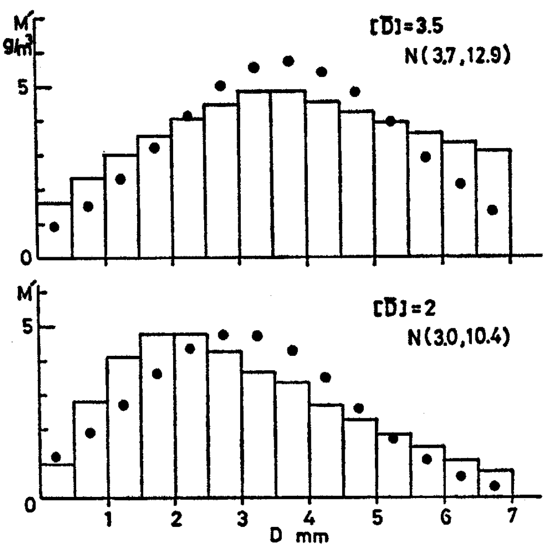

Fig. 11–b

Fig. 11 Comparison of the size distribution under the effect of the difference of falling velocity of drops (histogram, and obs.) with the truncated normal distribution of it (plot, and cal.). The initial distribution was given by the ideal distribution of L.W.C. = 5 g/m³.

size by the effects of various kinds of cloud physical processes such as evaporation, coalescence, breakup and so on. Therefore, if the peaked distribution near at D_0 and $\bar{D}$ was used for the initial distribution of liquid water content, the normalized distribution would become more narrower and its adaptability be better.

On the other hand, as described in Part 2, the effect of drop breakup and coalescence will be also responsible for normalization of liquid water amount distribution. In convective rainclouds, raindrops might be expected to be well mixed and liquid water supply to be much enough. Where, small drops always will be going to become larger by coalescence, and drops larger than 5 mm diameter can exist for only a short time to breakup to smaller drops. Thus population of medium size drops would be going to increase more and more, and the distribution will attain to be stationary. The process of raindrop coalescence and breakup will play a role for the high concentration of liquid water content around the mean diameter or the median volume diameter.

5. Summary

The raindrop distributions with size, observed and calculated to be well expected in nature, were studied from the view of share of liquid water content with drop-size. Most of them seemed to obey the truncated normal distribution of liquid water with size in the considered range of $0\sim7$ mm diameter. The adaptability of the normalized distribution for the actual is very high, and the derived rain parameters are available enough for practice, because they are in the region within $\pm 10\%$ of the obtained parameter values from the actual distribution. For such a rule of the normal distribution of liquid water content with size, some physical explanations were tried. They are due to the concentration effect of liquid water by the relative difference of fall-velocity of drops within a raincloud or bubble which is falling at the mean velocity, and also by the drop breakup and coalescence under the stochastic process.

The rain parameter diagram, which was based on the rule of the normal distribution of liquid water content, was proposed for practical works. If the parameters of Z (radar reflectivity), R (rainfall rate), and rain type (such as continuous, shower, and heavy) are known by observations, the variance σ^2 and the mean diameter $\bar{D}$ of the normal distribution $N(\bar{D}, \sigma^2)$, and $L.W.C.$ (liquid water content of rain per unit volume of air) can be obtained from the diagram. But, for this purpose, the more reliable relationship of $\sigma - \bar{D}$ which may be dependent on rain type will have to be established by observations in future.

Acknowledgement

The writer wishes to thank Prof. C. Magono of Hokkaido University for his steady consideration and advice. The computations were performed by the FACOM 230-60 at the computing center of Hokkaido University.

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