SINGULARITY AND DISTRIBUTION OF STRESSES AT AND AROUND TIP OF A SHALLOW CRACK

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Complex variable anlyses are applied to the plane elastic problems of semi-infinite sheet with a mode I edge crack of a cups-type morphology. Distributions of the stresses around the tip of the edge crack are derived and the stress intensity factor for the crack is determined. The surface effect upon the stress intensity K_I is shown to be given by a factor $\beta = 3 \cdot 2^{-3/2}$ in $K_I = \beta\ \sigma\ [\pi c]^{1/2}$, where σ and c are the mode I stress applied at infinity and the surface crack depth, respectively.

Keywords: *Stress intensity, Edge crack of a cusp type, Mode I, Surface effect, Complex potentials.*

1. INTRODUNCTION

There have been a few works referring to the significance of a surface crack, or an edge crack in a two dimensional representation, from its crack tip sigularity viewpoint[1][2]. Discussions developed up to date, however, have been confined to the problem of a classical or straight edge crack with well-defined square corners, even where a shallow crack problem remains difficult today, to which very limitted attempt of solution in regorous manner has been given[1]. From an engineering viewpoint, on the other hand, the surface or edge crack which plays an important role in controling crack propagation life and the integrity of structures shows itself many times as a rather shallow crack, especially in the initial stage of a stress-corrosion cracking process literally acompanied by a corrosive dissolution of the boundary, with not so well-defined or rounded corners. For this reason an understanding of the stress singularity behavior of the surface or edge crack is of a comparable or even greater importance than that of an internal crack from engineering as well as intellectual interests. Thus, in the edge crack mentioned here a shallow crack which is observed in stress-corrosion crackings, including those of a cusp-type morphology or with rounded corners, is involved.

Further, more importantly, the tip of such an edge crack might be expected to produce an enhanced stress intensity as compared with an internal crack of half crack length equal to the edge crack depth, if we assume that the physical volume around the edge crack tip should be in charge of the elastic energy relaxed due to the presence of the stress-free surface. Equating the relaxed energy with (K_I^2 - K_{Ii}^2)/E times the effective crack-affected volume will give an approximate estimate of the stress intensity, K_I, which is enhanced by the presence of the stress-free surface as compared with a stress intensity, K_{Ii}, of internal crack with a half crack length equal to the surface crack depth.

In this work the amplitude of singularity and distributions of the stresses at and around the tip of an edge crack of a cusp-type morphology will be examined in an analytically exact manner, and the influence of the stress-free surface upon them will be discussed.

2. FORMULATION OF THE PROBLEM

For the intended discussions the semi-infinite sheet under tension with an edge crack of a cusp-type morphology, delineated in Figure 1, is considered. Let the sheet lie in the upper portion of complex z-plane, z = x + iy, with the tip of the crack being located at z = ic. For analyses we make use of a couple of complex potentials, known as Gursat's functions of complex variable z, namely ϕ (z) and χ (z), which are arbitrarily chosen analytic functions and compose a bi-harmonic function, known as Airy's stress function,.

$$F(z) = Re[\bar{z}\,\phi(z) + \int^{z} dz\,\chi(z)], \qquad (1)$$

The boundary condition consideration will be facilitated, if we introduce such an auxiliary complex plane, the ζ -plane, $\zeta = \xi + i\eta$, illustrated in Figure 2, that the real axis and the upper-half plane map into the boundary and the upper physical region shown in Figure 1 by a mapping function $\omega(\zeta)$,

$$z = \omega(\zeta) = c\{\zeta - 1/(\zeta + i)\}. \quad (2)$$

In determining the function $\omega(\zeta)$, we provide the conditions,

$$\omega(0) = ic, \text{ and}$$

$$Im\ \omega(Re\ \zeta) \text{ vanishes at infinity.} \quad (3)$$

Differentiating equation(2) yields

$$\omega'(\zeta) = c\zeta(\zeta + i2)/(\zeta + i)^2, \quad (4)$$

from which we confirm that $\omega(\zeta)$ is analytic in the upper-half plane, $Im\ \zeta > 0$. Crack tip is described by $\omega'(\zeta) = 0$, which occurs at $\zeta = 0$ in the ζ-plane. Primes are used to denote differentiation by the variable shown in the parentheses. Thus, $f'(z) = f'(\zeta)/\omega'(\zeta)$, where, to minimize notation, we are designating $f(z) = f[\omega(\zeta)]$ as $f(\zeta)$. Stresses, σ_x, σ_y and τ_{xy}, and displacements, u and v, in rectangular coordinates can in this way be written as

$$\sigma_x + \sigma_y = 2\phi'(\zeta)/\omega'(\zeta) + \text{comp.conj.} \quad (5)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\ [\overline{\omega(\overline{\zeta})} \times d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta + \chi'(\zeta)]\ /\omega'(\zeta) \quad (6)$$

$$2\mu(u - iv) = \kappa\ \overline{\phi(\zeta)} - \overline{\omega(\zeta)}\,\overline{\phi'(\zeta)}/\overline{\omega'(\zeta)} - \overline{\chi(\zeta)}, \quad (7)$$

where μ and κ are elastic constants of the material, and bars denote complex conjugates.

In terms of the functions $\phi(\xi)$ and $\chi(\xi)$ the load-free boundary condition on the line, $\eta = 0$, delineated in Figure 1, can be written as

$$\phi(\xi) + \omega(\xi)\overline{\phi'(\xi)}/\overline{\omega'(\xi)} + \overline{\chi(\xi)} = \text{constant}, \quad (8)$$

since components, $\int_P^Q dsX$ and $\int_P^Q dsY$, of the resultant force on an arc PQ of an arbitrary curve with element ds is expressed in terms of $\phi(\zeta)$ and $\chi(\zeta)$ as[3]

$$\int_P^Q ds\ (X + iY) = -i\ [\phi(\zeta) + \omega(\zeta)\overline{\phi'(\zeta)}/\overline{\omega'(\zeta)} + \overline{\chi(\zeta)}]_P^Q, \quad (9)$$

and this equation is applicable to the boundary of Figure 1, where s can be considerd as a function of ξ. Thus, the solution of the problem is attributed to determining the functions $\phi(\zeta)$ and $\chi(\zeta)$ which are analytic in $Im\ \zeta > 0$ and satisfy the appropriate conditions.

If the z-plane is described by polar coordinates, r and θ, with pole at the notch tip and θ the counter-clockwise angle with the y-axis, then

$$z = \omega(\zeta) = i(c + re^{i\theta}). \quad (10)$$

Thus, from equation(2), ζ is related to r and θ as

$$c\zeta = -ire^{i\theta}/2 + [-re^{i\theta} - r^2e^{i2\theta}/4c]^{1/2}. \quad (11)$$

3. STRESS FIELDS AT AND AROUND TIP OF A SHALLOW CRACK

In terms of the formulations carried out in the previous section we will examine the essential character of the stresses, namely singularity and distributions of the stresses induced at and around the tip of the crack, which will be shown to be influenced by the presence of traction-free boundaries.

Examination of equation(8) shows that if the function $\chi(\zeta)$ is so chosen in the upper-half plane, $Im\ \zeta > 0$, that

$$\chi(\zeta) = -\overline{\phi}(\zeta) - \overline{\omega}(\zeta)\phi'(\zeta)/\omega'(\zeta), \quad \text{for } Im\ \zeta > 0, \quad (12)$$

then $\chi(\zeta)$ is analytic there and satisfies the boundary condition(8), disregarding the constant which is responsible for a rigid body displacement. Noticing that differentiation of equation(12) yields

$$\chi'(\zeta) = -\overline{\phi}'(\zeta) - \overline{\omega}'(\zeta)\phi'(\zeta)/\omega'(\zeta) + \overline{\omega}(\zeta)d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta, \quad (13)$$

we find that equation(6) reduces to

$$\sigma_y - \sigma_x + i2\tau_{xy} = 2\ [[\overline{\omega(\zeta)} - \overline{\omega}(\zeta)] \times d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta/\omega'(\zeta) - \overline{\phi}'(\zeta)/\omega'(\zeta) - \overline{\iota}(\zeta)\phi'(\zeta)/\omega'(\zeta)], \quad (14)$$

where $\iota(\zeta) = \omega'(\zeta)/\overline{\omega}'(\zeta)$, which tends to unity at infinity.

4. DETERMINATION OF COMPLEX POTENTIAL $\phi(\zeta)$

Since we now know from boundary condition consideration that $\chi(\zeta)$, which we introduced as an arbitrary analytic function, should be related to $\phi(\zeta)$ by equation(12), the problem reduces to the determination of $\phi(\zeta)$ which satisfies the loading conditions at infinity,

$$\sigma_x = \sigma,\ \sigma_y = \tau_{xy} = 0\ (\xi \to \infty). \quad (15)$$

These conditions will be fulfilled, if $\phi(\zeta)$ is so chosen as to satisfy equations(16) and (17) to follow.

$$\sigma_x + \sigma_y = 4Re[\phi'(\zeta)/\omega'(\zeta)] = \sigma, \quad (16)$$

$$\sigma_y - \sigma_x + i2\tau_{xy} = 2\ [[\overline{\omega(\zeta)} - \overline{\omega}(\zeta)] \times d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta/\omega'(\zeta) - \overline{\phi}'(\zeta)/\omega'(\zeta)$$

$$-\iota(\zeta)\phi'(\zeta)/\omega'(\zeta)] = -\sigma. \quad (17)$$

Here we assume that the function $\phi(\zeta)$ can be represented by

$$2\phi(\zeta) = Ac\zeta + B\omega(\zeta), \quad (18)$$

which is so chosen that $2\phi(\zeta)$ consist of $A_o\omega(\zeta)$, i.e., the elementary solution for the crack-free geometry, and a term, $B_o[c\zeta - \omega(\zeta)]$, vanishing at infinity, which sum up to the above representation(18), with $A = B_o$ and $B = A_o - B_o$. Here coefficients A_o and B_o, therefore A and B, are real from symmetry consideration in the loading conditions under consideration. Then it will be readily shown that the conditions expressed by equations(16) and (17) determines A + B, the sum of coefficients of the non-vanishing terms of $2\phi'(\zeta)/\omega'(\zeta)$ at infinity, as

$$A + B = \sigma/2, \quad (19)$$

by noticing that the term $2[\overline{\omega(\zeta)} - \overline{\omega(\zeta)}] \times d\{\phi'(\zeta)/\omega'(\zeta)\}/d\zeta/\overline{\omega'(\zeta)}$ vanishes and $2\phi'(\zeta)/\omega'(\zeta) = A_o + B_o[c/\omega'(\zeta) - 1]$ tends to A_o at infinity in equation(17). Other than the loading conditions, those for the displacements, u and v, to be bounded in the area $Im\,\zeta > 0$, would require

$$(4/3)A + 2B = 0, \quad (20)$$

which corresponds to the condition that a term $\{(4/3)A + 2B\}/(\zeta - i)$ of $2\mu(u - iv)$, equation(7), behaving as being singular in $Im\,\zeta > 0$, should vanish. It will also be shown that a term of $1/\zeta$, which consists of $2\mu(u - iv)$ and behaves singularly on the bounday $Im\,\zeta = 0$, has already vanished by satisfying the traction-free boundary condition(8), or equivalently in the definition of $\chi(\zeta)$, equation(12). From equations(19) and (20) A and B are obtained as

$$A = (3/2)\sigma \text{ and } B = -\sigma \quad (21)$$

Complete distributions of the stresses, σ_x, σ_y and τ_{xy}, can now be determined from equations(5), (14) and (18), with (21). To examine a crack tip singularity let attention be restricted to the domain $r \ll c$, where from equations(11) ζ is approximated to be

$$\zeta = -i[(r/c)^{1/2}e^{i\theta/2} - (r/c)e^{i\theta}/2 + (r/c)^{3/2}e^{i3\theta/2}/8]. \quad (22)$$

Now it is possible to write down the stresses in the vicinity of crack tip in terms of r and θ as

$$\sigma_x = k_I[2\pi r]^{-1/2}\cos(\theta/2)\times[1 + \sin(\theta/2)\sin(3\theta/2)], \quad (23)$$

$$\sigma_y = k_I[2\pi r]^{-1/2}\cos(\theta/2)\times[1 - \sin(\theta/2)\sin(3\theta/2)], \quad (24)$$

$$\tau_{xy} = -k_I[2\pi r]^{-1/2}\sin(\theta/2)\times\cos(\theta/2)\sin(3\theta/2), \quad (25)$$

where, it is to be noted, the factor k_I in the present crack morphology is derived as

$$k_I = 3\cdot 2^{-3/2}\sigma[\pi c]^{1/2}. \quad (26)$$

and the azimuth dependences of the singular terms compare identically to those with the internal straight crack.

5. DISCUSSIONS

It is first to be mentioned that the factor k_I, equation(26), is in agreement with that obtained in the direct evaluation procedure of the stress intensity factors K_I and K_{II}, namely that obtained by a general formula[4],

$$K_I - iK_{II} = \lim_{\zeta\to\zeta_o} 2\phi'(\zeta)/\omega'(\zeta)\times[2\pi e^{-i\delta}\{\omega(\zeta) - \omega(\zeta_o)\}]^{1/2}, \quad (27)$$

where an inessential modification of the definitions of K_I and K_{II} was made; accordingly the factor $2\pi e^{-i\delta}$ is introduced in equation(27) in place of 2 which appears in the original formula[4]. δ is the angle which the normal of the crack plane makes against the y-axis, and ζ_o the ζ-coordinate corresponding to the crack tip location. In the present example $\zeta_o = 0$, and $\phi(\zeta)$ is given by equation(18). Carrying out the actual algebra yields

$$2\phi'(\zeta)/\omega'(\zeta) = B_o c/\omega'(\zeta) + (A_o - B_o), \quad (28)$$

which reveals that factor $B_o = A$ contributes to singularity, while we already know that the factor $A_o = A + B$ depends on the uniformly applied stress σ at infinity as equation(19). Substituting the necessary quantities in equation(27) and letting ζ approarch to 0 render

$$K_I = 3\cdot 2^{-3/2}\sigma[\pi c]^{1/2}, \quad (29)$$

which is identical with k_I defined in equation(26). The present work treated an edge crack of a cusp-type morphology, analytically deriving the enhanced stress intensity due to the presence of the traction-free surface. The derived stress intensity is 6 percent greater, which means an effective crack length is 12.5 percent greater, than that of an internal crack with a half crack length equal to the surface crack depth, but compares inferiorly with $1.1215\sigma[\pi c]^{1/2}$ for an edge crack with rectangular corners and with depth c[1].

6. CONCLUSIONS

The stress intensity and distributions of the stresses at and around tip of an egde crack of a cusp-type morphology were derived. The derived stress intensity is expressed by $K_I = 3\cdot 2^{-3/2}\sigma[\pi c]^{1/2}$, with σ and c being mode I stress applied at infinity and surface crack depth, respectively. The stress intensity is 6 percent greater, which means an effective crack length is 12.5 percent greater, than that of an internal crack with a half crack length equal to the surface crack depth. The azimuth dependence of the carck-tip stresses compares identically with that of the internal straight crack.

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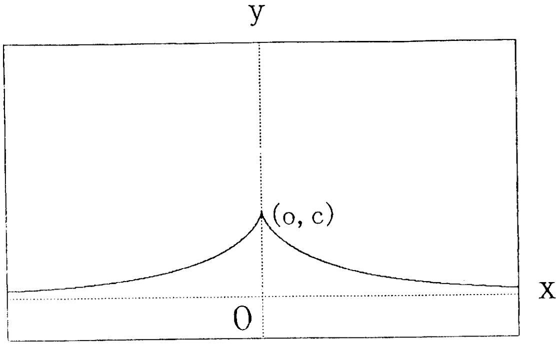

Fig. 1 Infinite sheet with shallow edge crack of a cusp type under tension in x direction

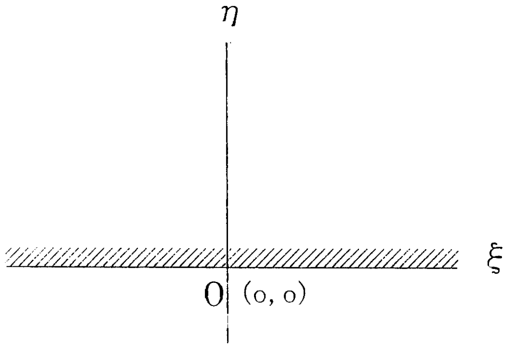

Fig. 2 Auxiliary complex plane, $\zeta = \xi + i\eta$

Cusp型亀裂先端のまわりの応力分布

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Cusp型モードI表面亀裂に誘起される応力場を、等角写像の手法を用いて解析的に導き、応力強度因子(stress intensity factor)の大きさを決定した。応力強度因子K_Iに及ぼす表面効果は$K_I = \beta\sigma[\pi c]^{1/2}$におけるファクター$\beta = 3\cdot 2^{-3/2}$により与えられることが示された。ここに、$\sigma$ および c は無限遠方に作用するモードI応力および表面亀裂深さである。