Finite Element Wave Propagation Analysis in Infinite Region Using Mixed Boundary Conditions

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Abstract

A numerical method for the elastic wave propagation problems involving infinite continuous systems has been proposed. In the method, the fictitious boundary consists from two kinds of mixed boundaries. Results obtined by the proposed method are compared with those with Lysmer's viscous boundaries. It is shown that the 2-step successive averaging method gives results almost the same as those with Lysmer's viscous boundaries.

1. Introduction

One of the analytical studies of blast effects, foundation vibrations and other dynamic problems is to consider them as wave propagation problems in an infinite solid. For the complicated geometrics encountered in practice it is not possible to find closed form solutions and, therefore, it is necessary to resort to numerical methods of the finite difference or finite element method.

With these methods only a finite number of nodal points can be considered; thus the numerical methods are not directly applicable to infinite systems. In a finite system, the reflected wave from the boundary may disturb the numerical results. Sometimes an infinite system may be approximated by a finite system with a special viscous boundary condition[1),2)]. This kind of boundary absorbs some part of the energy arriving at the boundary.

The purpose of this paper is to propose one of transmitting boundaries which consists from two kinds of mixed boundaries. The usefulness and the limitations are shown in some examples.

2. Reflection of waves at the boundary

2–1. One dimensional problem

The wave equation in one dimension is given by

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} \qquad c_0=\sqrt{\frac{E}{\rho}} \qquad \cdots\cdots\cdots(1)$$

The solution of (1) may be written using D'Alembert solution as

$$u=f(x-c_0t)+g(x+c_0t). \qquad \cdots\cdots\cdots(2)$$

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Thus longitudinal wave propagates at the celocity c_1 in a thin rod without distortion. Now consider the domain $x \geqq 0$. The stress field in the rod will be given by

$$\sigma(x, t) = E\{f'(x-c_0t)+g'(x+c_0t)\}$$
$$= \sigma_r(x-c_0t)+\sigma_i(x+c_0t). \qquad \cdots\cdots\cdots(3)$$

Here σ_r and σ_i are the incident and reflected stress waves respectively. Similarly, the velocity field in the rod will be given by

$$V(x, t) = c_0\{-f'(x-c_0t)+g'(x+c_0t)\}$$
$$= \frac{c_0}{E}\{-\sigma_r(x-c_0t)+\sigma_i(x+c_0t)\}. \qquad \cdots\cdots\cdots(4)$$

Now we suppose some force $F(t)$ to be acting on the end of the rod and the end to be moving at the velocity $V(t)$. Then, balancing of force at the end of the rod requires

$$F(t) = -A\{\sigma_r(0, t)+\sigma_i(0, t)\}. \qquad \cdots\cdots\cdots(5)$$

The velocity of the rod tip will be given by

$$V(t) = \frac{c_0}{E}\{-\sigma_r(0, t)+\sigma_i(0, t)\}. \qquad \cdots\cdots\cdots(6)$$

When the rod end is fixed, the boundary condition is given by $V(t)=0$, and the relation between incident and reflected waves is given by

$$\sigma_r(0, t) = \sigma_i(0, t) \qquad \cdots\cdots\cdots(7)$$

On the other hand, when the rod end is free, $F(t)$ becomes zero and we get the relation

$$\sigma_r(0, t) = -\sigma_i(0, t). \qquad \cdots\cdots\cdots(8)$$

2–2. Propagation and reflection of waves in a half space[3),4)]

We consider first the reflection of plane dilatation wave at a boundary $x=0$. The direction of propagation of the incident dilatation wave is taken to be in the xz-plane making an angle α_1 with the x-axis. If we consider a simple harmonic wave in which the displacement normal to the wave front is denoted by Φ_1, we may take

$$\Phi_1 = A_1 \sin(pt+f_1x-g_1z) \qquad \cdots\cdots\cdots(9)$$

where A_1 is the amplitude of the wave and

$$f_1 = p/c_1 \cos\alpha_1 \quad g_1 = p/c_1 \sin\alpha_1$$
$$u_1 = \Phi_1 \cos\alpha_1 \quad w_1 = -\Phi_1 \sin\alpha_1. \qquad \cdots\cdots\cdots(10)$$

The wave is here taken as travelling in the direction of decreasing x and z, (c_1 being the velocity of propagation of the wave).

Now, if dilatation and distortion waves are reflected at an angle α_2 and β_2 to the

x-axis, and their displacements normal to the wave front are Φ_2 and Φ_3 respectively, we have

$$\Phi_2 = A_2 \sin(pt - f_2 x - g_2 z + \delta_1)$$
$$\Phi_3 = A_3 \sin(pt - f_3 x - g_3 z + \delta_2) \quad \cdots\cdots\cdots(11)$$

and

$$f_2 = p/c_1 \cos\alpha_2 \qquad g_2 = p/c_1 \sin\alpha_2$$
$$f_3 = p/c_2 \cos\beta_2 \qquad g_3 = -p/c_2 \sin\beta_2$$
$$u_2 = -\Phi_2 \cos\alpha_2 \quad w_2 = -\Phi_3 \sin\alpha_2 \quad \cdots\cdots\cdots(12)$$
$$u_3 = \Phi_3 \sin\beta_2 \quad w_3 = -\Phi_3 \cos\beta_2.$$

Where c_2 is the velocity of propagation of distortion waves and δ_1, δ_2 allow for any phase change on reflection. Now, we consider the case of mixed boundary conditions, given by

$$u = \tau_{xz} = 0 \quad \text{at} \quad x = 0.$$

Such conditions correspond to an elastic half space constrained by a rigid lubricated boundary.

In order to satisfy the conditions, $\alpha_1 = \alpha_2$ is necessary and with $\delta_1 = \delta_2 = 0$ we have

$$u = u_1 + u_2 + u_3 = (A_1 - A_2)\cos\alpha_1 + A_3 \sin\beta_2 = 0$$
$$\tau_{xz} = 2(A_1 - A_2)\cos\alpha_1 \sin\beta_2 - A_3 \cos 2\beta_2 = 0. \quad \cdots\cdots\cdots(13)$$

Since these equations apply to harmonic wave with any frequency and any angle of incident, $A_1 = A_2$ and $A_3 = 0$ must be satisfied.

Consider the case of another mixed boundary condition given by

$$w = \sigma_x = 0 \qquad \text{at} \quad x = 0$$

Similarly we have

$$(A_1 + A_2)\cos 2\beta_2 \sin\alpha_1 - A_3 \sin\beta_2 \sin 2\beta_2 = 0$$
$$(A_1 + A_2)\sin\alpha_1 + A_3 \cos\beta_2 = 0 \quad \cdots\cdots\cdots(14)$$

In order to apply to the wave with any frequency and any angle of incident, we have $A_2 = -A_1$ and $A_3 = 0$. It is seen that no mode conversion occurs for the dilatation waves incident to these mixed boundary. Now, consider a plane distortion wave impinges on the boundary, we have the amplitudes of incident distortion wave, reflected distortion and dilatation waves as B_1, B_2, B_3 respectively. Then for the case of the boundary conditions $u = \tau_{xz} = 0$ at $x = 0$, we have $B_2 = -B_1$, $B_3 = 0$. And for the case of the boundary conditions $\sigma_x = w = 0$ at $x = 0$, we have $B_1 = B_2$ and $B_3 = 0$. It is seen that no mode conversion occurs for incident distortion wave too.

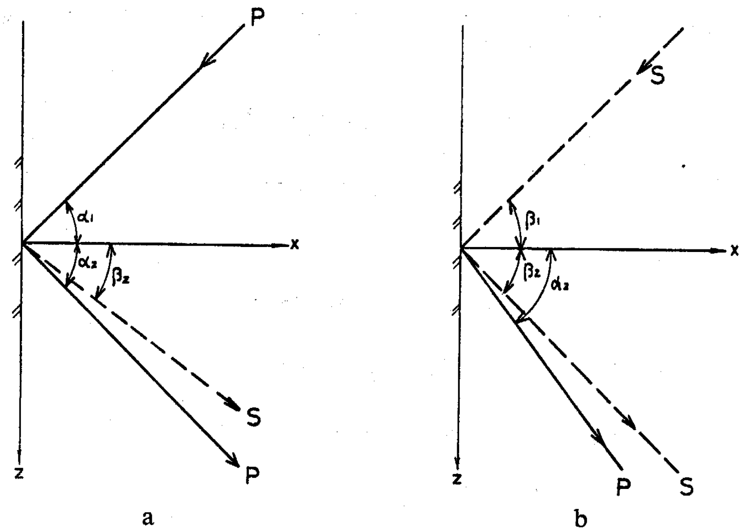


Fig. 1a Reflection of dilatation wave at a boundary.
1b Reflection of distortion wave at a boundary.

3. Transmitting boundaries

3–1. Viscous boundary

In order to get the solution for infinite region through a finite system it is necessary to use the appropriate boundary conditions at the fictitious boundary. We consider a plane dilatation wave travelling in xz-plane making an angle α with the x-axis. The stress field at (x, z) is given by

$$\begin{aligned}\sigma_x&=(\lambda+2\mu)\cos\alpha\frac{\dot{u}}{c_1}-\lambda\sin\alpha\frac{\dot{w}}{c_1}\\ \sigma_z&=\lambda\cos\alpha\frac{\dot{u}}{c_1}-(\lambda+2\mu)\sin\alpha\frac{\dot{w}}{c_1}\\ \tau_{xz}&=-\mu\sin\alpha\frac{\dot{u}}{c_1}+\mu\cos\alpha\frac{\dot{w}}{c_1}\end{aligned}\qquad\cdots\cdots\cdots(15)$$

Where λ and μ are the Lame's constants; $\dot{u}$ and $\dot{w}$ are the velocities in x and z directions respectively. In order to settle a boundary for numerical analysis at (x, z), it is enough to give the stress boundary condition at (x, z) by (15) to pass all energy arriving at the boundary to the exterior region. Therefore, at a boundary x=constant, it is necessary to give above σ_x, τ_{xz} as boundary conditions. For the case of a boundary z=constant, it is necessary to give above σ_z, τ_{xz} as boundary conditions.

When the propagating wave is distortional one, the stress field is given as

$$\begin{aligned}\sigma_x&=(\lambda+2\mu)\cos\beta\frac{\dot{u}}{c_2}-\lambda\sin\beta\frac{\dot{w}}{c_2}\\ \sigma_z&=\lambda\cos\beta\frac{\dot{u}}{c_2}-(\lambda+2\mu)\sin\beta\frac{\dot{w}}{c_2}\end{aligned}\qquad\cdots\cdots\cdots(16)$$

$$\tau_{xz} = -\mu \sin \beta \frac{\dot{u}}{c_2} + \mu \cos \beta \frac{\dot{w}}{c_2}$$

and the boundary conditions are given in similar ways.

It is seen that the boundary stresses σ_x, σ_z, τ_{xz} at transmitting boundary are given as the functions of $\dot{u}$, $\dot{w}$ and α, β. This corresponds to a situation in which the boundary is supported on infinitesimal dashpots making angles given by the functions of α, β, c_1, c_2. As an instance, we take the direction of propagation of incident dilatation wave to make rectangle with z-axis ($\alpha = 0$) and the fictitions boundary to be given by x=constant. Then the velocity $\dot{u}$ only remains and $\dot{w}$ disappears. The boundary stresses are given by

$$\sigma_x = \frac{(\lambda + 2\mu)}{c_1} \dot{u} = c_1 \rho \dot{u}$$
$$\tau_{xz} = 0 \qquad \cdots\cdots\cdots(17)$$

When the incident wave is distortional, the boundary stresses are given by

$$\sigma_x = 0$$
$$\tau_{xz} = \frac{\mu}{c_2} \dot{w} = c_2 \rho \dot{w} \qquad \cdots\cdots\cdots(18)$$

These conditions are the Lysmer's viscous boundary conditions with $a = b = 1$.

3–2. Method of average of reflected waves at two different mixed boundaries

Consider a one-dimensional semi-infinite elastic rod ($x \geqq 0$). When the end of the rod ($x = 0$) is free, the stress field is given by

$$\sigma_1 = \sigma_i(x + c_0 t) + \sigma_{r1}(x - c_0 t)$$

and when the end is fixed

$$\sigma_2 = \sigma_i(x + c_0 t) + \sigma_{r2}(x - c_0 t)$$

σ_{r1} and σ_{r2} are the reflected stresses at free and fixed boundaries respectively. The boundary condition at each end requires

$$\sigma_{r1}(0, t) + \sigma_{r2}(0, t) = 0$$

So, when we superpose the stress fields for rods with free and fixed end, stress field becomes equal to the double of the incident stress.

$$\sigma(x, t) = \sigma_1(x, t) + \sigma_2(x, t) = 2\sigma_i(x + c_0 t) \qquad \cdots\cdots\cdots(19)$$

This means the average of stress fields with fixed and free ends gives the stress field in infinite rod.

Now we extend the similar method into two-dimensional stress propagation problem. Consider a plane dilatation or distortion wave travelling in xz-plane impinges on a boundary. Now, the boundary is a rigid lubricated or another kind of mixed boundaries. The reflected waves at each kind of mixed boundary are eliminated by averaging process.

The method seems to give a transmitting boundary. After travelling some distance, however, the reflected wave at one mixed boundary may incident on another or the same mixed boundary. Then the signs of two reflected waves at two different kind of mixed boundaries coincide again and the averaged stress field contains the effect of reflected waves. So this averaging method is available only until the reflected wave is reflected secondly, and the averaging process corresponds to a extension of the analysing region.

Now, the stress field given by the averaging process of two stress fields is the exact solution of the problem before the second reflection occurs at one of the mixed boundaries. So the stress field may be used as an initial stress field in successive iteration analysis. If we iterate the process, averaging successively the two stress fields before the second reflection occurs, we may get the successive exact solution. The averaged stress field is a part of stress field in infinite stress field and at the boundary the shape of the pulse is step state. So in numerical analysis some modification is necessary in order to simulate the continuous wave train by this step pulse. One of the method to do this is to modify the displacement at wave front. In one dimensional central difference scheme

$$u_j^{n+1}=2u_j^n-u_j^{n-1}+R_j\frac{\Delta t^2}{M_j}$$

the displacement u_j^{n-1} may be modified to

$$u_j^{n-1}\longleftarrow u_j^n-\left(1.0-c_1\frac{\Delta t}{\Delta x}\right)(u_j^n-u_j^{n-1})\,. \qquad \cdots\cdots\cdots(20)$$

Here, M_j is the lumped mass of node j and R_j is the nodal force.

In two dimensional problem, the modifications are given by

$$\left.\begin{aligned} u_j^{n-1}&\longleftarrow u_j^n-\left(1.0-c_2\frac{\Delta t}{\Delta x}\right)(u_j^n-u_j^{n-1})\\ w_j^{n-1}&\longleftarrow w_j^n-\left(1.0-c_1\frac{\Delta t}{\Delta x}\right)(w_j^n-w_j^{n-1})\end{aligned}\right. \qquad \cdots\cdots\cdots(21)$$

for $x=$constant boundary.

This procedure corresponds to the use of a stress pulse with a sloping wave front instead of step like pulse front.

4. Numerical results and discussions

In the numerical analysis of wave propagation problem, Finite Element method of spatial discritization and central difference method for time integration were employed. In the finite element analysis the plane stress rectagular elements and lumped mass method were used. The numerical constants are Young's modulus $E=100.0$, Poission's ratio $\nu=0.25$, density $\rho=1.0$ and time step for numerical integration $\Delta t=0.07$. The space mesh sizes are $\Delta x=\Delta y=1.0$.

4–1. One-dimensional problem

As one-dimensional example, we consider a rod with length$=10.0$ and width $b=1.0$ subjected to a short duration of impulse loading applied at one end. The opposite end is the fictitious one and its condition is free or fixed. We represent the load by

$$P(n)=250.0\left(1.0-\cos\frac{2\pi}{20}n\right) \qquad 0\leqq n\leqq 20$$

$$=0 \qquad n\geqq 20$$

Now, the lateral node displacements are constrained and the problem is one-dimensional. The stress at the centre of the element 5 is shown in Fig. 2. The compression wave impinging on the free end (fictitious end) is reflected as tension wave, and after travelling along the rod it is reflected again at the loading end (free end) as compression wave. The compression wave impinging on the fixed end (fictitious end) is reflected as compression wave, and then reflected as tension wave again at the loading end. These two stress state are opposite in sign between the first and the second reflection at the fictitious end. Average of two stress states in element 5 is shown in the figure and it shows the almost complete elimination of reflected waves in the interval.

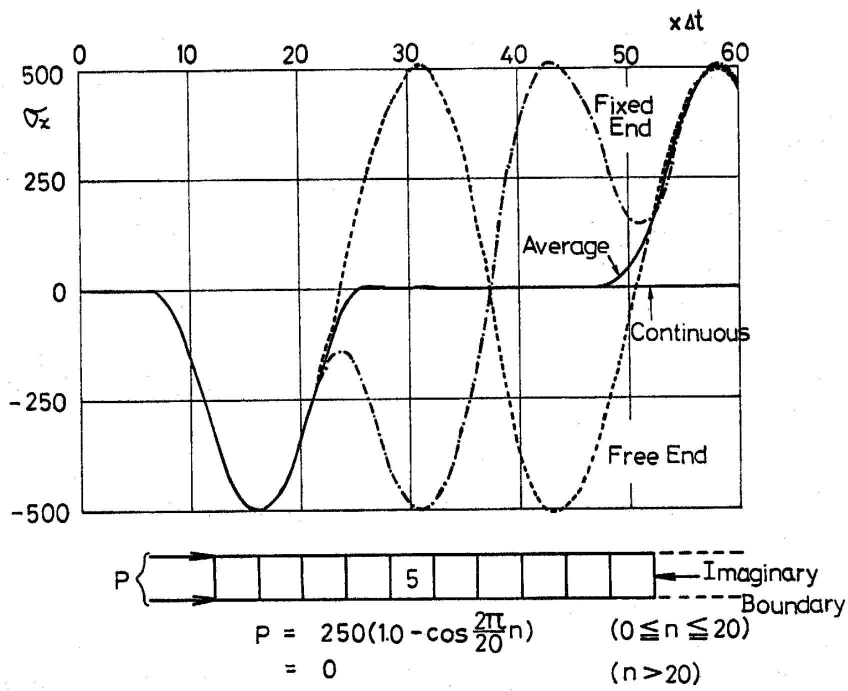


Fig. 2 Stress wave in rod with free or fixed boundary.
($E=100.0$, $\rho=1.0$, $\Delta t=0.07$, $\Delta x=1.0$)

After the second reflection at the same end, however, the reflected wave becomes again visible.

In Fig. 3, the result of successive averaging method, in which the reflected waves by free and fixed boundaries are averaged successively before the second reflections at the fictitious boundaries occur, is shown. In this method, the averaging process may be excuted at any integrating time steps between first and second reflections at the mixed boundaries. 2, 5, 15 in the figure mean the number of time integrating steps between the successive averaging processes. The exact solution is the numericl result in the sufficiently long rod in which the reflected waves from the fictitious boundaries do not come into the result. In one-dimensional problem, the Lymser's method with $a=b=1$ is theoretically exact and the numerical result almost coincides with the exact solution. The result of successive averaging method with every 2 steps iteration is better than the result with every 5 or 15 steps iteration.

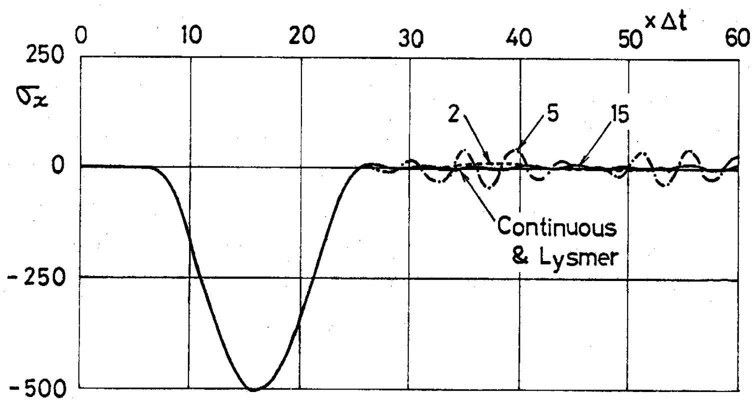


Fig. 3 Comparison of the numerical results of successive averaging method in a rod.

4–2. Two-dimensional problem

As two-dimensional examples followings were examined.

(1) Semi-infinite rectangular plate under concentrated in-plane pulse loading along its symmetrical axis

(2) Semi-infinite plate under concentrated in-plane pulse loading along its symmetrical axis

The difference of above two examples is the number of fictitious boundary planes to be settled. The load is represented by

$$P(n)=500.0\left(1.0-\cos\frac{2\pi}{20}n\right) \qquad 0\leqq n\leqq 20$$

$$=0 \qquad n\geqq 20$$

(1) Semi-infinite rectangular plate; Problem with one fictitious boundary.

In Fig 4, AB, BD, are free and CD is fictitious boundaries respectively. The numerical results are discussed on (i) the displacement u_z at point C, (ii) σ_z in element (a), (iii) τ_{xz} in element (b), (iv) σ_x in element (c). Exact solutions are the numerical results in a long rectangular plate where the reflected waves from the fictitious boundaries do not come into the results.

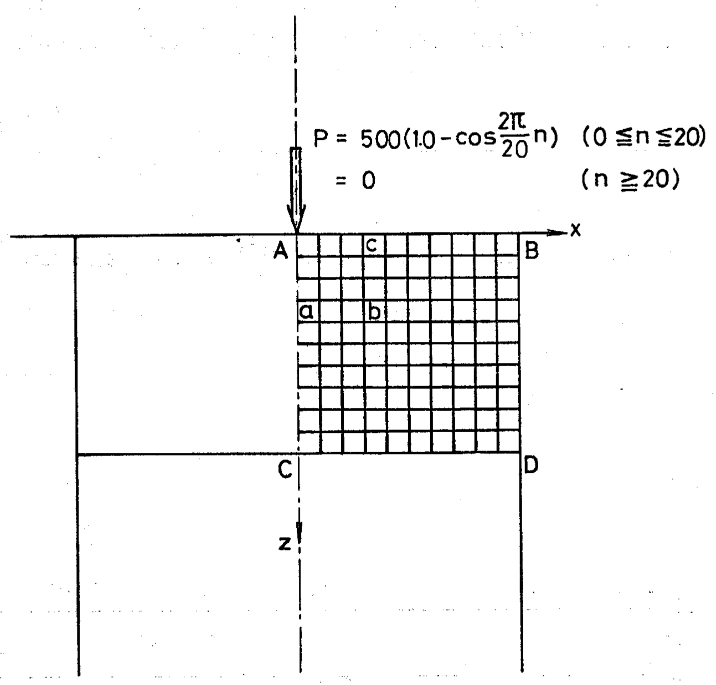


Fig. 4 Semi-infinite rectangular plate and semi-infinite plate subjected to impulse loads.

In Fig 5, σ_z in element (a) by the direct average method is shown. With each mixed boundary condition (A and B), the stress in the element shows complicated wave train. And with the averaging process, the stress coincides with the exact solution until the secondly reflected waves at the mixed boundaries come into the element. This means the direct averaging method corresponds to the double extension of the analysing region[5].

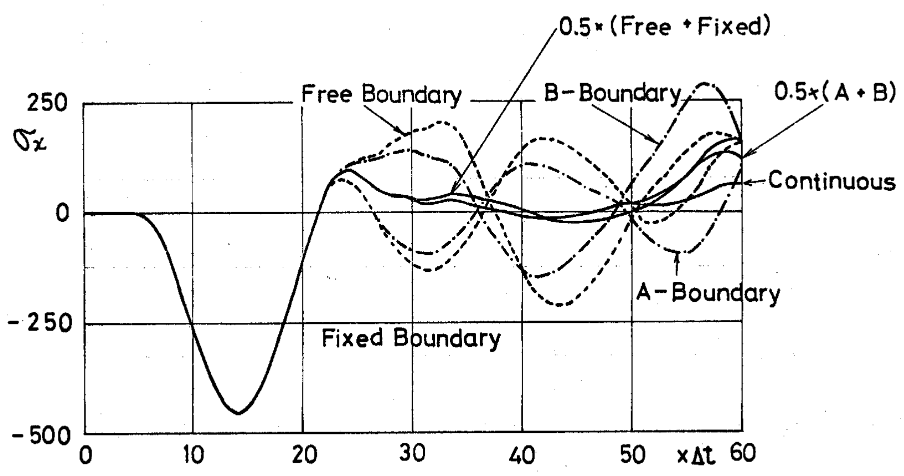
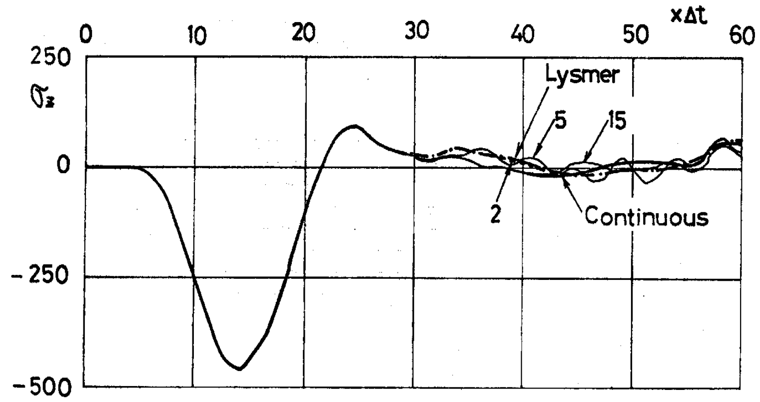
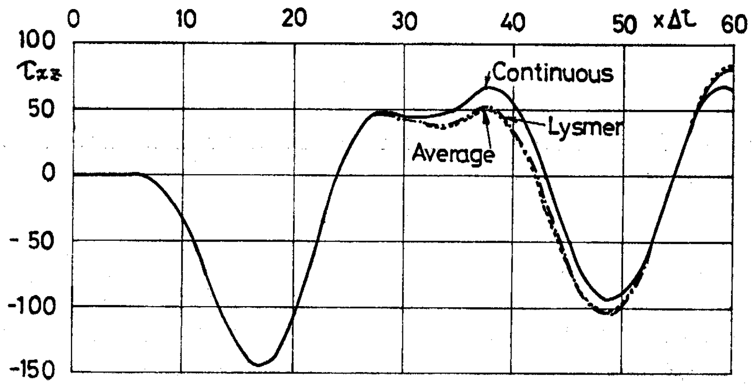


Fig. 5 Stress wave propagation and reflection in a semi-infinite rectangular plate and the result by direct average method.

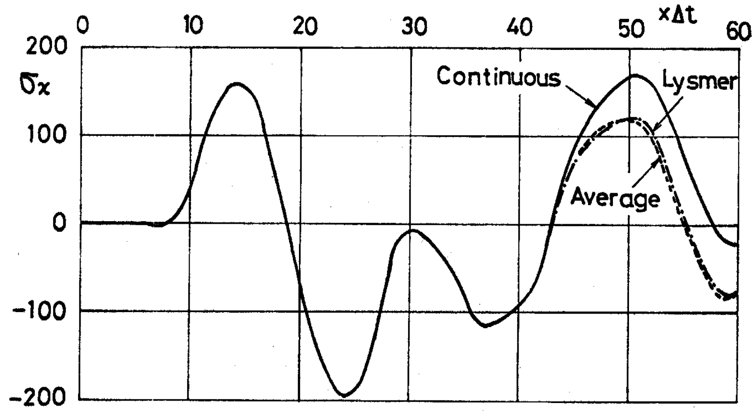
In Fig 6a, the stresses in element (a) are compared. With every approximete method there seem some deviations of numerical results from the exact result. It is interesting that the result of successive averaging method shows a similar result as the Lysmer's result. The results of successive averaging method with 5 or 15 iteration steps



6a



6b



6c

Fig. 6 Comparison of stresses in a semi-infinite rectangular plate.

a. σ_z in element (a)

b. τ_{xz} in element (b)

c. σ_x in element (c)

show some vibration and it seem that the reflection wave eliminating process by averaging does not work well. In Fig 6b, 6c and in Fig 7, we can see the same inclinations as in Fig 6a. These results say that the 2-step successive averaging method gives a almost the same result as Lysmer's one.

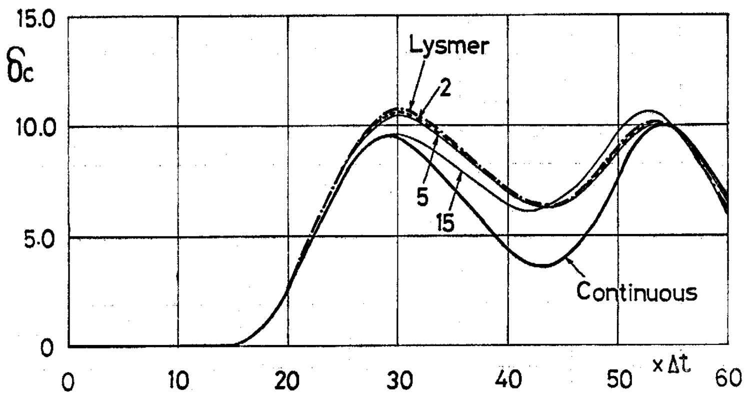


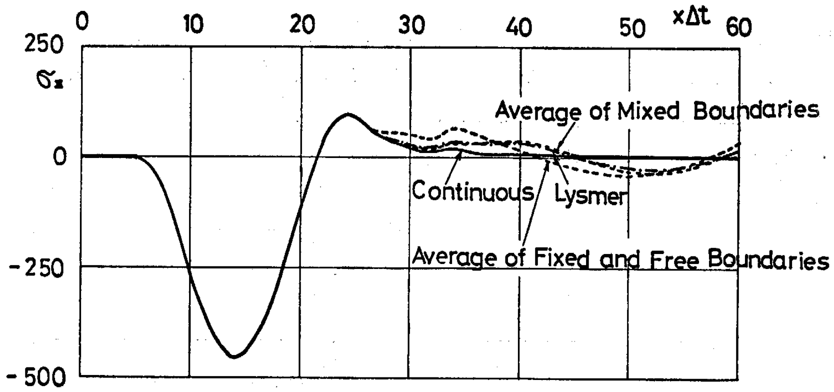
Fig. 7 . Time dependent displacement of node C in Fig. 4.

(2) Semi-infinite plate; Problem with two fictitious boundaries.

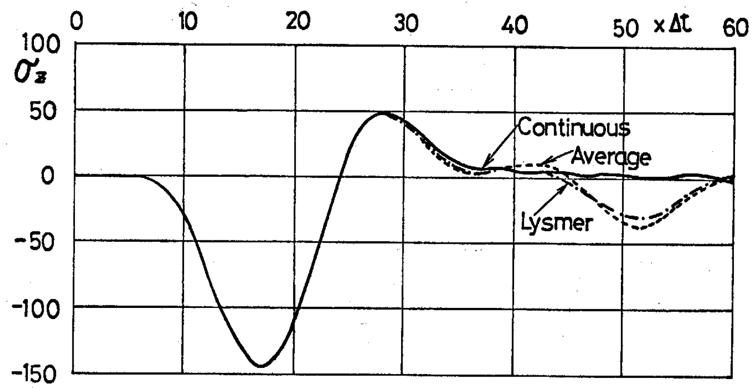
In this problem the boundary BD is the fictitious one in Fig 4. Now, the wave reflection by fictitious boundary occurs at CD and BD. This means a reflected wave at CD at a integrating time step may incident on BD at the next step. In successive averaging method, the reflected waves at the fictitious boundaries must be averaged before they incident on the same or another fictitious boundaries. This needs the averaging process at every time integrating step. If the averaging process is executed at every two or more steps, the non-eliminated wave may come into the results. In this problem, therefore, successive averaging method with one and two time integrating steps are discussed.

In Fig 8a, σ_z in element (a) is compared. In this problem the result of 2-step successive averaging method and Lysmer method give the almost same results. The 1-step successive averaging method means the averaging method at every time integration step[6]. By the central difference time integrating method, wave travels only 1 space mesh during 1 step time interval. This means all node displacements except those on the fictitious boundaries coincide with for two regions with different mixed boundary conditions. The successive averaging method with free and fixed boundaries give the same result as those of 1-step successive averaging method with two kinds of mixed boundaries. The result of this method shows a little larger error than that of 2-step successive averaging method.

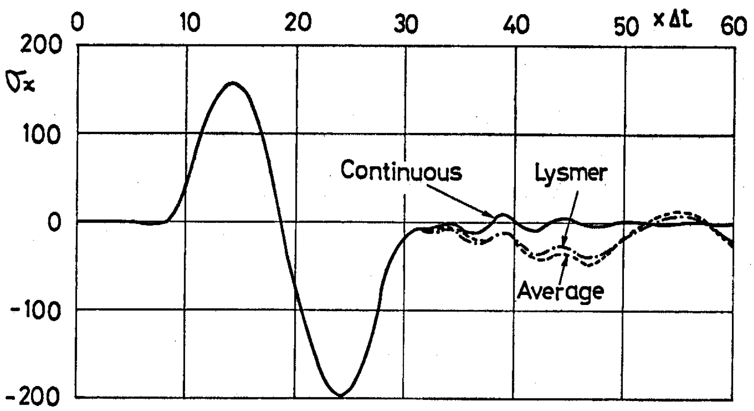
In Fig 8b, 8c τ_{xz} in element (b) and σ_x in element (c) are shown. They show the resemblance between the results of 2-step successive averaging method and those of Lysmer's method.



8a



8b



8c

Fig. 8 Comparison of stresses in a semi-infinite plate subjected to impulse load.
a. σ_z in element (a)
b. τ_{xz} in element (b)
c. σ_x in element (c)

5. Conclusions

A numerical method for the elastic wave propagation problems involving infinite continuous systems has been proposed. In the method, the fictitious boundary consists from two kinds of mixed boundaries. Results obtained by the proposed method are compared with those with Lysmer's viscous boundaries.

It is shown that the proposed averaging method eliminates the energy of waves arriving at the fictitious boundary. In the case when the reflected waves at the fictitious boundary may impinge on the same or another fictitious boundary, the method needs a successive averaging process. The 2-step successive averaging method gives results almost the same as those with Lysmer's viscous boundaries.

In the successive averaging process, the shape of the wave in the analysed finite region is a stress pulse with a step state wave front and it is a part of continuous wave train. In order to simulate a infinite wave train by a part of it, it is necessary to modify the shape of the pulse. In the modification of this study, the pulse with sloping wave front is used. With a improvement of this modification, the better results will be expected.

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