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Three Dimensional Eddy Current Analysis by Finite Element Method Using Vector and Scalar Elements

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Abstract

This paper describes a finite element formulation of full Maxwell's equations in terms of a vector potential and a scalar potential and its application to eddy current problems. The vector potential and the scalar potential are approximated by vector basis functions and scalar basis functions, respectively. A gauge fixing of the potentials is translated into the regularization of the indeterminate linear equation finally obtained by the Galerkin procedure. Two convenient gauge conditions for three dimensional eddy current problems are presented. By applying the computer code to test problems, this finite element formulation and the gauge conditions are proved to be applicable to linear electromagnetic problems.

1. Introduction

Finite element methods have been widely applied to solutions of electromagnetic problems. In the finite element analysis, the electromagnetic field in the static or low-frequency range usually is expressed in terms of magnetic vector potential **A** and electric scalar potential ϕ [1-3] or in terms of current vector potential **T** and magnetic scalar potential Ω[4], while the electromagnetic field in the high-frequency range is expressed in term of electric or magnetic field [5,6]. If the vector field or the vector potential is approximated by traditional elements satisfying continuity of three components of the vector field, the number of unknowns increases in the low-frequency eddy current analysis [3] and spurious solutions appear in the high-frequency eigen value analysis [5,6]. On the other hand, it has been reported that the vector elements satisfying continuity of tangential components of the vector field suppress the spurious solutions [7,8]. These elements too are applied to solve the magnetostatic field problems [9].

In this paper, the finite element formulation of full Maxwell's equations in terms of the magnetic vector potential and the electric scalar potential is presented. In the present formulation, the vector potential and the scalar potential are discretized by the vector elements and the usual scalar elements, respectively. The gauge fixing of the electromagnetic potentials is discussed and two convenient gauge conditions are proposed. The computer code for static and sinusoidal time dependent problems has been developed and applied to low-frequency eddy current problems.

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2. Full Maxwell's Equations and Electromagnetic Potentials

The domain, Ω, bounded by the electric wall, Γ_e and/or the magnetic wall, Γ_m , is considered. Full Maxwell's equations for sinusoidal time dependence with electric current sources
($\mathbf{J}$, ρ) in Ω are given by

$$rot\ \mathbf{H}= j\omega\mathbf{D}+\mathbf{J} \tag{1}$$
$$rot\ \mathbf{E}= -j\omega\mathbf{B} \tag{2}$$
$$div\ \mathbf{B}= 0 \tag{3}$$
$$div\ \mathbf{D}= \rho \tag{4}$$

where
- $\mathbf{E}$ is the electric field in V/m,
- $\mathbf{D}$ is the displacement flux density in C/m²,
- $\mathbf{H}$ is the magnetic field in A/m,
- $\mathbf{B}$ is the magnetic flux density in Wb/m²,
- $\mathbf{J}$ is the electric current density in A/m², and
- ρ is the electric charge density in C/m³.

The variables are related by the following linear constitutive relationships

$$\mathbf{D}=(\varepsilon-j\sigma/\omega)\mathbf{E} \tag{5}$$
$$\mathbf{B}=\mu\mathbf{H} \tag{6}$$

where
- ε is the real permittivity in F/m,
- μ is the permeability in H/m, and
- σ is the conductivity in S/m.

The term of induced current density

$$\mathbf{J}_e=\sigma\mathbf{E} \tag{7}$$

appears on the right side of Eq.(1), and the term of induced electric charge density

$$\rho_e=(j/\omega)\ div\ \mathbf{J}_e \tag{8}$$

appears on the left side of Eq.(4). Therefore, $\mathbf{J}_e$ and ρ_e satisfy the law of charge.

The magnetic flux density can be expressed in term of the magnetic vector potential $\mathbf{A}$.

$$\mathbf{B}=rot\ \mathbf{A} \tag{9}$$

From Eq.(2), it follows that the electric field of Eq.(2) differs from the magnetic vector potential by an irrotational vector gradϕ, ϕ being the electric scalar potential.

$$\mathbf{E}=-j\omega\mathbf{A}-\mathrm{grad}\phi \tag{10}$$

It is well known that the potentials $\mathbf{A}$ and ϕ, satisfy Eqs.(9) and (10) for the given $\mathbf{B}$ and $\mathbf{E}$, are not unique. Eqs.(9) and (10) are invariant to the following gauge transformation.

$$\mathbf{A}'=\mathbf{A}-\mathrm{grad}\ \chi \tag{11}$$

$$\phi' = \phi - j\omega\chi \tag{12}$$

where χ is a scalar potential similar to ϕ.

At the surface of material discontinuity, the following four differential equations are obtained from Eqs. (1) - (4)

$$\mathbf{n}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0} \tag{13a}$$
$$\mathbf{n}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{13b}$$
$$\mathbf{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{14a}$$
$$\mathbf{n}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s \tag{14b}$$

Then, on Γ_e

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \tag{15a}$$
$$\mathbf{n} \cdot \mathbf{B} = 0 \tag{15b}$$

and, on Γ_m

$$\mathbf{n} \times \mathbf{H} = \mathbf{J}_s \tag{16a}$$
$$\mathbf{n} \cdot \mathbf{D} = \rho_s. \tag{16b}$$

where

$\mathbf{J}_s$ is the surface current density in A/m, and

ρ_s is the surface charge density in C/m².

The boundary condition of Eq. (13) for the potentials of $\mathbf{A}$ and ϕ becomes

$$\mathbf{n}_{12} \times (\mathbf{A}_2 - \mathbf{A}_1) = \mathbf{0} \tag{17a}$$
$$\phi_2 - \phi_1 = 0\,. \tag{17b}$$

Then, on Γ_e

$$\mathbf{n} \times \mathbf{A} = \mathbf{0} \tag{18a}$$
$$\phi = 0\,. \tag{18b}$$

3. Hilbelt Spaces for Electromagnetic Potentials

Now, let us take a view of the gauge transformation from an angle on function space [10]. Since the rotation operator act to both $\mathbf{A}$ and $\mathbf{A'}$ of Eq. (11), they must be different functions of the same space. According to the linearity of Eq. (11), the space must contain gradϕ in itself. Let us define two function spaces with the L_2 norm

$$H_B\ (rot, \Omega) = \{\mathbf{A} \in \{L_2\}^3;\ rot\ \mathbf{A} \in \{L_2\}^3,\ \mathbf{n} \times \mathbf{A} = \mathbf{0}\ \text{on}\ \Gamma_e\} \tag{19}$$
$$H_B\ (grad, \Omega) = \{\phi \in L_2;\ grad\phi \in \{L_2\}^3,\ \phi = 0\ \text{on}\ \Gamma_e\} \tag{20}$$

where L_2 and $\{L_2\}^3$ are the spaces of square integrable scalar field and vector field, respectively, and the subscript B denotes the given boundary conditions. The following null space is a subspace of $H_B\ (rot, \Omega)$.

$$H_B\ (rot^0, \Omega) = \{\mathbf{A} \in \{L_2\}^3;\ rot\mathbf{A} = 0,\ \mathbf{n} \times \mathbf{A} = \mathbf{0}\ \text{on}\ \Gamma_e\} \tag{21}$$

According to the vector identity of $rot(grad\phi) = 0$, the spaces of $H^B\ (rot, \Omega)$, $H_B\ (grad, \Omega)$ and $H_B\ (rot^0, \Omega)$ satisfy the following relationship

$$\text{grad}H_B\ (grad, \Omega) = H_B\ (rot^0, \Omega) \in H_B\ (rot, \Omega) \tag{22}$$

For the finite element formulation of full Maxwell's equation, it is necessary to choose the finite dimensional subspaces $\bar{H}_B$ (rot, Ω), $\bar{H}_B$ $(grad, \Omega)$ and $\bar{H}_B$ (rot^0, Ω) from the spaces H_B (rot, Ω), H_B $(grad, \Omega)$ and H_B (rot^0, Ω), respectively. These subspaces must be carefully selected so as to satisfy the following relationship such as Eq. (22).

$$\mathrm{grad}\bar{H}_B\ (grad, \Omega) = \bar{H}_B\ (rot^0, \Omega) \in \bar{H}_B\ (rot, \Omega) \tag{23}$$

Several examples of novel vector elements are shown in Fig.1.

In Fig.1, the dark points denote the positions of unknowns of the scalar potential in the usual scalar elements and the arrows denotes the position and direction of unknowns of the vector potential in the novel vector elements. These elements always satisfy the essential physical boundary conditions, $\mathbf{n}\times\mathbf{E}$=continuous and $\mathbf{n}\cdot\mathbf{B}$= continuous, over Ω.

4. Finite Element Formulation

Let us approximate $\mathbf{A}$ and ϕ over Ω as following forms

$$\mathbf{A} = \sum_{i=1}^{N_v} \mathbf{a}_i A_i \tag{24}$$

$$\phi = \sum_{m=1}^{N_s} \nu_m \Phi_m \tag{25}$$

where $\mathbf{a}_i$ and ν_m are the vector and scalar basis functions, A_i and Φ_m are unknown parameters, and N_v and N_s are the number of unknowns of vector and scalar potentials. To obtain the finite element formulation by the Galerkin procedure, Eqs. (1) and

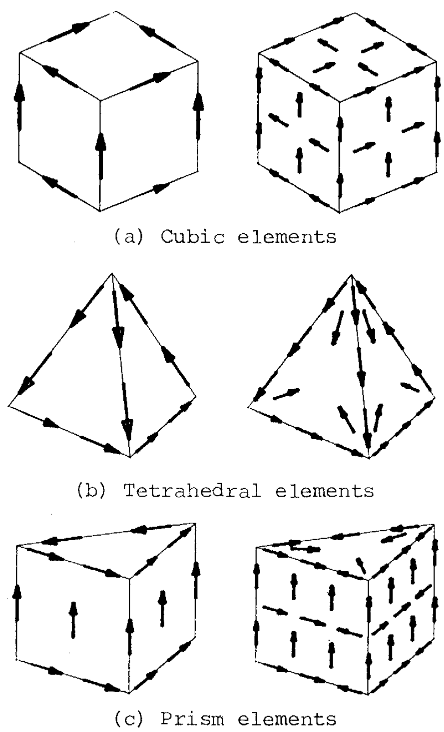

Fig. 1 Vector elements.

(4) rewritten with Eqs. (5), (6), (9) and (10) are multiplied by the vector test function $\mathbf{a}_i$ are the scalar test function ν_m, respectively, and integrated over Ω. After the partial integration, the final linear equation can be obtained as a following form

$$\begin{bmatrix} K-\omega^2 M & j\omega C \\ -j\omega C^T & -S \end{bmatrix} \begin{Bmatrix} A \\ \Phi \end{Bmatrix} = \begin{Bmatrix} P \\ -Q \end{Bmatrix} \tag{26}$$

where the elements of the matrices are given by

$$\begin{aligned} K_{ij} &= \int rot\ \mathbf{a}_i \cdot \mu^{-1}\ rot\ \mathbf{a}_j\ d\Omega \\ M_{ij} &= \int \mathbf{a}_i \cdot (\varepsilon - j\sigma/\omega)\ \mathbf{a}_j\ d\Omega \\ C_{in} &= \int rot\ \mathbf{a}_i \cdot (\varepsilon - j\sigma/\omega)\ grad\ \nu_n\ d\Omega \\ S_{mn} &= \int grad\ \nu_m \cdot (\varepsilon - j\sigma/\omega)\ grad\ \nu_n\ d\Omega \\ P_i &= \int \mathbf{a}_i \cdot \mathbf{J}\ d\Omega \\ Q_m &= \int \nu_m\ \rho\ d\Omega \end{aligned} \tag{27}$$

According to the Eq. (23), the linear equation of Eq. (26) is always singular in the case of $\sigma=0$ over Ω, and it has the N_s-fold indeterminate solution if and if only the sources, $\mathbf{J}$ and ρ, satisfy the law of conservation of electric charge.

5. Convenient Gauge Conditions

The formulation in the previous section can translate the gauge fixing of the differential equations of Eqs. (1) and (4) rewritten by $\mathbf{A}$ and ϕ into the regularization of the indeterminate linear equation of Eq. (26).

When $\omega>0$, two efficient gauge conditions can be found in order to uniquely solve Eq. (26). The first gauge condition is

$$\{\Phi\} = \{0\}\ . \tag{28}$$

The condition of Eq. (28) is convenient to analyze problems with only source $\mathbf{J}$. Another gauge condition is

$$\{A^*\} = \{0\} \tag{29}$$

which number of element is N_s. The elements of Eq. (29) are chosen by the pivot selection of Eq. (26). The condition of Eq. (29) is effective to analyze problems with scalar potential source. In the eddy current problems of $f\sim100$Hz and somewhere $\sigma\sim10^7$ S/m, the condition of Eq. (29) as the approximate gauge condition can be imposed on Eq. (26) with the gauge condition of Eq. (28).

When $\omega=0$, Eq. (26) is separated into two linear equations, namely, magnetostatic and electrostatic ones. Then, the magneto-static equation need the gauge condition of Eq. (29) for uniqueness of solution.

6. Examples of Application

Some results on certain problems proposed in TEAM WORKSHOPS [11] are presented below.

6.1 *Asymmetrical conductor with a hole*

The model is shown in Fig.2. A thick aluminum plate with a hole, which is placed eccentrically, is set asymmetrically in a non-uniform magnetic field. The field is produced by the exciting current which varies sinusoidally with time. The analyzed region is subdivided into $29\times 27\times 14$ cubic elements. The outermost boundary condition of the analyzed region is assumed to be Γ_e. The linear equation of Eq.(26) with the gauge condition of Eq.(28) is solved by ICCG method.

The eddy current distribution in the aluminum conductor at the frequency of 50Hz is shown in Fig.3. It can be seen that the eddy current flows on avoiding the hole. In the $\mathbf{T}-\Omega$ formulation, the hole must be filled by a medium of low conductivity and it is always difficult to determine what conductivity should be used. The present method does not cause this kind of difficulty. The weak skin effect can be seen in the Fig.3. The z-component of the magnetic flux density along the line of $y=72$mm and $z=34$mm between the exciting coil and the aluminum plate at the frequency of 50Hz is shown in Fig.4. The magnetostatic flux density is shown in Fig.4, too. The flux density at 50

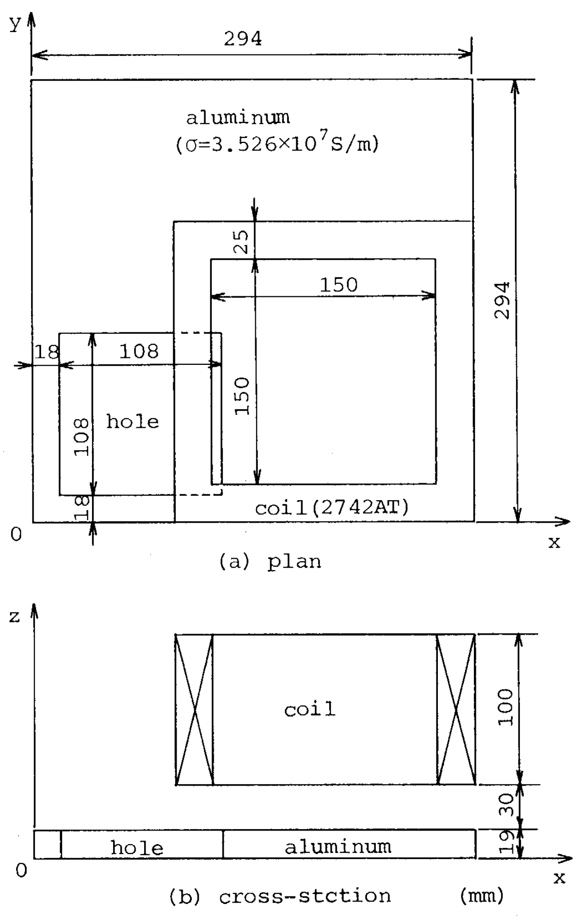

Fig. 2 Asymmetrical conductor with a hole.

Hz decreases by approximately one-half against the static one because of the eddy current in the conductor. Figure 5 shows the flux density at 200Hz. The flux density at 200Hz decreases further.

6.2 *The Bath cube problem*

The model is shown in Fig. 6 . Four aluminum blocks situated under a laminated iron pole are placed in a laminated iron box. The excitation is realized by sinusoidal magnetic mortive force (MMF) with a frequency of 50Hz. In order to solve the

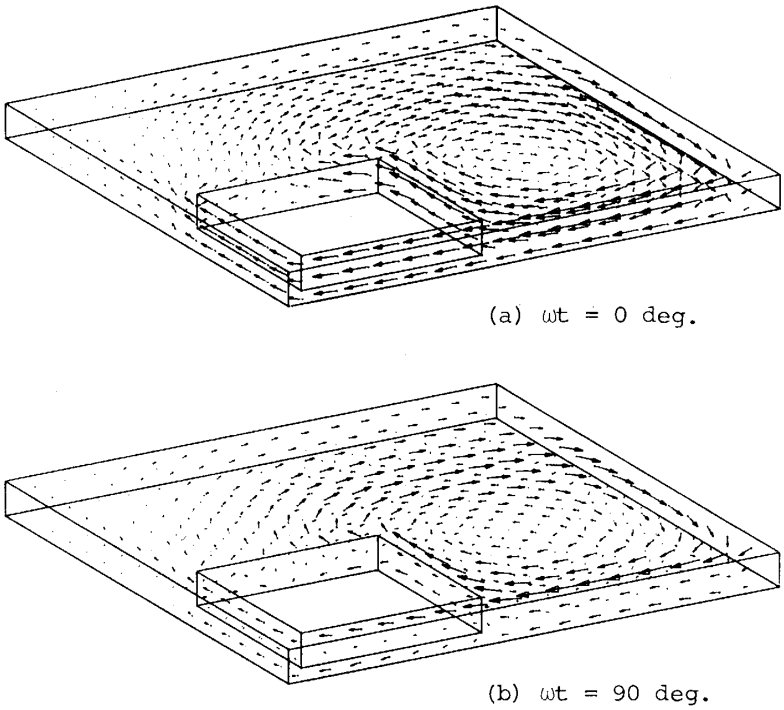

Fig. 3 Eddy current distribution in an asymmetrical conductor with a hole at f=50Hz. $J_{max}=1.57\times 10^6$ A/m^2.

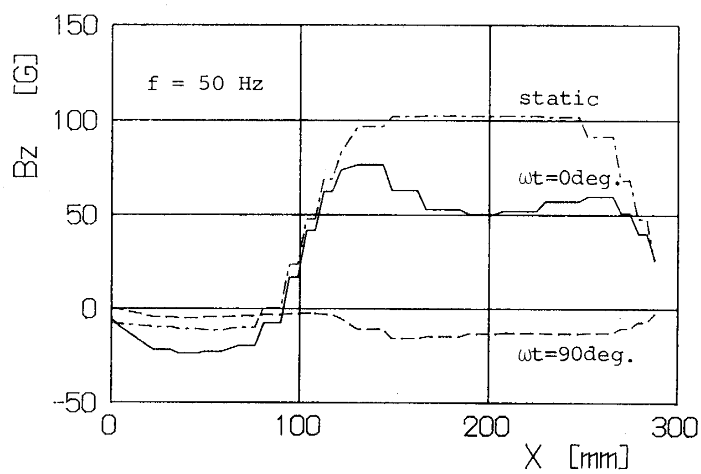

Fig. 4 B_z along the line of y=72mm and z=34mm at f=50Hz.

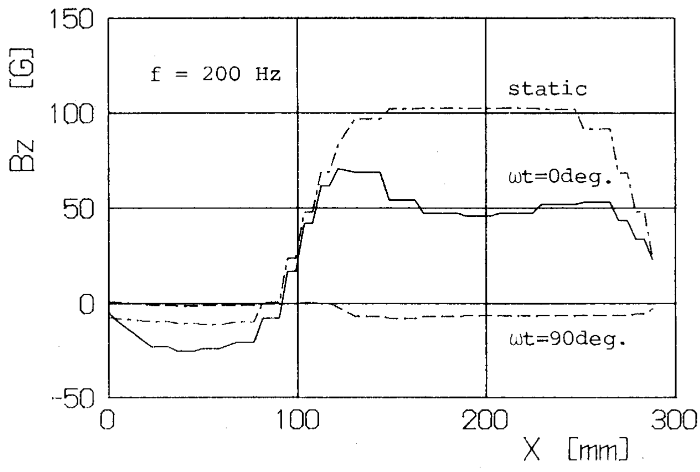

Fig. 5 B_z along the line of y=72mm and z=34mm at f=200Hz.

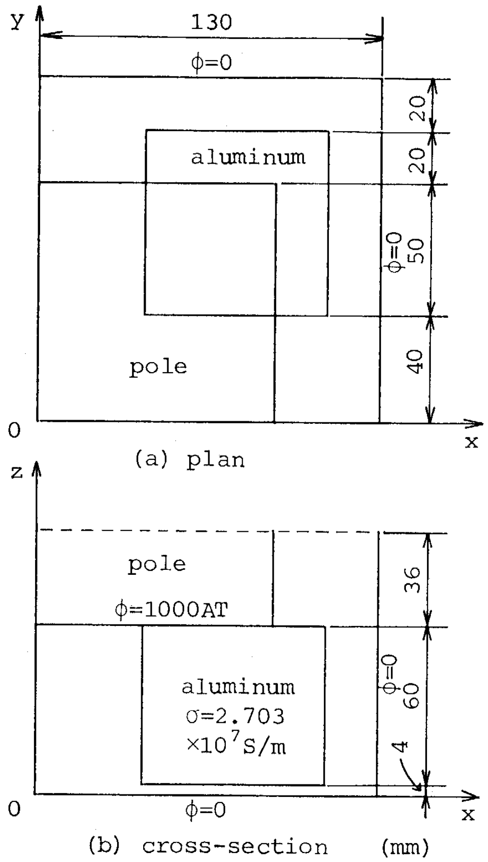

Fig. 6 The Bath cube problem.

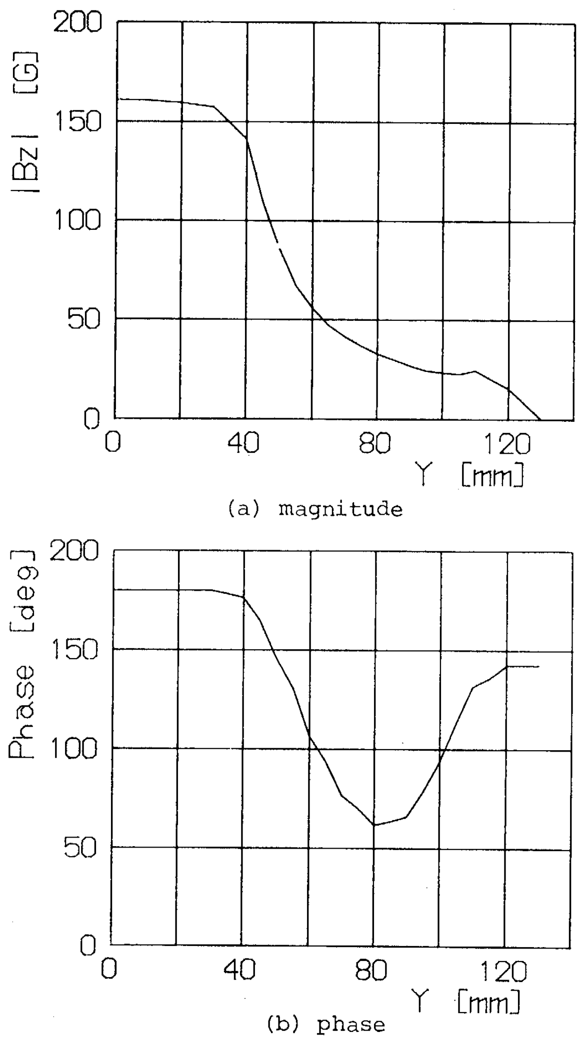

Fig. 7 The field component B_z along the line of x=70mm and z=2mm at 50Hz.

problem exciting with MMF, another formulation in terms of a electric vector potential $\mathbf{A}_e$ and a magnetic scalar potential ϕ_m is performed. However, it is very similar to the formulation of Eq.(26). The analyzed region is subdivided into 13×13×11 elements. The laminated iron box is magnetically grounded and the MMF of 1000AT is applied to the laminated iron pole. Two planes of symmetry of Fig. 6 is assumed to be Γ_m. The total number of unknowns without any gauge condition becomes 5636.

The z-component of the magnetic flux density along the line of x=70mm and z=2 mm at the frequency of 50Hz is shown in Fig. 7. These agree well with the results obtained by $\mathbf{T}-\Omega$ method and the experiment [4]. The $\mathbf{T}-\Omega$ formulation is restricted to the low frequency problem, but the present formulation can be utilized to solve high frequency problems.

7. Conclusion

The finite element formulation of full Maxwall's equations interms of the vector potential and the scalar potential has been presented. The vector elements and the scalar elements are used to approximate the vector potential and the scalar potential, respectively. Two gauge conditions are proposed to solve problems uniquely or economically. The only one computer code of present formulation can make itself solve from static problems to high frequency problems.

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