State Space Model Identification From Actual Input-Output Data Using Minimal Realization Algorithm

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Abstract

A procedure is presented for identifying the system triple $\{A, B, C\}$ and the system order for linear, constant, multivariable, discrete-time plants from actual input-output data.

The approach is based on the minimal realization and the cross-correlation technique between the input sequence and the output measurement sequence. The dimension of the system is determined from checking the rank of the Hankel matrix which is constructed by the Markov parameter estimates obtained by the auto-correlation of the input sequence and the cross-correlation between the input sequence and the output sequence.

The estimates of the triple $\{A, B, C\}$ are calculated by Markov parameter estimates using minimal realization algorithm.

I. Introduction

In recent years, a considerable effort has been made in the area of identifying the parameters of linear constant, discrete time systems given input-output data records. This development has been stimulated by the desire to apply the elegant results of modern control and filtering theory to practical problems. In most cases, these problems assume the perfect knowledge about the underlying systems. In practice, however we can seldom expect a priori information regarding them. In such cases, system identification methods have to be employed to obtain the necessary information.

For the practical usefulness of identification procedures, the state space representation is considered in this paper, since many available modern control theories regard this representation.

Some recent papers have discussed the system identification from input-output data.

In single input-output systems, Saridis and Stein[1)], and Wong and Polak[2)] considered some techniques for the identification in the case of the known system order. The work of Mehra[3)] uses the system model in an adaptive Kalman filter.

In multiple input-output systems, Gopinath[4)] presented the identification method of systems with unknown size. Budin[5)] improved Gopinath's work and proposed the more efficient procedure. Furthermore, Passeri and Herget[6)], and Gupta and Fairman[7)] gave identification procedures in the framework discussed in the references 4) and 5). In these methods, determining a selector matrix has played an important role, while it has given considerable difficulties.

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Guidorzi[8] proposed a unified approach for linear multivariable system identification based on a canonical representation of a model without restricting special inputs. In resent paper by Tes and Weinert[9], the systematic procedure for estimating the system structure and parameter of an innovation model was constructed by a special canonical form via the correlation technique.

The approach presented in this paper is based on the correlation technique between actual input-output data, and the minimal relaization concept. The most popular application of the cross-correlation technique is well known to be the determination of the impulse response of a plant by correlating the white input sequence with the observed output sequence. The comparison of some identification algorithms has been presented by Saridis[10], who reported that the modified cross-correlation method proposed by Hill and McMurty[11] gave good results as far as examined. In another works, the correlation techniques have been used by Wouters[12], Funahashi and Nakamura[13], and Chow[14]. In the area of minimal realization, Ho and Kalman[15] were the first to provide an algorithm from the sequence of Markov parameters, and Rissanen[16] reformulated Ho-Kalman's algorithm. Furthermore, Silverman[17] proposed an algorithm based on Gauss elimination, and Gupta and Fairman[18] treated the Markov parameter matrices as the elements of vectors in an inner product space. Tether[19] discussed an algorithm for constructing minimal linear partial realization of an unknown system from an external description as given by its Markov parameter. On the other hand, Rissanen and Kailath[20] have given an algorithm for recursive determination of linear models driven by a white noise sequence, whose approach is to construct a partial realization from the estimates of the covariance matrices obtained by the output data which are stationary processes.

In this paper, the first step is to get the estimates of Markov parameters from correlation techniques by using input-output data and the second step is to identify the system order and the triple $\{A, B, C\}$ by the use of the minimal realization algorithm from Markov parameters.

II. Problem Statement

It is assumed that measured imputs and outputs are modeled by the following discrete-time system:

$$\mathbf{x}(k+1)=A\mathbf{x}(k)+B\mathbf{u}(k) \tag{1}$$

$$\mathbf{y}(k)=C\mathbf{x}(k)+\mathbf{v}(k) \tag{2}$$

where, $\mathbf{x}(k)\in R^n$, $\mathbf{u}(k)\in R^r$ and $\mathbf{y}(k)\in R^m$.

The sequence $\{\mathbf{u}(k)\}$ represents the perfectly measurable control input. In this paper, $\{\mathbf{u}(k)\}$ is assumed to be a zero-mean white input sequence independent of the state. Furthermore, it is assumed that the sequence $\{\mathbf{v}(k)\}$ is a stationary noise independent of the input sequence $\{\mathbf{u}(k)\}$. The problem is then to obtain the estimates of the system order n and the system triple $\{A, B, C\}$ from the actual input-output data.

III. The Markov Parameter from Output Data

Consider iterations of the state equation and the measurement equation given by (1) and (2) so that

$$\begin{aligned}
\mathbf{y}(k) &= C\mathbf{x}(k)+\mathbf{v}(k) \\
\mathbf{y}(k+1) &= CA\mathbf{x}(k)+CB\mathbf{u}(k)+\mathbf{v}(k+1) \\
&\vdots \\
\mathbf{y}(k+p) &= CA^p\mathbf{x}(k)+\sum_{j=0}^{p-1} CA^jB\mathbf{u}(k+p-j-1)+\mathbf{v}(k+p).
\end{aligned} \tag{3}$$

Taking a look at the cross-correlation function between the output and the input, we have, for our real process,

$$\begin{aligned}
R_{yu}(p) &\triangleq E\{\mathbf{y}(k+p)\mathbf{u}^T(k)\} \\
&= CA^pE\{\mathbf{x}(k)\mathbf{u}^T(k)\}+\sum_{j=0}^{p-1} CA^jBE\{\mathbf{u}(k+p-j-1)\mathbf{u}^T(k)\} \\
&\quad +E\{\mathbf{v}(k+p)\mathbf{u}^T(k)\}.
\end{aligned} \tag{4}$$

By the assumption, it is easily seen that

$$E\{\mathbf{x}(k)\mathbf{u}^T(k)\}=0 \tag{5}$$

$$E\{\mathbf{v}(k+p)\mathbf{u}^T(k)\}=0, \qquad \text{for all } p \tag{6}$$

$$E\{\mathbf{u}(k+p)\mathbf{u}^T(k)\}=0, \qquad \text{for } p\neq 0. \tag{7}$$

Using (5), (6) and (7) yields

$$R_{yu}(p)=CA^{p-1}BR_{uu}(0), \qquad \text{for } p=1, 2,\ldots \tag{8}$$

where,

$$R_{uu}(0)\triangleq E\{\mathbf{u}(k)\mathbf{u}^T(k)\} \tag{9}$$

is a nonsingular matrix. Then (8) gives

$$R(p)\triangleq R_{yu}(p)\,[R_{uu}(0)]^{-1}=CA^{p-1}B, \qquad \text{for} \quad p=1, 2,\ldots. \tag{10}$$

The right hand side of (10), $CA^{p-1}B$ is called the Markov parameter. Therefore, the problem identifying the parameters A, B and C is reduced to the minimal realization discussed by Ho and Kalman[15] or Rissanen[16] when the Markov parameters are given.

IV. The Minimal Realization from The Markov Parameter

Ho and Kalman[15] have presented the following algorithm to get the minimal realization from Markov parameters. This procedure starts from the determination of the minimal system order.

(1) Determination of the minimal system order:

Construct the following Hankel matrix $S_p(mp\times rp)$ from the Matkov parameter.

$$S_p \triangleq \begin{bmatrix} R(1) & R(2) & \cdots R(p) \\ R(2) & R(3) & \cdots R(p+1) \\ \vdots & & \\ R(p) & R(p+1) & \cdots R(2p-1) \end{bmatrix} \tag{11}$$

where, m and r are the dimensions of the output vector and the input vector respectively. The minimal dimension n of the system is chosen as the rank such that

$$n=\operatorname{rank}(S_q)=\operatorname{rank}(S_{q+1})=\cdots \tag{12}$$

hold.

(2) Realization of the triple $\{A, B, C\}$:

Find a nonsingular $mq\times mq$ matrix P and a nonsingular $rq\times rq$ matrix Q that satisfy

$$PS_qQ=\left[\begin{array}{c:c} I_{n,n} & O_{n,rq-n} \\ \hdashline O_{mq-n,n} & O_{mq-n,rq-n} \end{array}\right]=E_{mq,n}E_{n,rq} \tag{13}$$

where,

$$E_{i,j}=\begin{cases} [I_{i,i} \quad O_{i,j-1}] & (i\times j) \quad \text{if } i<j \\ \begin{bmatrix} I_{j,j} \\ O_{i-j,j} \end{bmatrix} & (i\times j) \quad \text{if } i>j \\ [I_{i,i}] & (i\times i) \quad \text{if } i=j \end{cases} \tag{14}$$

$I_{i,i}$ and $O_{i,j}$ are the $i\times i$ unit and $i\times j$ zero matrices respectively. P and Q are obtained by applying the elementary operation as follows.

$$\left[\begin{array}{c:c} S_q & I_{mq,mq} \\ \hdashline I_{rq,rq} & O_{rq,mq} \end{array}\right] \Longrightarrow \left[\begin{array}{c:c} I_{mq,rm} & P \\ \hdashline Q & O_{rq,mq} \end{array}\right] \tag{15}$$

The Gauss-Jordan sweep-out method is effective regarding the computational aspect. Then the minimal realization is obtained as follows:

$$A=E_{n,mq}P[\sigma S_q]E_{rq,n} \tag{16}$$

$$B=E_{n,mq}PS_qE_{rq,r} \tag{17}$$

$$C=E_{m,mq}S_qE_{rq,n} \tag{18}$$

where,

$$\sigma^k S_q = \begin{pmatrix} R(1+k) & R(2+k) & \cdots R(q+k) \\ R(2+k) & R(3+k) & \cdots R(q+1+k) \\ \vdots & & \vdots \\ R(q+k) & R(q+1+k) & \cdots R(2q+k) \end{pmatrix} \tag{19}$$

V. Practical Approach for Estimation of Markov Parameter

The auto-correlation $R_{uu}(0)$ and the cross-correlation $R_{yu}(p)$, $p=1, 2,\dots$, which are used for getting the Markov parameter $R(p)$, are theoretical values as far. Therefore, their estimates have to be required from input-output data for practical purpose.

When $\mathbf{u}(k)$ and $\mathbf{y}(k)$, $k=1, 2,\dots, N$ are obtained, the estimates of $R_{uu}(0)$ and $R_{yu}(p)$, $p=1, 2,\dots$ are described as follows respectively:

$$\hat{R}_{uu}(0) = \frac{1}{N}\sum_{k=1}^{N} \mathbf{u}(k)\mathbf{u}^T(k) \tag{20}$$

$$\hat{R}_{yu}(p) = \frac{1}{N}\sum_{k=1}^{N-p} \mathbf{y}(k+p)\mathbf{u}^T(k), \qquad p=1, 2,\dots. \tag{21}$$

Due to the prescribed assumption, the following relation may be valid.

$$R_{uu}(0) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} \mathbf{u}(k)\mathbf{u}^T(k) \tag{22}$$

$$R_{yu}(p) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N-p} \mathbf{y}(k+p)\mathbf{u}^T(k), \qquad p=1, 2,\dots. \tag{23}$$

Defining the estimate of the Markov parameter $R(p)$ from N observations of input and output data yields

$$\hat{R}^N(p) = \left[\frac{1}{N}\sum_{k=1}^{N-p} \mathbf{y}(k+p)\mathbf{u}^T(k)\right]\left[\frac{1}{N}\sum_{k=1}^{N} \mathbf{u}(k)\mathbf{u}^T(k)\right]. \tag{24}$$

Now, it is clearly seen that

$$\lim_{N\to\infty} \hat{R}^N(p) = R(p). \tag{25}$$

From the above discussion, it is possible to obtain consistent estimates of Markov parameters $R(p)$, $p=1, 2,\dots$ for sufficiently large N.

VI. Extension of On-Line Estimation

Let denote following notations:

$$\hat{R}^N_{uu}(0) = \sum_{k=1}^{N} \mathbf{u}(k)\mathbf{u}^T(k) \tag{26}$$

$$\hat{R}^N_{yu}(p) = \sum_{k=1}^{N-p} \mathbf{y}(k+p)\mathbf{u}^T(k) \tag{27}$$

Then $\hat{R}^N(p)$ is described from (25) as

$$\hat{R}^N(p)=R^N_{yu}(p)[R^N_{uu}(0)]^{-1}. \tag{28}$$

Putting

$$R^N_{uu}(0)=U_N \tag{29}$$

$$V_N=U_N^{-1}, \tag{30}$$

we then have,

$$V_N^{-1}=U_N=U_{N-1}+\mathbf{u}(N)\mathbf{u}^T(N) \tag{31}$$

$$=V_{N-1}^{-1}+\mathbf{u}(N)\mathbf{u}^T(N) \tag{32}$$

Applying a matrix inversion lemma to (32) yields,

$$V_N=V_{N-1}-V_{N-1}\mathbf{u}(N)[\mathbf{u}^T(N)V_{N-1}\mathbf{u}(N)+1]^{-1}\mathbf{u}^T(N)V_{N-1} \tag{33}$$

On the other hand, (28) is written as

$$\hat{R}^N(p)=W_NV_N=\left(\frac{1}{N}W_N\right)(NV_N)=W_N^*V_N^* \tag{34}$$

where,

$$W_N=\sum_{k=1}^{N-p}W(k) \tag{35}$$

$$W(k)=\mathbf{y}(k+p)\mathbf{u}^T(k) \tag{36}$$

$$V_N^*=NV_N \tag{37}$$

$$W_N^*=\frac{1}{N}W_N \tag{38}$$

It is easily seen that V_N^* and W_N^* give the following recursive relationships:

$$V_N^*=\frac{1}{N-1}\{V_{N-1}^*-V_{N-1}^*\mathbf{u}(N)[\mathbf{u}^T(N)V_{N-1}^*\mathbf{u}(N)+N-1]^{-1}\mathbf{u}^T(N)V_{N-1}^*\} \tag{39}$$

$$W_N^*=\frac{N-1}{N}W_{N-1}^*+\frac{1}{N}W(N), \qquad \text{for} \quad N\geq 2 \tag{40}$$

An on-line estimator of the Markov parameter is defined by (34), (39) and (40). In this case an initial value V_1^* is given by (41) approximately, because V_1^* does not exist for the singularity of $\mathbf{u}(1)\mathbf{u}^T(1)$.

$$\hat{V}_1^*=\text{diag}\{[u_1(1)^2]^{-1},[u_2(1)^2]^{-1},\dots,[u_r(1)^2]^{-1}\} \tag{41}$$

where $u_i(1)$ is the i-th element of $\mathbf{u}(1)$. This initial value $\hat{V}_1^*$ defines a sequence $\{\hat{V}_N^*\}$, for which the following property is seen by (39).

$$\lim_{N\to\infty} \hat{V}_N^* = \lim_{N\to\infty} V_N^* = [R_{uu}(0)]^{-1} \tag{42}$$

So the actual estimates of Markov parameters are obtained as

$$\hat{R}^N(p) = \hat{W}_N^* \hat{V}_N^*, \quad p=1, 2,\ldots \tag{43}$$

with the property

$$\lim_{N\to\infty} \hat{R}^N(p) = R(p), \quad p=1, 2,\ldots \tag{44}$$

VII. A Numerical Example

A numerical example is presented in this section to illustrate the proposed identification procedure. Consider the following two dimensional input vector, two dimensional output vector, fourth order system whose pulse transfer function is given by

$$G(z) = \frac{\begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix}}{G_d(z)}$$

where,

$$G_{11}(z) = z^3 - 0.3z^2 + 0.74z + 0.418$$

$$G_{12}(z) = z^3 - 1.3z^2 + 1.54z - 0.674$$

$$G_{21}(z) = 1.1z^3 - 0.68z^2 - 0.059z + 0.062$$

$$G_{22}(z) = z^3 - 1.5z^2 + 0.74z - 0.12$$

$$G_d(z) = z^4 - 1.8z^3 + 1.19z^2 - 0.342z + 0.036$$

As an input signal the Gaussian sequence with zero mean value and unity variance is employed and a measurement noise is implimented by the Gaussian noise $N(0, 10^{-2})$.

The Markov parameter estimates are given in Fig. 2–5 and the test for determining the system order is illustrated in Fig. 1.

Before discussing the determination of system order, we shall now discuss about the implications of computer arithmetic. At the first step of Gaussian elimination of the Hankel matrix, the maximum absolute value of all elements is employed as the pivot element. The dimension of the system is decided by checking the absolute ratio ρ_k of the new pivot element s_{kk} at the k-th step per the pivot element s_{11} at the frist step.

$$\rho_k \triangleq \left| \frac{s_{kk}}{s_{11}} \right|$$

The results of the structural identification are shown in Fig. 1, which presents significant sharp decreases at $k=4$ on an average. Therefore, it is reasonable to determine the system order as 4.

The estimates $\{\hat{A}, \hat{B}, \hat{C}\}$ of system triple are presented in Table 1.

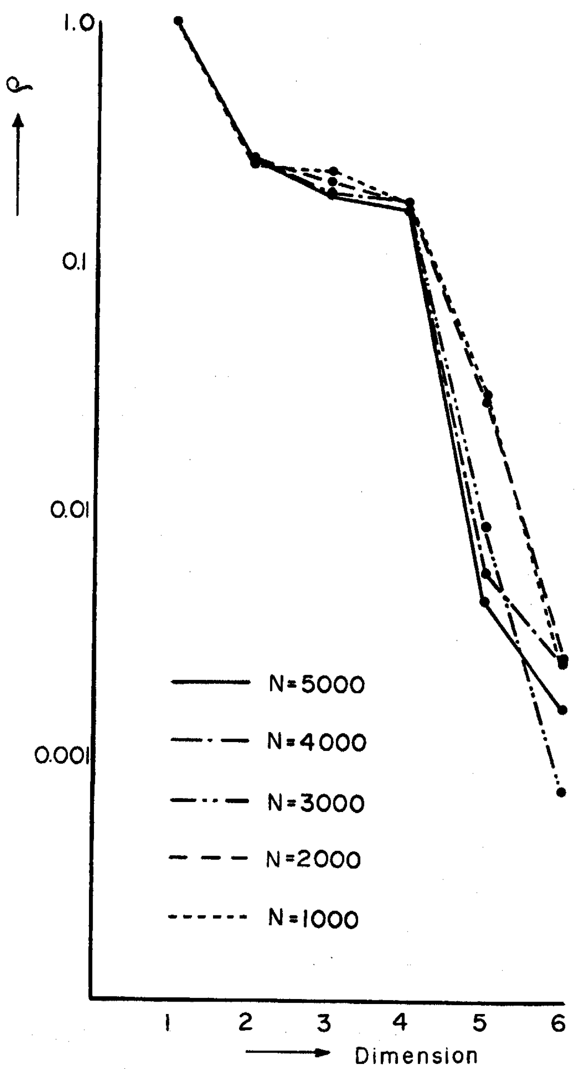

Fig. 1 Determination of system dimension.

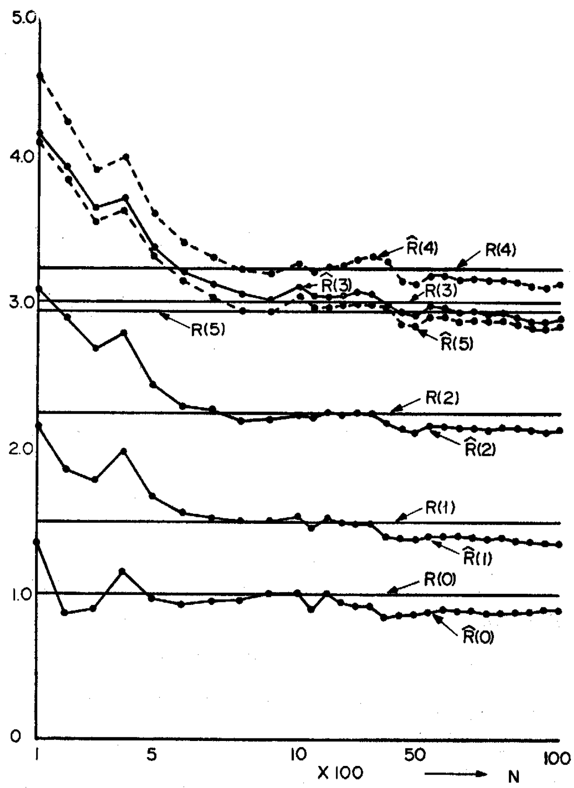

Fig. 2 Estimates of 11-elements of Markov parameters.

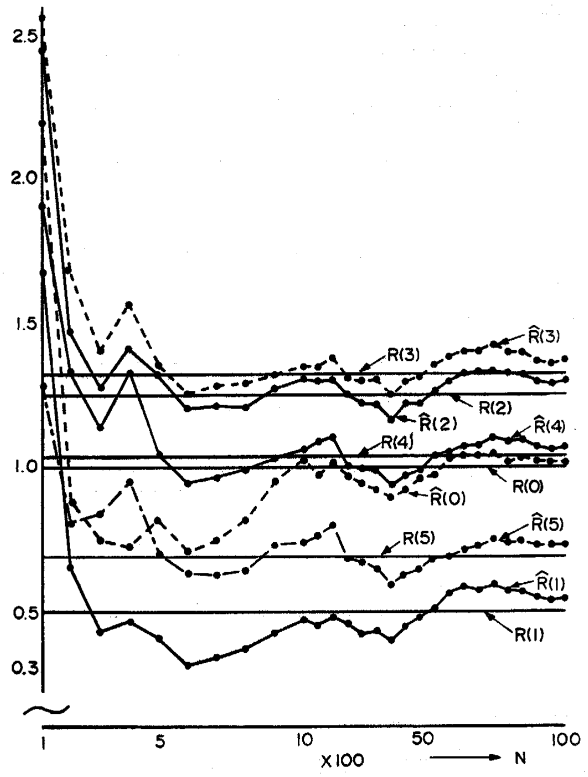

Fig. 3 Estimates of 12-elements of Markov parameters.

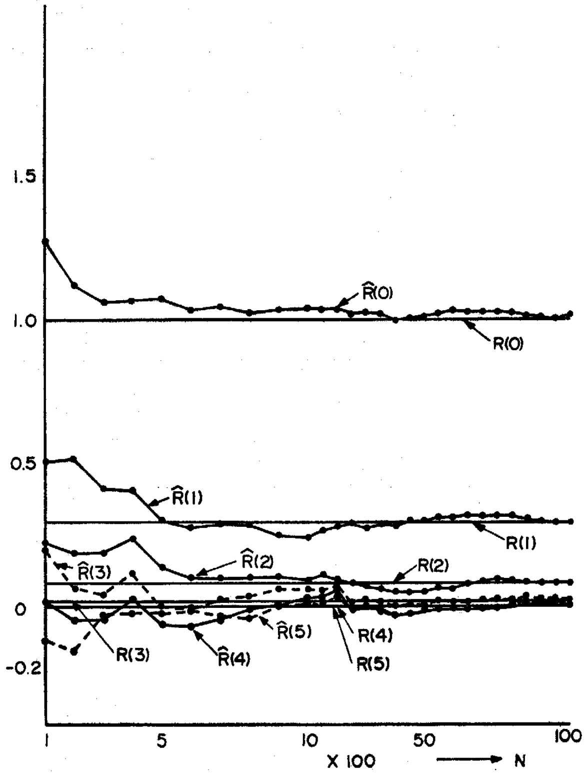

Fig. 4 Estimates of 21-elements of Markov parameters.

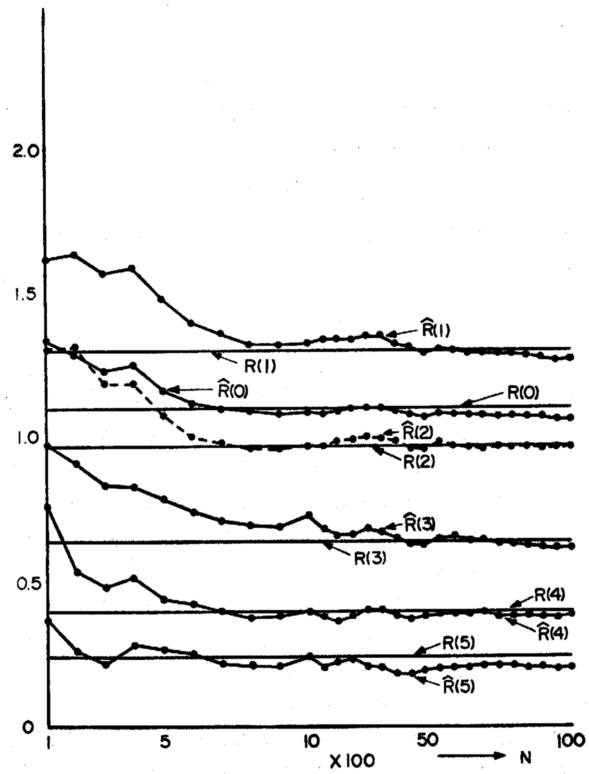

Fig. 5 Estimates of 22-elements of Markov parameters.

Table 1. Estimates of System Triple

N	$\hat{A}$	$\hat{B}$	$\hat{C}$
500	$\begin{pmatrix} 0.92 & 0.32 & 0.03 & -0.01 \\ -0.31 & 0.56 & -0.06 & -0.02 \\ -0.34 & 0.88 & 0.18 & -0.02 \\ -1.51 & 1.59 & 1.52 & -0.01 \end{pmatrix}$	$\begin{pmatrix} 0.67 & 0.37 \\ 1.0 & 0.02 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$	$\begin{pmatrix} 2.44 & -0.67 & -0.07 & -0.61 \\ 1.08 & 0.43 & 0.67 & -0.07 \end{pmatrix}$
1000	$\begin{pmatrix} 0.93 & 0.32 & 0.06 & 0.01 \\ -0.34 & 0.56 & -0.07 & 0.08 \\ -0.20 & 0.81 & 0.27 & -0.03 \\ -1.49 & 1.88 & 1.68 & -0.05 \end{pmatrix}$	$\begin{pmatrix} 0.68 & 0.40 \\ 1.0 & -0.05 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$	$\begin{pmatrix} 2.24 & -0.53 & 0.11 & -0.59 \\ 0.97 & 0.43 & 0.68 & -0.15 \end{pmatrix}$
5000	$\begin{pmatrix} 0.90 & 0.32 & 0.04 & 0.01 \\ -0.22 & 0.58 & -0.19 & 0.02 \\ -0.23 & 0.51 & 0.06 & -0.03 \\ -1.72 & 1.95 & 1.53 & 0.27 \end{pmatrix}$	$\begin{pmatrix} 0.67 & 0.39 \\ 1.0 & 0.07 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$	$\begin{pmatrix} 2.21 & -0.57 & 0.18 & -0.54 \\ 0.96 & 0.43 & 0.62 & 0.0 \end{pmatrix}$
10000	$\begin{pmatrix} 0.90 & 0.31 & 0.05 & 0.0 \\ -0.20 & 0.60 & -0.16 & 0.03 \\ -0.66 & 0.86 & 0.36 & -0.02 \\ -1.82 & 2.00 & 1.63 & -0.03 \end{pmatrix}$	$\begin{pmatrix} 0.68 & 0.41 \\ 1.0 & 0.06 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$	$\begin{pmatrix} 2.14 & -0.57 & 0.17 & -0.53 \\ 0.97 & 0.4 & 0.59 & -0.12 \end{pmatrix}$

VIII. Conclusions

The over-all identification procedure has been developed from input-output data by the aid of the correlation technique and the minimal realization algorithm. As the first step, correlation technique has played an important role in order to estimate Markov parameters by using a known zero mean white random sequence as an input control signal. The final step uses the minimal realization algorithm from Markov parameters, which is called the Ho-Kalman algorithm. Furthermore, an on-line identification scheme has been considered for the practical purpose so that it avoids the large computational burden of an off-line scheme.

As a numerical example, the fourth order system with two inputs and two outputs was considered to show the proposed identification procedure.

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