Gauge Theory on the Hilbert Manifold

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Abstract

The mathematical foundation of the gauge theory is built up systematically. The H-valued differential form bundle is constructed on the Hilbert manifold and the exterior covariant differentiation is defined on the H-valued differential form bundle. The connection form and the curvature form are introduced by means of the exterior covariant differentiation. The structure equation and the Bianchi identity are derived.

§ 1. Introduction

In his attempt [2] to unify the gravitational field and the electromagnetic field, Weyl developed the concept of the gauge transformation. After that, using the Dirac theory of electron, in [3] he succeeded to formulate the theory which unifies the gravitational field, the electromagnetic field and the wave field of the electron, and this is the initiation of the gauge theory. Utiyama noticed that the gravitational field, and also the electromagnetic field, are represented in connection fields, and evolved a general gauge theory in this viewpoint. In particular, in [7] he showed that, in introducing the gauge field connection, the Lagrangian is still invariant even when the global transformation group is replaced by the local transformation group. After Utiyama, Kibble [8] constructed a theory which unified the gravity and torsion fields by formulating the equation of motion of the gauge field torsions. These works strongly suggests that four kinds of forces, electromagnetic, weak, strong, gravity, may be treated within a common framework of the gauge theory.

In the gauge theory, the gauge field is a connection in the covariant derivation of the matter fields, and it is introduced by replacing the global symmetry by the local symmetry. Matter fields space has representations as fibre bundles whose fibres are the representation spaces of transformation groups of vector fields, tensor fields, spinor fields etc., where the representation spaces are finite or infinite dimensional Hilbert spaces.

In such standpoint, we consider a Hilbert manifold M and construct an attaching bundle whose fibres are Hilbert spaces $H_{\mathfrak{x}}(M)$ defined at each point $\mathfrak{x}$ of M and have the same structures with a Hilbert space H, and the attaching bundle keeps the local symmetry. The transformation group is $GL(H)$. We also construct the $H(M)$-valued differential form which corresponds to a matter field. This form is expressed in terms of the moving frame $\mathfrak{b}(\mathfrak{x})$ on M.

The $H(M)$-valued differential form of class $C^{(n)}$ on M is defined with the help of the H-valued differential form of class $C^{(n)}$ on M. We investigate the H-valued differ-

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ential forms, and generalize the definitions of the exterior multiplication and the exterior differentiation. We obtain analogous results with the case of real valued differential forms on differentiable manifolds. The set $\Omega_p^{(n)}(M, H(M))$ of all $H(M)$-valued differential forms of class $C^{(n)}$ on M is a $C^{(n)}(M, R)$-module. The exterior differential of $\Omega_p \in \Omega_p^{(n)}(M, H(M))$ with respect to $\mathfrak{b}(\mathfrak{x})$ is defined with the help of the exterior differential of the H-valued differential form and we obtain familiar properties of the exterior differentiation.

The exterior covariant differentiation is a linear continuous map $D_H\colon \Omega_p^{(n)}(M, H(M)) \to \Omega_{p+1}^{(n-1)}(M, H(M))$ such that $D_H(\omega_q \wedge \Omega_p) = d\omega_q \wedge \Omega_p + (-1)^q \omega_q \wedge D_H \Omega_p$ for $\omega_q \in \Omega_q^{(n)}(M, R)$, $\Omega_p \in \Omega_p^{(n)}(M, H(M))$. The connection form ω_H of H-type with respect to $\mathfrak{b}(\mathfrak{x})$ is defined by $D_H \mathfrak{b}(\mathfrak{x}) = \mathfrak{b}(\mathfrak{x})\omega_H(\mathfrak{x})$, and the covariant derivation ∇, the connection Γ are introduced. The curvature form R_H is defined by $D_H^2 \mathfrak{b}(\mathfrak{x}) = \mathfrak{b}(\mathfrak{x}) R_H(\mathfrak{x})$. The Lagrangian determining the motion equation of the gauge field Γ is constituted form R_H. The structure equation $R_H = d\omega_H + \omega_H \wedge \omega_H$ and the Bianchi identity is derived.

Throughout this paper it is assumed that Hilbert spaces are separable. By a base of Hilbert space we mean the base $Ae = (Ae_1, \ldots, Ae_n, \ldots)$ which is made by an orthonormal base e and an invertible bounded linear operator A.

Table of Contents

§2. Differential forms on the Hilbert manifold

1. $C^{(n)}$-diffeomorphisms

Let E, F be two Banach spaces over the real field R. We denote the set of all p-linear continuous mappings

$$f: \underbrace{E\times\cdots\times E}_{p\text{-times}} \longrightarrow F$$

and the set of all p-linear alternating continuous mappings

$$f: \underbrace{E\times\cdots\times E}_{p\text{-times}} \longrightarrow F$$

by $\mathscr{L}_p(E, F)$ and $\mathscr{A}_p(E, F)$ respectively. In particular, $\mathscr{L}_0(E, F)=\mathscr{A}_0(E, F)$ is the set of all continuous mappings $f: E\to F$ and $\mathscr{L}_1(E, F)=\mathscr{A}_1(E, F)$ is the set of all linear continuous mappings $f: E\to F$. In what follows $\mathscr{L}_1(E, F)$ is denoted by $\mathscr{L}(E, F)$. These sets are all vector spaces and form Banach spaces over the real field R with the norms defined by the usual manner. As example, the norm of an arbitrary vector $f\in\mathscr{A}_p(E, F)$ is given by

$$\|f\|=\sup_{\|x_1\|\leqq 1,\dots,\|x_p\|\leqq 1}\|f(x_1,\dots, x_p)\|\,,$$

where $x_1,\dots, x_p\in E$.

Let V be an open not empty set of E and consider a mapping $f: V\to F$. Now, we say that a mapping f is differentiable at the point $a\in V$ if the following conditions are satisfied: (1) f is continuous at the point a, (2) there exists a linear continuous mapping $f'(a)\in\mathscr{L}(E, F)$ such that

$$\lim_{\substack{x\to a\\(x\neq a)}}\frac{\|f(x)-f(a)-f'(x)(x-a)\|}{\|x-a\|}=0,$$

that is, $f(x)-f(a)=f'(a)(x-a)+o(\|x-a\|)$.

A mapping $f'(a)\in\mathscr{L}(E, F)$ is called the derivative of the mapping f at the point $a\in V$.

By the differentiability of a mapping $f: V\to F$ in V we shall mean a differentiable mapping f at every point of V. We say that a mapping $f: V\to F$ is of class $C^{(1)}$ in V if f is differentiable in V and if the derived mapping $f': V\to\mathscr{L}(E, F)$ is continuous in V.

Furthermore, a mapping $f: V\to F$ is said to be twice differentiable in V if the mapping f and the mapping $f': V\to\mathscr{L}(E, F)$ are differentiable in V. The derivative of f' at the point $a\in V$ is denoted by $f''(a)$ and we see $f''(a)\in\mathscr{L}_2(E, F)$. We say that a mapping $f: V\to F$ is of class $C^{(2)}$ in V if the mapping f is twice differentiable in V and if the mapping $f'': V\to\mathscr{L}_2(E, F)$ is continuous in V.

In general, by induction, a mapping $f: V\to F$ is said to be of class $C^{(n)}$ in V if the mapping f is n times differentiable in V and if the mapping $f^{(n)}: V\to\mathscr{L}_n(E, F)$ is continuous in V.

Now we shall give the definition of $C^{(n)}$-diffeomorphism.

Definition.

Let E, F be two Banach spaces and let V, W be open sets of E, F respectively. Then we say that a mapping $f: V\to W$ is $C^{(n)}$*-diffeomorphism* if the following conditions are satisfied:

(1) f is a bijection of V onto W,

(2) the mappings f, f^{-1} are of class $C^{(n)}$.

In what follows, by the base of the Hilbert space E we shall mean the base $Ae=(Ae_1,\dots, Ae_n,\dots)$ where $e=(e_1,\dots, e_n,\dots)$ is an orthonormal base of E and the mapping $A: E\to E$ is a bounded linear operator which has the bounded inverse operator A^{-1}.

Now, let $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ and $\mathfrak{e}'=(\mathfrak{e}'_1, \mathfrak{e}'_2,\dots, \mathfrak{e}'_n,\dots)$ be two bases of E, and let B, C be bounded linear operators on E. We expand a vector X of E in terms of two bases $\mathfrak{e}$ and $\mathfrak{e}'$ in the form

$$X=\sum_{i=1}^{\infty} X^i\mathfrak{e}_i=\sum_{i=1}^{\infty} X'^i\mathfrak{e}'_i.$$

Moreover we assume that two matrices

$$\begin{pmatrix} b_1^1 & b_2^1\cdots b_n^1\cdots \\ b_1^2 & b_2^2\cdots b_n^2\cdots \\ \vdots & \vdots \quad\ \vdots \\ b_1^n & b_2^n\cdots b_n^n\cdots \\ \vdots & \vdots \quad\ \vdots \end{pmatrix}, \quad \begin{pmatrix} c_1^1 & c_2^1\cdots c_n^1\cdots \\ c_1^2 & c_2^2\cdots c_n^2\cdots \\ \vdots & \vdots \quad\ \vdots \\ c_1^n & c_2^n\cdots c_n^n\cdots \\ \vdots & \vdots \quad\ \vdots \end{pmatrix}$$

are the matrix representations of B, C with respect to a base $\mathfrak{e}$ respectively. It is easy to show that, when a transformation from a base $\mathfrak{e}$ to a base $\mathfrak{e}'$ is given by

$$\begin{pmatrix} \mathfrak{e}'_1 \\ \mathfrak{e}'_2 \\ \vdots \\ \mathfrak{e}'_n \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2\cdots a_1^n\cdots \\ a_2^1 & a_2^2\cdots a_2^n\cdots \\ \vdots & \vdots \quad\ \vdots \\ a_n^1 & a_n^2\cdots a_n^n\cdots \\ \vdots & \vdots \quad\ \vdots \end{pmatrix} \begin{pmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \vdots \\ \mathfrak{e}_n \\ \vdots \end{pmatrix},$$

the components of a vector X are transformed in the following manner

$$\begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^n \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1\cdots a_n^1\cdots \\ a_1^2 & a_2^2\cdots a_n^2\cdots \\ \vdots & \vdots \quad\ \vdots \\ a_1^n & a_2^n\cdots a_n^n\cdots \\ \vdots & \vdots \quad\ \vdots \end{pmatrix} \begin{pmatrix} X'^1 \\ X'^2 \\ \vdots \\ X'^n \\ \vdots \end{pmatrix},$$

and furthermore we have

$$BX=\sum_{i=1}^{\infty}\{\sum_{j=1}^{\infty} b_j^i x^j\}\mathfrak{e}_i,$$

$$(BC)X=\sum_{i=1}^{\infty}\{\sum_{j=1}^{\infty} b_j^i(\sum_{k=1}^{\infty} c_k^j X^k)\}\mathfrak{e}_i$$

$$=\sum_{i=1}^{\infty}\{\sum_{k=1}^{\infty}(\sum_{j=1}^{\infty} b_j^i c_k^j)x^k)\mathfrak{e}_i.$$

Henceforth the above results shall be used without reference in case of necessity.

Subsequently, we shall introduce the partial derivatives of the function. Let V

be an open set of a Hilbert space E and let $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ be a base of E. Then, for each point $x \in V$, we have $x=\sum_{i=1}^{\infty} x^i\mathfrak{e}_i$. Considering a function $f\colon V\to R$, we shall define the partial derivatives of f as follows: a function $f\colon V\to R$ is said to have the partial derivative with respect to x^i at a point $x \in V$ if the function $g_i(t)=f(x+t\mathfrak{e}_i)\colon R\to R$ is differentiable at $t=0$.

We denote the partial derivative with respect to x^i at a point $x \in V$ by $\partial_i f(x)$ or $\partial f(x)/\partial x^i$.

Furthermore, we can define the partial derivatives of higher order of a function $f\colon V\to R$, in analogy to what is defined for the ordinary function.

In what follows, we assume that E, F are Hilbert spaces. Let $\mathfrak{e}=(\mathfrak{e}_1,\dots, \mathfrak{e}_n,\dots)$ and $\mathfrak{f}=(\mathfrak{f}_1,\dots, \mathfrak{f}_n,\dots)$ be bases of E and F respectively. Giving a mapping $f\colon E\to F$, we expand this mapping f in terms of a base $\mathfrak{f}$ as follows.

$$f(x)=\sum_{i=1}^{\infty} f^i(x)\mathfrak{f}_i, \quad \text{for each point} \quad x \in E.$$

Now we shall investigate the relation between the derivative of a mapping $f\colon E\to F$ and the derivatives of the functions $f^i\colon E\to R$ $(i=1, 2,\dots)$.

Proposition 1.

Let V be an open set of E. Suppose that a function $f\colon V\to R$ is of class $C^{(n)}$ in V. Then f has the partial derivatives $\partial_{j_1}\cdots\partial_{j_m}f(x)$ $(m\leqq n;\ j_1,\dots, j_m=1, 2,\dots)$ which are continuous in V, and we have, for each point $x \in V$ and for any vectors $y_1,\dots, y_m \in E$,

$$f^{(m)}(x)(y_1,\dots, y_m)=\sum_{j_1,\dots,j_m=1}^{\infty} \partial_{j_1}\cdots\partial_{j_m}f(x)y_1^{j_1}\cdots y_m^{j_m} \tag{1.1}$$

where

$$y_1=\sum_{j=1}^{\infty} y_1^j\mathfrak{e}_j,\dots, y_m=\sum_{j=1}^{\infty} y_m^j\mathfrak{e}_j$$

and

$$\partial_{j_1}\cdots\partial_{j_m}f(x)=\frac{\partial^m f(x)}{\partial x^{j_1}\cdots\partial x^{j_m}}.$$

Proof. We prove this proposition by induction on m.

First, we shall show that f has $\partial_j f(x)$ $(j=1, 2,\dots)$ which are continuous in V and (1.1) is held in case $m=1$.

Since f is of class $C^{(n)}$ in V, there exists $f'(x) \in \mathscr{L}(E, R)$ at each point $x \in V$ such that

$$\|f(x+\Delta x)-f(x)-f'(x)\Delta x\|=o(\|\Delta x\|). \tag{1.2}$$

Setting $f'_j(x)=f'(x)\mathfrak{e}_j$, we see

$$f'(x)y=\sum_{j=1}^{\infty} f'_j(x)y^j$$

where $y=\sum_{j=1}^{\infty} y^j\mathfrak{e}_j$. Especially, in case $\Delta x=t\mathfrak{e}_j$, we have $f'(x)\Delta x=f'_j(x)t$. Thus (1.2) becomes

$$\|f(x+te_j)-f(x)-f'_j(x)t\|=o(|t|).$$

This show that f has $\partial_j f(x)$ at $x\in V$ and $f'_j(x)=\partial_j f(x)$. Therefore we get

$$f'(x)y=\sum_{j=1}^{\infty}\partial_j f(x)y^j. \tag{1.3}$$

Now, in order to see that $\partial_j f(x)$ $(j=1, 2,\ldots)$ are continuous in V, we start with

$$\|f'(x+\Delta x)-f'(x)\|=\sup_{\|y\|\leqq 1}|\{f'(x+\Delta x)-f'(x)\}y|.$$

By (1.3), we have

$$\|f'(x+\Delta x)-f'(x)\|=\sup_{\|y\|\leqq 1}|\sum_{j=1}^{\infty}\{\partial_j f(x+\Delta x)-\partial_j f(x)\}y^j|.$$

Furthermore, by the continuity of $f'(x)$ in V, we get, for $y\in E$ such that $\|y\|\leqq 1$,

$$\lim_{\Delta x\to 0}|\sum_{j=1}^{\infty}\{\partial_j f(x+\Delta x)-\partial_j f(x)\}y^j|=0.$$

Especially, setting $y=e_j/\|e_j\|$, we infer

$$\lim_{\Delta x\to 0}|\partial_j f(x+\Delta x)-\partial_j f(x)|=0.$$

Namely, $\partial_j f(x)$ $(j=1, 2,\ldots)$ are continuous in V.

Next, suppose that f has $\partial_{j_1}\cdots\partial_{j_m}f(x)$ $(m<n; j_1,\ldots, j_m=1, 2,\ldots)$ which are continuous in V and (1.1) is held. Our purpose is to show that there exist $\partial_{j_1}\cdots\partial_{j_m}\partial_{j_{m+1}}f(x)$ $(j_1,\ldots, j_m, j_{m+1}=1, 2,\ldots)$, moreover these are continuous in V and (1.1) is also held.

Since f is of class $C^{(n)}$ in V, there exists $f^{(m+1)}(x)\in\mathscr{L}_{m+1}(E, R)$ at each point $x\in V$ such that

$$\|f^{(m)}(x+\Delta x)-f^{(m)}(x)-f^{(m+1)}(x)\Delta x\|=o(\|\Delta x\|),$$

that is,

$$\sup_{\|y_1\|\leqq 1,\ldots,\|y_m\|\leqq 1}|\{f^{(m)}(x+\Delta x)-f^{(m)}(x)-f^{(m+1)}(x)\Delta x\}(y_1,\ldots, y_m)|=o(\|\Delta x\|), \tag{1.4}$$

where $y_1,\ldots, y_m\in E$. Writing $f^{(m+1)}_{j_1\cdots j_{m+1}}(x)=f^{(m+1)}(x)(e_{j_1},\ldots, e_{j_{m+1}})$, we have

$$f^{(m+1)}(x)(y_1,\ldots, y_{m+1})=\sum_{j_1,\ldots,j_{m+1}=1}^{\infty}f^{(m+1)}_{j_1\cdots j_{m+1}}(x)y_1^{j_1}\cdots y_{m+1}^{j_{m+1}}, \tag{1.5}$$

where $$y_1=\sum_{j_1=1}^{\infty}y_1^{j_1}e_{j_1},\ldots, y_{m+1}=\sum_{j_{m+1}=1}^{\infty}y_{m+1}^{j_{m+1}}e_{j_{m+1}}.$$

By hypothesis of induction, (1.4) becomes

$$\sup_{\|y_1\|\leqq 1,\ldots,\|y_m\|\leqq 1}|\sum_{j_1,\ldots,j_m=1}^{\infty}\{\partial_{j_1}\cdots\partial_{j_m}f(x+\Delta x)-\partial_{j_1}\cdots\partial_{j_m}f(x)$$

$$-\sum_{j_{m+1}=1}^{\infty} f^{(m+1)}_{j_{m+1}j_1\cdots j_m}(x)\Delta x^{j_{m+1}}\}y_1^{j_1}\cdots y_m^{j_m}=o(\|\Delta x\|), \tag{1.6}$$

where $\Delta x=\sum_{j_{m+1}=1}^{\infty}\Delta x^{j_{m+1}}e_{j_{m+1}}$. Now, especially setting in (1.6)

$$\Delta x=te_{j_{m+1}},\quad y_1=e_{j_1}/\|e_{j_1}\|,\ldots, y_m=e_{j_m}/\|e_{j_m}\|,$$

we obtain

$$|\partial_{j_1}\cdots\partial_{j_m}f(x+te_{j_{m+1}})-\partial_{j_1}\cdots\partial_{j_m}f(x)-f^{(m+1)}_{j_{m+1}j_1\cdots j_m}(x)t|=o(|t|).$$

This show that there exists $\partial_{j_{m+1}}\partial_{j_1}\cdots\partial_{j_m}f(x)$ and $f^{(m+1)}_{j_{m+1}j_1\cdots j_m}(x)=\partial_{j_{m+1}}\partial_{j_1}\cdots\partial_{j_m}f(x)$. Therefore, from (1.5), we obtain

$$f^{(m+1)}(x)(y_1,\ldots, y_{m+1})=\sum_{j_1,\ldots,j_{m+1}=1}^{\infty}\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)y_1^{j_1}\cdots y_{m+1}^{j_{m+1}}. \tag{1.7}$$

Successively, to see the continuity of $\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)$ $(j_1,\ldots, j_{m+1}=1, 2,\ldots)$ in V, we start with

$$\|f^{(m+1)}(x+\Delta x)-f^{(m+1)}(x)\|$$
$$=\sup_{\|y_1\|\leqq 1,\ldots,\|y_{m+1}\|\leqq 1}|\{f^{(m+1)}(x+\Delta x)-f^{(m+1)}(x)\}(y_1,\ldots, y_{m+1})|$$

By (1.7), we have

$$\|f^{(m+1)}(x+\Delta x)-f^{(m+1)}(x)\|$$
$$=\sup_{\|y_1\|\leqq 1,\ldots,\|y_{m+1}\|\leqq 1}\Big|\sum_{j_1,\ldots,j_{m+1}=1}^{\infty}\{\partial_{j_1}\cdots\partial_{j_{m+1}}f(x+\Delta x)-\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)\}y_1^{j_1}\cdots y_{m+1}^{j_{m+1}}\Big|.$$

Furthermore, by the continuity of $f^{(m+1)}(x)$ in V, we get,

$$\lim_{\Delta x\to 0}\Big|\sum_{j_1,\ldots,j_{m+1}=1}^{\infty}\{\partial_{j_1}\cdots\partial_{j_{m+1}}f(x+\Delta x)-\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)\}y_1^{j_1}\cdots y_{m+1}^{j_{m+1}}\Big|=0$$

where $\|y_1\|\leqq 1,\ldots, \|y_{m+1}\|\leqq 1$. Especially, putting

$$y_1=e_{j_1}/\|e_{j_1}\|,\ldots, y_{m+1}=e_{j_{m+1}}/\|e_{j_{m+1}}\|$$

in above formula, we infer

$$\lim_{\Delta x\to 0}|\partial_{j_1}\cdots\partial_{j_{m+1}}f(x+\Delta x)-\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)|=0.$$

Namely, $\partial_{j_1}\cdots\partial_{j_{m+1}}f(x)$ is continuous in V.

Lemma.

Let E, F and H be three Hilbert spaces. For each point x of an open set V of H, let us consider a mapping $f(x)\in\mathscr{L}(E, F)$ which is represented in terms of a base $\mathfrak{f}=(\mathfrak{f}_1, \mathfrak{f}_2,\ldots, \mathfrak{f}_n,\ldots)$ of F in the form

$$f(x)=\sum_{i=1}^{\infty}f^i(x)\mathfrak{f}_i,$$

i.e., for any $y \in E$,

$$f(x: y) = \sum_{i=1}^{\infty} f^i(x: y)\mathfrak{f}_i$$

where $f(x: y) = f(x)y$, $f^i(x: y) = f^i(x)y$.

Then, there exist constants C^i such that

$$\|f^i(x)\| \leqq C^i \|f(x)\| \qquad (i=1, 2, \ldots).$$

Proof.

Let $e = (e_1, e_2, \ldots, e_n, \ldots)$ be an orthonormal base of F. Using this base, we express $f(x)$ as follows

$$f(x) = \sum_{i=1}^{\infty} \tilde{f}^i(x) e_i.$$

Then we have

$$\|f(x)\| = \sup_{\|y\| \leqq 1} \|f(x: y)\| = \sup_{\|y\| \leqq 1} \{\sum_{i=1}^{\infty} |\tilde{f}^i(x: y)|^2\}^{\frac{1}{2}}.$$

Now, there exists a bounded linear operator A for a base $\mathfrak{f}$ such that its inverse operator A^{-1} is bounded and $\mathfrak{f}_i = Ae_i$ $(i=1, 2, \ldots)$. Moreover we see

$$f^i(x: y) = \sum_{j=1}^{\infty} A_j^{-1^i} \tilde{f}^j(x: y)$$

where $A_j^{-1^i} = \langle e_i, A^{-1}e_j \rangle$ (by the notation $\langle\ ,\ \rangle$, we express the inner product of the Hilbert space). Applying the Schwarz's inequality to the above formula, we get

$$|f^i(x: y)| \leqq \{\sum_{j=1}^{\infty} |A_j^{-1^i}|^2\}^{\frac{1}{2}} \{\sum_{j=1}^{\infty} |\tilde{f}^j(x: y)|^2\}^{\frac{1}{2}}$$

and therefore we have

$$\begin{aligned}\|f^i(x)\| &= \sup_{\|y\| \leqq 1} |f^i(x: y)| \\ &\leqq \{\sum_{j=1}^{\infty} |A_j^{-1^i}|^2\}^{\frac{1}{2}} \sup_{\|y\| \leqq 1} \{\sum_{j=1}^{\infty} |\tilde{f}^j(x: y)|^2\}^{\frac{1}{2}} \\ &= \{\sum_{j=1}^{\infty} |A_j^{-1^i}|^2\}^{\frac{1}{2}} \|f(x)\|.\end{aligned}$$

Since A^{-1} is a bounded operator, we see

$$\{\sum_{j=1}^{\infty} |A_j^{-1^i}|^2\}^{\frac{1}{2}} < \infty.$$

Thus, we have accomplished our purpose.

Proposition 2.

Let E, F be Hilbert spaces and V be an arbitrary open set of E. Suppose that a *map $f: V \to F$ is represented in terms of a base $\mathfrak{f}=(\mathfrak{f}_1, \mathfrak{f}_2,\dots, \mathfrak{f}_n,\dots)$ of F in the form*

$$f(x)=\sum_{i=1}^{\infty} f^i(x)\mathfrak{f}_i\,.$$

Then, if a map f is of class $C^{(n)}$ in V, every function $f^i: V\to R$ is also of class $C^{(n)}$ in V and we have

$$f^{(m)}(x)=\sum_{i=1}^{\infty} f^{i(m)}(x)\mathfrak{f}_i\,, \tag{1.8}$$

i.e.,

$$f^{(m)}(x)(y_1, y_2,\dots, y_m)=\sum_{i=1}^{\infty} f^{i(m)}(x)(y_1, y_2,\dots, y_m)\mathfrak{f}_i$$

where $y_1, y_2,\dots, y_m\in E$, $1\leqq m\leqq n$.

Proof. We shall prove this proposition by induction on m.

First, we verify that $f^i(x)$ $(i=1, 2,\dots)$ have the continuous derivatives of the first order in V and (1.8) is held in case $m=1$.

Since a map f is of class $C^{(n)}$ in V, there exists $f'(x)\in\mathscr{L}(E, F)$ at each point $x\in V$ such that

$$\|f(x+\Delta x)-f(x)-f'(x)\Delta x\|=o(\|x\|)\,.$$

Using an expression $f'(x)=\sum_{i=1}^{\infty} g^i(x)\mathfrak{f}_i$, the above relation becomes

$$\|\sum_{i=1}^{\infty}\{f^i(x+\Delta x)-f^i(x)-g^i(x)\Delta x\}\mathfrak{f}_i\|=o(\|\Delta x\|)$$

where $g^i(x)\in\mathscr{L}(E, R)$. From this, we have

$$|f^i(x+\Delta x)-f^i(x)-g^i(x)\Delta x|=o(\|\Delta x\|), \qquad (i=1, 2,\dots)\,.$$

This show that every function $f^i(x)$ has the derivative $f^{i\prime}(x)$ and $f^{i\prime}(x)=g^i(x)$. Thus we get

$$f'(x)=\sum_{i=1}^{\infty} f^{i\prime}(x)\mathfrak{f}_i\,.$$

Furthermore, every derivative $f^{i\prime}(x)$ is continuous in V. In fact, there exist the constants C^i by lemma such that

$$\|f^{i\prime}(x+\Delta x)-f^{i\prime}(x)\|\leqq C^i\|f'(x+\Delta x)-f'(x)\|\,.$$

From

$$\lim_{\Delta x\to 0}\|f'(x+\Delta x)-f'(x)\|=0,$$

we see

$$\lim_{\Delta x\to 0}\|f^{i\prime}(x+\Delta x)-f^{i\prime}(x)\|=0\,.$$

Next, we assume that every function $f^i(x)$ has the continuous derivative of order m and (1.8) is held in the case of m.

Since f is of class $C^{(n)}$ in V, there exists $f^{(m+1)}(x) \in \mathscr{L}_{m+1}(E, F)$ at each point $x \in V$ such that

$$\|f^{(m)}(x+\Delta x)-f^{(m)}(x)-f^{(m+1)}(x)\Delta x\| = o(\|\Delta x\|).$$

By hypothesis of induction and using an expression $f^{(m+1)}(x)=\sum_{i=1}^{\infty} g^i(x)\mathfrak{f}_i$, we have

$$\|\sum_{i=1}^{\infty}\{f^{i(m)}(x+\Delta x)-f^{i(m)}(x)-g^i(x)\Delta x\}\mathfrak{f}_i\| = o(\|\Delta x\|)$$

where $g^i(x) \in \mathscr{L}_{m+1}(E, R)$. From this, we have

$$\|f^{i(m)}(x+\Delta x)-f^{i(m)}(x)-g^i(x)\Delta x\| = o(\|\Delta x\|), \qquad (i=1, 2,\dots).$$

Thus, every function $f^i(x)$ has the derivative of order $m+1$ and $g^i(x)=f^{i(m+1)}(x)$. Hence we have

$$f^{(m+1)}(x)=\sum_{i=1}^{\infty} f^{i(m+1)}(x)\mathfrak{f}_i.$$

Finally, the continuity of $f^{i(m+1)}(x)$ $(i=1, 2,\dots)$ can be proved in the same manner as in the case of $f^{i\prime}(x)$.

As an immediate consequence from Proposition 1 and 2, we have the following:

Proposition 3.

Suppose that a map $f\colon V \to F$ is expanded in terms of a base $\mathfrak{f}=(\mathfrak{f}_1, \mathfrak{f}_2,\dots, \mathfrak{f}_n,\dots)$ of F in the form

$$f(x)=\sum_{i=1}^{\infty} f^i(x)\mathfrak{f}_i.$$

Then, if a map f is of class $C^{(n)}$ in V, we have

$$f^{(m)}(x)(y_1,\dots, y_m)=\sum_{i=1}^{\infty}\{\sum_{j_1,\dots,j_m=1}^{\infty} \partial_{j_1}\cdots\partial_{j_m} f^i(x) y_1^{j_1}\cdots y_m^{j_m}\}\mathfrak{f}_i,$$

where, $y_1=\sum_{j=1}^{\infty} y_1^j \mathfrak{e}_j,\dots, y_m=\sum_{j=1}^{\infty} y_m^j \mathfrak{e}_j$ are m vectors of E and $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ is a base of E.

2. Tangent spaces

Let M be a Hausdorff space and E be a Hilbert space. By the atlas of class $C^{(n)}$ on M we mean a set of pairs $(U_\alpha, \varphi_\alpha)$ $(\alpha \in I)$ which satisfies the following conditions:

(1) each U_α is an open set of M and $\bigcup_{\alpha\in I} U_\alpha = M$,

(2) each φ_α is a homeomorphism of U_α onto an open subset $\varphi_\alpha(U_\alpha)$ of E,

(3) the map $\varphi_{\beta\alpha}=\varphi_\beta\circ\varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta)\to\varphi_\beta(U_\alpha \cap U_\beta)$ is a $C^{(n)}$-diffeomorphism for each pair of indices α, β.

Each pair $(U_\alpha, \varphi_\alpha)$ is called a chart or a local coordinate system of the atlas. If a point $x \in M$ lies in U_α, then we say that $(U_\alpha, \varphi_\alpha)$ is a chart at x and φ_α is a local coordinate system at x.

Let $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\ldots, \mathfrak{e}_n,\ldots)$ be a base of E. Expressed the image $x_\alpha=\varphi_\alpha(x)$ of $x \in U_\alpha$ in terms of $\mathfrak{e}$ in the form $x_\alpha=\sum_{i=1}^{\infty} x_\alpha^i \mathfrak{e}_i$, its components $(x_\alpha^1, x_\alpha^2,\ldots, x_\alpha^n,\ldots)$ are called the coordinates of x_α with respect to $\mathfrak{e}$.

Suppose that an open subset U of M and a homeomorphism φ: $U\to V$ onto an open subset V of E are given. We say that a pair (U, φ) is compatible with the atlas $\{(U_\alpha, \varphi_\alpha)_{\alpha\in I}\}$ if every map $\varphi_\alpha\circ\varphi^{-1}$ is a $C^{(n)}$-diffeomorphism. Two atlases are said to be compatible if every chart of one atlas is compatible with an another atlas. It is verified immediately that the relation of compatibility between atlases is an equivalence relation. An equivalence class of atlases of class $C^{(n)}$ on M is said to define a structure of $C^{(n)}$-manifold on M. Then we say that M is a Hilbert manifold of class $C^{(n)}$ or a $C^{(n)}$-Hilbert manifold and E is a base space of M. In what follows, we shall assume that the base space of a Hilbert manifold M is the Hilbert space E.

From now on, we shall introduce the definition of tangent space. Let M be a Hilbert manifold of class $C^{(n)}$ and let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be the atlas of class $C^{(n)}$ on M. By I_x we shall denote the index set of all charts at $x \in M$. Considering a product set $E\times I_x$, we denote by X_α an element (X, α) of $E\times I_x$ and moreover, for a bounded linear operator $A\in \mathscr{L}(E, E)$, we define AX_α by $(AX, \alpha)\in E\times I_x$.

We say that two elements X_α, Y_β of $E\times I_x$ are equivalent if and only if there exists a $C^{(n)}$-diffeomorphism $\varphi_{\beta\alpha}$ such that $Y_\beta=\varphi'_{\beta\alpha}(x_\alpha)X_\alpha$ where $\varphi'_{\beta\alpha}(x_\alpha)=\varphi'_{\beta\alpha}(\varphi_\alpha(x))\in \mathscr{L}(E, E)$ is the derivative of ${}_{\beta\alpha}\varphi$at $x_\alpha=\varphi_\alpha(x)\in E$ and henceforth this derivative $\varphi'_{\beta\alpha}(x_\alpha)$ is said to be the derivative of $\varphi_{\beta\alpha}$ at a point x of M. Since $\varphi_{\beta\alpha}$ is a $C^{(n)}$-diffeomorphism and $\varphi'_{\gamma\alpha}(x_\alpha)=\varphi'_{\gamma\beta}(x_\beta)\circ\varphi'_{\beta\alpha}(x_\alpha)$, we infer readily that the equivalence thus defined satisfies the axiom of equivalence relation.

By a tangent vector $\mathfrak{X}$ at $x\in M$ we mean an equivalence class $\{X_\alpha\}$ of elements of $E\times I_x$.

Moreover we introduce the addition of two tangent vectors $\mathfrak{X}=\{X_\alpha\}$ and $\mathfrak{Y}=\{Y_\alpha\}$ at x, and the multiplication of a scalar $\lambda\in R$ to a tangent vector $\mathfrak{X}=\{X_\alpha\}$ at x as follows:

$$\mathfrak{X}+\mathfrak{Y}=\{(X+Y)_\alpha\}, \quad \lambda\mathfrak{X}=\{(\lambda X)_\alpha\}.$$

It can be seen easily that these addition and multiplication thus defined are irrelevant to the representatives of equivalence classes.

Thus, the set of all tangent vectors $\mathfrak{X}$ at $x\in M$ which has the above introduced addition and multiplication, is clearly a vector space over the real field R.

This vector space is said to be the tangent space of M at $x\in M$ and we denote this tangent space by $T_x(M)$.

Next, our purpose is to make the tangent space $T_x(M)$ to a Hilbert space which is isomorphic to the Hilbert space E.

Given an index $\alpha\in I_x$ and a tangent vector $\mathfrak{X}\in T_x(M)$, then there exists a unique vector $X\in E$ such that $X_\alpha=(X, \alpha)\in \mathfrak{X}$. From now on, we shall identify a representative X_α of $\mathfrak{X}$ with this unique vector $X\in E$. Thus we can consider a map $\phi_\alpha(x)$: $\mathfrak{X}\to X_\alpha$ of the tangent space $T_x(M)$ into the Hilbert space E.

Now we define an inner product $\langle \mathfrak{X}, \mathfrak{Y}\rangle$ of two tangent vectors $\mathfrak{X}, \mathfrak{Y} \in T_x(M)$ as follows:

$$\langle \mathfrak{X}, \mathfrak{Y}\rangle = \langle \phi_{\alpha_0}(x)(\mathfrak{X}), \phi_{\alpha_0}(x)(\mathfrak{Y})\rangle ,$$

where α_0 is an arbitrary fixed index belonging to I_x. We infer readily that the inner product thus defined satisfies the axiom of inner product. Of course, the norm $\|\mathfrak{X}\|$ of a tangent vector $\mathfrak{X}$ is given by $\sqrt{\langle \mathfrak{X}, \mathfrak{X}\rangle}$. Moreover, it is seen easily that the tangent space $T_x(M)$ is complete with respect to the above norm. Consequently, we get a Hilbert space $T_x(M)$. In what follows, the tangent spaces is assumed to be the Hilbert space.

Roughly speaking, the tangent space $T_x(M)$ at $x \in M$ is the Hilbert space E attached to a point x of M. As a matter fact, we have

Proposition.

Let M be a Hilbert manifold of class $C^{(n)}$ and $T_x(M)$ be the tangent space at a point x of M.

Then a map $\phi_{\alpha_0}(x)$: $T_x(M) \to E$ given by $X_{\alpha_0} = \phi_{\alpha_0}(x)(\mathfrak{X})$ for an arbitrary tangent vector $\mathfrak{X}$ of $T_x(M)$, is an isomorphism of the tangent space $T_x(M)$ onto the Hilbert space E where X_{α_0} is the representative of $\mathfrak{X}$ identified with a vector of E and α_0 is an arbitrary fixed index belonging to I_x.

Proof. Let $\mathfrak{X}, \mathfrak{Y}$ be two arbitrary tangent vectors at a point x of M. To prove this Proposition it suffices to show that a map $\phi_{\alpha_0}(x)$ is linear, injective and surjective, because we have

$$\langle \mathfrak{X}, \mathfrak{Y}\rangle = \langle \phi_{\alpha_0}(x)(\mathfrak{X}), \phi_{\alpha_0}(x)(\mathfrak{Y})\rangle .$$

We see immediately from definitions of addition and multiplications in the tangent space $T_x(M)$ that

$$\phi_{\alpha_0}(x)(\mathfrak{X}+\mathfrak{Y}) = X_{\alpha_0} + Y_{\alpha_0} = \phi_{\alpha_0}(x)(\mathfrak{X}) + \phi_{\alpha_0}(x)(\mathfrak{Y}),$$

$$\phi_{\alpha_0}(x)(\lambda\mathfrak{X}) = \lambda X_{\alpha_0} = \lambda\phi_{\alpha_0}(x)(\mathfrak{X}),$$

where $X_{\alpha_0} = \phi_{\alpha_0}(x)(\mathfrak{X})$, $Y_{\alpha_0} = \phi_{\alpha_0}(x)(\mathfrak{Y})$ and λ is an arbitrary real number. Thus $\phi_{\alpha_0}(x)$ is linear.

Now it is easy to see that $\phi_{\alpha_0}(x)$ is injective. Suppose that $\phi_{\alpha_0}(x)(\mathfrak{X})=0$ for an tangent vector $\mathfrak{X} \in T_x(M)$. Then $X_{\alpha_0}=0$ and since $\varphi'_{\alpha\alpha_0}(x) \in \mathscr{L}(E, E)$ we have $X_\alpha = \varphi'_{\alpha\alpha_0}(x)\, X_{\alpha_0} = 0$ for every index $\alpha \in I_x$. Hence $\mathfrak{X}=0$ and therefore $\phi_{\alpha_0}(x)$ is injective.

Next, we prove that $\phi_{\alpha_0}(x)$ is surjective. In fact, given an arbitrary element $X \in E$, we infer easily that there exists $\mathfrak{X} = \{X_\alpha\} \in T_x(M)$ maked by $X_\alpha = \varphi'_{\alpha\alpha_0}(x)\, X$ for every index $\alpha \in I_x$. Then we have $\phi_{\alpha_0}(x)(\mathfrak{X}) = X$. Thus $\phi_{\alpha_0}(x)$ is surjective.

Henceforth we shall denote by X or X_x, instead of $\mathfrak{X}$, the tangent vector at a point x, if there is no possibility of confusion.

Let X_α, X_β be two representatives of a tangent vector X at a point x of M. Then it is known already that we have a relation $X_\beta=\varphi'_{\beta\alpha}(x_\alpha)X_\alpha$. Here we investigate a coordinate representation of the above transformation law.

Letting $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ be a base of the Hilbert space E, we expand a relation $x_\beta=\varphi_{\beta\alpha}(x_\alpha)$, the above two representatives X_α and X_β in terms of $\mathfrak{e}$ respectively in the following form

$$x_\beta=\sum_{i=1}^{\infty} x_\beta^i(x_\alpha)\mathfrak{e}_i, \quad X_\alpha=\sum_{i=1}^{\infty} X_\alpha^i\mathfrak{e}_i,$$

$$X_\beta=\sum_{i=1}^{\infty} X_\alpha^i\mathfrak{e}_i.$$

Then, in virtue of Proposition 3 in § 2.1, we have

$$\varphi'_{\beta\alpha}(x_\alpha)X_\alpha=\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{\infty}\frac{\partial x_\beta^i}{\partial x_\alpha^j}(x_\alpha)X_\alpha^j\right\}\mathfrak{e}_i,$$

and therefore we get, from $X_\beta=\varphi'_{\beta\alpha}(x_\alpha)X_\alpha$,

$$X_\beta^i=\sum_{j=1}^{\infty}\frac{\partial x_\beta^i}{\partial x_\alpha^j}(x_\alpha)X_\alpha^j.$$

Thus, as a coordinate representation of the transformation law

$$X_\beta=\varphi'_{\beta\alpha}(x_\alpha)X_\alpha,$$

we obtain in the matrix form

$$\begin{pmatrix} X_\beta^1 \\ \\ X_\beta^2 \\ \vdots \\ X_\beta^i \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\partial x_\beta^1}{\partial x_\alpha^1} & \frac{\partial x_\beta^1}{\partial x_\alpha^2} \cdots\cdots & \frac{\partial x_\beta^1}{\partial x_\alpha^j} \cdots \\ \frac{\partial x_\beta^2}{\partial x_\alpha^1} & \frac{\partial x_\beta^2}{\partial x_\alpha^2} \cdots\cdots & \frac{\partial x_\beta^2}{\partial x_\alpha^j} \cdots \\ \vdots & \vdots & \vdots \\ \frac{\partial x_\beta^i}{\partial x_\alpha^1} & \frac{\partial x_\beta^i}{\partial x_\alpha^2} \cdots\cdots & \frac{\partial x_\beta^i}{\partial x_\alpha^j} \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} X_\alpha^1 \\ \\ X_\alpha^2 \\ \vdots \\ X_\alpha^j \\ \vdots \end{pmatrix} \tag{2.1}$$

where we omit x_α.

We denote for simplicity the matrix obtained above by $(\partial x_\beta^i/\partial x_\alpha^j)$ and we call it the matrix of coordinates transformation $x_\beta=\varphi_{\beta\alpha}(x_\alpha)$ with respect to a base $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ of E.

Here we give a remark on the matrix of coordinate transformation. Let $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be two charts at a point $x\in M$ and let be $x_\alpha=\varphi_\alpha(x)$, $x_\beta=\varphi_\beta(x)$. Then we have two coordinate transformations $x_\alpha=\varphi_{\alpha\beta}(x_\beta)$ and $x_\beta=\varphi_{\beta\alpha}(x_\alpha)$ which are inverse with each other, where $\varphi_{\alpha\beta}=\varphi_\alpha\circ\varphi_\beta^{-1}$ and $\varphi_{\beta\alpha}=\varphi_\beta\circ\varphi_\alpha^{-1}$.
Hence we get immediately $\varphi_{\alpha\beta}\circ\varphi_{\beta\alpha}=\varphi_{\beta\alpha}\circ\varphi_{\alpha\beta}=$identity. Thus, by using the rule for the derivative of a compound map, we see

$$\varphi'_{\alpha\beta}(x_\beta)\circ\varphi'_{\beta\alpha}(x_\alpha)=1, \quad \varphi'_{\beta\alpha}(x_\alpha)\circ\varphi'_{\alpha\beta}(x_\beta)=1 \tag{2.2}$$

where 1 is the identity operator on E. Moreover, letting $(\partial x_\alpha^i/\partial x_\beta^j)$ and $(\partial x_\beta^i/\partial x_\alpha^j)$ be respectively the matrices of coordinate transformations $x_\alpha=\varphi_{\alpha\beta}(x_\beta)$ and $x_\beta=\varphi_{\beta\alpha}(x_\alpha)$ with respect to a base $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ of E, then we have from (2.2)

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{\partial x_\alpha^i}{\partial x_\beta^k}(x_\beta)\frac{\partial x_\beta^k}{\partial x_\alpha^j}(x_\alpha)=\delta_j^i,\\&\sum_{k=1}^{\infty}\frac{\partial x_\beta^i}{\partial x_\alpha^k}(x_\alpha)\frac{\partial x_\alpha^k}{\partial x_\beta^j}(x_\beta)=\delta_j^i\end{aligned}\tag{2.3}$$

where δ_j^i is the Kronecker's symbol. This shows that matrices $(\partial x_\alpha^i/\partial x_\beta^j)$ and $(\partial x_\beta^i/\partial x_\alpha^j)$ is mutually reciprocal.

We conclude this paragraph with representing the tangent vector as the derivation in analogy to the differential geometry. Let M be a Hilbert manifold of class $C^{(n)}$ as it has been and U be an open set of M. Giving a chart $(U_\alpha, \varphi_\alpha)$ at an arbitrary point x of U, we denote by $C^{(n)}(U, R)$ a set of all functions $f\colon U\to R$ such that the function $f_\alpha=f\circ\varphi_\alpha^{-1}\colon \varphi_\alpha(U\cap U_\alpha)\to R$ is of class $C^{(n)}$ at x. We say that a function $f\in C^{(n)}(U, R)$ is of class $C^{(n)}$ on an open set U of M. Since, in case of another chart (U_β, φ_β) at a point x of U, we have a relation $f_\beta=f_\alpha\circ\varphi_{\alpha\beta}$, the above definition is clearly independent of charts at $x\in U$.

Now, for an arbitrary real number $\lambda\in R$ and arbitrary two functions f, $g\in C^{(n)}(U, R)$, we introduce its addition and multiplication as usual by the following manner;

$$\begin{aligned}&(\lambda f)(x)=\lambda f(x),\quad (f+g)(x)=f(x)+g(x),\\&(fg)(x)=f(x)g(x),\end{aligned}$$

where x is an arbitrary point of U. Obviously λf, $f+g$ and fg belong to $C^{(n)}(U, R)$. Therefore $C^{(n)}(U, R)$ is an algebra over the real field R.

We shall go here to express the tangent vector by the derivation. Let $(U_\alpha, \varphi_\alpha)$ be a chart at a point x of M and X_α be a representative of an arbitrary tangent vector $X\in T_x(M)$. Moreover, let U be an open set containing the point x. By $X(f)$ or $X_x(f)$ we shall mean the real value as follows:

$$X(f)=f'_\alpha(x_\alpha)X_\alpha\tag{2.4}$$

where f is an arbitrary function of $C^{(n)}(U, R)$, $f_\alpha=f\circ\varphi_\alpha^{-1}$ and $x_\alpha=\varphi_\alpha(x)$. This definition is independent of the choice of representative X_α. In fact, let X_β be another representative of X. We have $f_\beta=f_\alpha\circ\varphi_{\alpha\beta}$ and therefore $f'_\beta(x_\beta)=f'_\alpha(x_\alpha)\circ\varphi'_{\alpha\beta}(x_\beta)$. By using the transformation law

$$X_\alpha=\varphi'_{\alpha\beta}(x_\beta)X_\beta,$$

we see immediately

$$X(f)=f'_\alpha(x_\alpha)X_\alpha=f'_\beta(x_\beta)X_\beta.\tag{2.5}$$

Thus we may denote by $f'(x)X$, instead of $f'_\alpha(x_\alpha)X_\alpha$, in the above definition (2.4).

We can infer easily that $X_x(f)$ satisfies the following properties:

$$X_x(f+g)=X_x(f)+X_x(g),$$
$$X_x(\lambda f)=\lambda X_x(f), \tag{2.6}$$
$$X_x(fg)=X_x(f)g(x)+f(x)X_x(g),$$

where f, g are any functions of $C^{(n)}(U, R)$ and λ is an arbitrary real number. Furthermore, in virtue of $f'_\alpha(x_\alpha)\in\mathscr{L}(E, R)$, we have clearly

$$(X+Y)(f)=X(f)+Y(f),$$
$$(\lambda X)(f)=\lambda(X(f)). \tag{2.7}$$

Next, let $\mathfrak{e}=(\mathfrak{e}_1, \mathfrak{e}_2,\dots, \mathfrak{e}_n,\dots)$ be a base of E and let X^i_α, X^i_β, x^i_α, x^i_β be respectively components of X_α, X_β, x_α, x_β with respect to the base $\mathfrak{e}$. Then, by applying Proposition 1 of § 2.1 to the formulas (2.5) and (2.7), we obtain

$$X(f)=\sum_{i=1}^\infty X^i_\alpha\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)=\sum_{i=1}^\infty X^i_\beta\frac{\partial f_\beta}{\partial x^i_\beta}(x_\beta),$$
$$\sum_{i=1}^\infty (X^i_\alpha+Y^i_\alpha)\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)=\sum_{i=1}^\infty X^i_\alpha\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)+\sum_{i=1}^\infty Y^i_\alpha\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha), \tag{2.8}$$
$$\sum_{i=1}^\infty (\lambda X^i_\alpha)\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)=\lambda\sum_{i=1}^\infty X^i_\alpha\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha).$$

These facts show that a tangent vector X may be expressed by $X=\sum_{i=1}^\infty X^i\frac{\partial}{\partial x^i}$ or $X_x=\sum_{i=1}^\infty X^i_x\left(\frac{\partial}{\partial x^i}\right)_x$ and that $X+Y$, λX may be executed as follows:

$$\sum_{i=1}^\infty X^i\frac{\partial}{\partial x^i}+\sum_{i=1}^\infty Y^i\frac{\partial}{\partial x^i}=\sum_{i=1}^\infty (X^i+Y^i)\frac{\partial}{\partial x^i},$$
$$\lambda\sum_{i=1}^\infty X^i\frac{\partial}{\partial x^i}=\sum_{i=1}^\infty (\lambda X^i)\frac{\partial}{\partial x^i}.$$

With the aid of the transformation law (2.1), we get from (2.8)

$$\sum_{i=1}^\infty X^i_\alpha\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)=\sum_{i=1}^\infty X^i_\alpha\sum_{j=1}^\infty\frac{\partial x^j_\beta}{\partial x^i_\alpha}(x_\alpha)\frac{\partial f_\beta}{\partial x^j_\beta}(x_\beta).$$

In particular, putting $(X^1_\alpha,\dots, X^i_\alpha,\dots)=(0,\dots, 0, \overset{i}{\breve{1}}, 0,\dots)$ in the above formula, we have

$$\frac{\partial f_\alpha}{\partial x^i_\alpha}(x_\alpha)=\sum_{j=1}^\infty\frac{\partial x^j_\beta}{\partial x^i_\alpha}(x_\alpha)\frac{\partial f_\beta}{\partial x^j_\beta}(x_\beta), \tag{2.9}$$

and moreover taking $f=\varphi^k_\alpha$ in the above equality (2.9) where φ^k_α is a coordinate function mapping a point x to k-coordinate x^k_α of $x_\alpha=\varphi_\alpha(x)$, we get

$$\sum_{j=1}^\infty\frac{\partial x^k_\alpha}{\partial x^j_\beta}(x_\beta)\frac{\partial x^j_\beta}{\partial x^i_\alpha}(x_\alpha)=\delta^k_i.$$

This is a formula (2.3) obtained formerly and the relation (2.9) gives a differential rule for the transformation of variables, namely,

$$\frac{\partial}{\partial x_\alpha^i}=\sum_{j=1}^{\infty}\frac{\partial x_\beta^j}{\partial x_\alpha^i}(x_\alpha)\frac{\partial}{\partial x_\beta^j}\,. \tag{2.10}$$

Now let $e^\alpha=(e_1^\alpha, e_2^\alpha,\dots, e_n^\alpha,\dots)$ and $e^\beta=(e_1^\beta, e_2^\beta,\dots, e_n^\beta,\dots)$ be two bases of E such that they are binded by a transformation law as follows:

$$e_i^\alpha=\sum_{i=1}^{\infty}\frac{\partial x_\beta^j}{\partial x_\alpha^i}(x_\alpha)e_j^\beta. \tag{2.11}$$

Comparing (2.10) with (2.11), we see that $\partial/\partial x_\alpha=(\partial/\partial x_\alpha^1, \partial/\partial x_\alpha^2,\dots, \partial/\partial x_\alpha^n,\dots)$ obey the same transformation law as the base $e^\alpha=(e_1^\alpha, e_2^\alpha,\dots, e_n^\alpha,\dots)$.

Consequently, we may identify a tangent vector $X=\sum_{i=1}^{\infty} X^i e_i$ with a corresponding derivation $\sum_{i=1}^{\infty} X^i\frac{\partial}{\partial x^i}$ and henceforth we write a tangent vector X in the form

$$X=\sum_{i=1}^{\infty} X^i\frac{\partial}{\partial x^i}\,. \tag{2.12}$$

Finally, we shall define the vector field of class $C^{(n)}$ and derive the Jacobi's identity.

Consider the tangent space $T_x(M)$ at each point x of a $C^{(n)}$-Hilbert manifold M and put $T(M)=\bigcup_{x\in M} T_x(M)$.

Definition.

By *a vector field* on M we mean a map $X\colon M\to T(M)$ such that $X(x)$ belongs to the tangent space $T_x(M)$ for each point $x\in M$. Let x_0 be an arbitrary point of M and U be an open set of M containing a point x_0. Then, by *a vector field of class* $C^{(n)}$ on M we mean a vector field X on M such that a map $\phi_{\alpha_0}(X)\colon U\to E$ given by $\phi_{\alpha_0}(X)\colon x\to \phi_{\alpha_0}(x)\,(X(x))\in E$, belongs to $C^{(n)}(U, E)$ where $\phi_{\alpha_0}(x)$ is the isomorphism of $T_x(M)$ onto E given in Proposition of § 2.2.

In what follows we write

$$X(x)=\sum_{i=1}^{\infty} X^i(x)\left(\frac{\partial}{\partial x^i}\right)_x.$$

By $\mathfrak{X}^{(n)}(M)$ we denote a set of all vector fields of class $C^{(n)}$ on M. Clearly $\mathfrak{X}^{(n)}(M)$ is a $C^{(n)}(M, R)$-module.

Next, letting X and Y be in $\mathfrak{X}^{(n)}(M)$, we shall introduce the bracket $[X, Y]$. For this purpose, choose a base $e=(e_1, e_2,\dots, e_n,\dots)$ of E and set $\bar{X}(x)=\phi_{\alpha_0}(x)\,(X(x))=\sum_{i=1}^{\infty}\bar{X}^i(x)e_i$, similarly, $\bar{Y}(x)=\phi_{\alpha_0}(x)(Y(x))=\sum_{i=1}^{\infty}\bar{Y}^i(x)e_i$ where we omit α_0 for simplicity. Then we define the bracket $[X, Y]$ as a map from $C^{(n)}(M, R)$ into $C^{(n-2)}(M, R)$ as follows:

$$[X, Y](f)=\bar{X}(\bar{Y}(f))-\bar{Y}(\bar{X}(f)).$$

We shall show that $[X, Y]$ is a vector field of class $C^{(n-2)}$ on M. In fact, by the law for the derivative of a product and Proposition 1, 3 in § 2.1, we have

$$\begin{aligned}[X, Y](f(x)) &= \{f'(x)(\bar{Y}(x))\}'(\bar{X}(x)) - \{f'(x)(\bar{X}(x))\}'(\bar{Y}(x)) \\ &= f''(x)(\bar{Y}(x), \bar{X}(x)) + f'(x)\{\bar{Y}'(x)(\bar{X}(x))\} \\ &\quad - f''(x)(\bar{X}(x), \bar{Y}(x)) - f'(x)\{\bar{X}'(x)(\bar{Y}(x))\} \\ &= f'(x)\{\bar{Y}'(x)(\bar{X}(x)) - \bar{X}'(x)(\bar{Y}(x))\} \\ &= \sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\left\{\bar{X}^j(x)\frac{\partial \bar{Y}^i(x)}{\partial x^j} - \bar{Y}^j(x)\frac{\partial \bar{X}^i(x)}{\partial x^j}\right\}\right]\frac{\partial f(x)}{\partial x^i}\end{aligned}$$

where, since $f''(x) \in \mathscr{L}_2(E, R)$, the two terms involving $f''(x)$ cancel. This expression for $[X, Y](f)$ show that $[X, Y]$ is a vector field.

Finally, let X, Y, Z be in $\mathfrak{X}^{(n)}(M)$, then it is easy to prove that the above bracket satisfies the following identities:

$$[Y, X] = -[X, Y],$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(Jacobi's identity)

3. The differential of functions of class $C^{(n)}$

Let U be an open neighborhood of an arbitrary point x of M and f be a function of class $C^{(n)}$ defined on U. By the differential of f at x we mean a map $df(x)$ of $T_x(M)$ into R given by $df(x)X = X(f)$ where X is an arbitrary tangent vector of M at x.

We infer readily that the differential $df(x)$ is a linear continuous mapping, since we have

$$|df(x)(X-Y)| = |f'(x)(X-Y)| \leqq \|f'(x)\| \, \|X-Y\|, \quad \text{for} \quad X, Y \in T_x(M).$$

Thus the differential $df(x)$ of a function f is a linear continuous functional on the tangent space $T_x(M)$.

Moreover, we conclude immediately from (2.6) of § 2.2 that for $X \in T_x(M)$ and $f, g \in C^{(n)}(U, R)$,

$$\begin{aligned} d(f+g)(x)X &= df(x)X + dg(x)X, \\ d(\lambda f)(x)X &= \lambda(df(x)X), \\ d(fg)(x)X &= (df(x)X)g(x) + f(x)(dg(x)X), \end{aligned} \tag{3.1}$$

where λ is an arbitrary real number.

Now, let (U, φ) be a chart at a point x_0 of M and $(x^1, x^2, \ldots, x^n, \ldots)$ be the coordinates of a point x of U with respect to a base $\mathfrak{e}$ of E. Then we have the following expression:

$$df(x_0) = \sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)\, dx^i(x_0), \tag{3.2}$$

where the right-hand side of this expression means that,

$$\left\{\sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)\, dx^i(x_0)\right\} X = \sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)(dx^i(x_0)X),$$

for $X \in T_{x_0}(M)$, and dx^i are the differential of coordinate functions $\varphi^i: U \to R$ given by $x^i = \varphi^i(x)$.

To prove this it suffices to show that

$$\sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)(dx^i(x_0)X) = X(f).$$

Letting $X = \sum_{i=1}^{\infty} X^i(x_0) \frac{\partial}{\partial x^i}$, we have the following relation from the definition of the differential $df(x_0)$,

$$df(x_0)\left(\sum_{i=1}^{\infty} X^i(x_0) \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{\infty} X^i(x_0) \frac{\partial f}{\partial x^i}(x_0),$$

in particular, we see

$$df(x_0) \frac{\partial}{\partial x^i} = \frac{\partial f}{\partial x^i}(x_0).$$

Taking up a coordinate function φ^j as function f in this equality, we have

$$dx^j(x_0) \frac{\partial}{\partial x^i} = \frac{\partial x^j}{\partial x^i}(x_0) = \delta_i^j. \tag{3.3}$$

By (3.3) and the continuity of the differential, we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)(dx^i(x_0)X) &= \sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)\left\{dx^i(x_0)\left(\sum_{i=1}^{\infty} X^j(x_0) \frac{\partial}{\partial x^j}\right)\right\} \\
&= \sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0)\left\{\sum_{j=1}^{\infty} X^j(x_0)\, dx^i(x_0) \frac{\partial}{\partial x^j}\right\} \\
&= \sum_{i=1}^{\infty} \frac{\partial f}{\partial x^i}(x_0) X^i(x_0) \\
&= X(f).
\end{aligned}$$

In what follows we conventionally write the differential of f at x in the form

$$df = \sum_{i=1}^{\infty} \partial_i f\, dx^i.$$

4. *H*-valued differential forms

Let M be a Hilbert manifold of class $C^{(n)}$ as one has been and H be a Hilbert space. We denote $\bigcup_{x \in M} \mathscr{A}_p(T_x(M), H)$ by $\mathscr{A}_p(T(M), H)$ where $\mathscr{A}_p(T_x(M), H)$ is the set of all p-linear alternating continuous mappings of $T_x(M)$ into H.

Definition.

By *a H-valued differential p-form on M* we shall mean a map Ω_p of M into $\mathscr{A}_p(T(M), H)$ such that, for each point x of M, $\Omega_p(x)$ belongs to $\mathscr{A}_p(T_x(M), H)$. Moreover, let U be an open neighborhood of an arbitrary point x_0 of M. Then, by *a H-valued differential p-form of class $C^{(n)}$ on M* we shall mean a H-valued differential p-form Ω_p on M such that, for any p vector fields $X_1,\dots, X_p \in \mathfrak{X}^{(n)}(M)$, a map $\Omega_p(X_1,\dots, X_p)$ of U into H given by $\Omega_p(X_1,\dots, X_p)(x)=\Omega_p(x)(X_1(x),\dots, X_p(x))$ for an arbitrary point x of U belongs to $C^{(n)}(U, H)$. Here $C^{(n)}(U, H)$ is a set of all mappings of U into H of class $C^{(n)}$.

For simplicity we write $\Omega_p(x\colon X_1,\dots, X_p)$ for $\Omega_p(x)(X_1(x),\dots, X_p(x))$. We denote by $\Omega_p^{(n)}(M, H)$ a set of all H-valued differential p-forms of class $C^{(n)}$ on M. This $\Omega_p^{(n)}(M, H)$ is clearly a real vector space under the natural addition and scalar multiplication. Moreover we define the multiplication of $f\in C^{(n)}(M, R)$ and $\Omega_p \in \Omega_p^{(n)}(M, H)$ by $(f\Omega_p)(x)=f(x)\Omega_p(x)$ for each point x of M. Thus a vector space $\Omega_p^{(n)}(M, H)$ becomes a $C^{(n)}(M, R)$-module.

Now let $\mathfrak{y}=(\mathfrak{y}_1, \mathfrak{y}_2,\dots, \mathfrak{y}_n,\dots)$ be a base of H. Then a H-valued differential p-form Ω_p of class $C^{(n)}$ on M is uniquely represented in terms of $\mathfrak{Y}$ in the following form:

$$\Omega_p(x\colon X_1,\dots, X_p)=\sum_{i=1}^{\infty} \omega_p^i(x\colon X_1,\dots, X_p)\mathfrak{y}_i, \tag{4.1}$$

where $X_1,\dots, X_p \in \mathfrak{X}^{(n)}(M)$.

It is clear that ω_p^i $(i=1, 2,\dots)$ are the real valued differential p-forms on M. By Proposition 2 of § 2.1, we see readily that ω_p^i $(i=1, 2,\dots)$ are of class $C^{(n)}$. For simplicity we write (4.1) in the form

$$\Omega_p(x)=\sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{y}_i. \tag{4.2}$$

Thus we have the following proposition:

Proposition.

Let $\mathfrak{y}=(\mathfrak{y}_1, \mathfrak{y}_2,\dots, \mathfrak{y}_n,\dots)$ be a base of H. Then, a H-valued differential p-form Ω_p of class $C^{(n)}$ on M is uniquely represented in terms of $\mathfrak{y}$ as follows:

$$\Omega_p(x)=\sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{y}_i,$$

where ω_p^i $(i=1, 2,\dots)$ are the real valued differential p-forms of class $C^{(n)}$ on M.

As particular case, a H-valued differential 0-form is none other than a H-valued function, namely, $\Omega_0^{(n)}(M, H)=C^{(n)}(M, H)$, and a H-valued differential 1-form is a map Ω_1 such that $\Omega_1(x)\in \mathscr{L}(T_x(M), H)$.

5. Exterior multiplication

Let H_1, H_2 be Hilbert spaces and $H_1\otimes H_2$ be the direct product of H_1 and H_2 (see [19], Chapter II). We put $H=H_1\otimes H_2$. It is well known that (1) H is also a Hilbert space, (2) the direct product is continuous with respect to each factor and (3)

$\{\mathfrak{y}_i^{(1)}\otimes\mathfrak{y}_j^{(2)}\}_{i,j=1,2,\dots}$ is a base of H where $\mathfrak{y}^{(1)}=(\mathfrak{y}_1^{(1)}, \mathfrak{y}_2^{(1)},\dots, \mathfrak{y}_n^{(1)},\dots)$, $\mathfrak{y}^{(2)}=(\mathfrak{y}_1^{(2)}, \mathfrak{y}_2^{(2)}, \dots, \mathfrak{y}_n^{(2)},\dots)$ are the bases of H_1, H_2 respectively.

Next, let U be an open set of a Hilbert manifold M of class $C^{(n)}$ and consider a map f of U into H such that $f(x)=f_1(x)\otimes f_2(x)$ for an arbitrary point x of U where f_1, f_2 are the mappings of U into H_1, H_2 respectively. Then it is easy to show that we have the law for the derivative of a product, namely,

$$f'(x)y=f_1'(x)y\otimes f_2(x)+f_1(x)\otimes f_2'(x)y \tag{5.1}$$

where y is an arbitrary vector of the base space E of M.

Now we shall define the exterior multiplication in the following manner.

Definition.

Let Φ_p, Ψ_q be respectively a H_1-valued differential p-form and a H_2-valued differential q-form on M. By *the exterior product* of two differential forms Φ_p and Ψ_q, denoted by $\Phi_p\wedge\Psi_q$, we shall mean a H-valued differential $(p+q)$-form such that, for an arbitrary point x of M and any $(p+q)$ vector fields $X_1,\dots, X_{p+q}$ of $\mathfrak{X}^{(n)}(M)$,

$$\begin{aligned}(\Phi_p\wedge\Psi_q)(x: X_1,\dots, X_{p+q})\\ =\sum_\sigma \varepsilon(\sigma)\Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)})\otimes\Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\end{aligned}$$

where the summation is done over all permutations σ of $\{1, 2,\dots, p+q\}$ satisfying $\sigma(1)<\cdots<\sigma(p)$ and $\sigma(p+1)<\cdots<\sigma(p+q)$, and

$$\varepsilon(\sigma)=\begin{cases}+1, & \text{for even permutation } \sigma,\\ -1, & \text{for odd permutation } \sigma.\end{cases}$$

Now suppose that Φ_p, Ψ_q are two differential forms of class $C^{(n)}$ on M, then the exterior product $\Phi_p\wedge\Psi_q$ is also of class $C^{(n)}$ on M. In fact, letting U be an open set of an arbitrary point x of M and $X_1,\dots, X_{p+q}$ be vector fields of $\mathfrak{X}^{(n)}(M)$, we have a mapping $(\Phi_p(X_1,\dots, X_p), \Psi_q(X_{p+1},\dots, X_{p+q}))$ of U into a product space $H_1\times H_2$ of class $C^{(n)}$, because $\Phi_p(X_1,\dots, X_p)$ is a mapping of U into H_1 of class $C^{(n)}$ and $\Psi_q(X_{p+1},\dots, X_{p+q})$ is a mapping of U into H_2 of class $C^{(n)}$. Moreover, since $\Phi_p\wedge\Psi_q$ is a bilinear continuous mapping of $H_1\times H_2$ into $H=H_1\otimes H_2$, this mapping $\Phi_p\wedge\Psi_q$ is of class $C^{(\infty)}$. Thus a mapping $(\Phi_p\wedge\Psi_q)(X_1,\dots, X_{p+q})$ of U into H is of class $C^{(n)}$.

Next, let us consider two differential forms $\Phi_p\in\Omega_p^{(n)}(M, H_1)$, $\Psi_q\in\Omega_q^{(n)}(M, H_2)$ which are represented by

$$\Phi_p(x: X_1,\dots, X_p)=\sum_{i=1}^{\infty}\varphi_p^i(x: X_1,\dots, X_p)\mathfrak{y}_i^{(1)},$$

$$\Psi_q(x: X_{p+1},\dots, X_{p+q})=\sum_{i=1}^{\infty}\psi_q^i(x: X_{p+1},\dots, X_{p+q})\mathfrak{y}_i^{(2)},$$

where $\varphi_p^i\in\Omega_p^{(n)}(M, R)$, $\psi_q^i\in\Omega_q^{(n)}(M, R)$ for $i=1, 2,\dots$ and $\mathfrak{y}^{(1)}=(\mathfrak{y}_1^{(1)}, \mathfrak{y}_2^{(1)},\dots, \mathfrak{y}_n^{(1)},\dots)$, $\mathfrak{y}^{(2)}=(\mathfrak{y}_1^{(2)}, \mathfrak{y}_2^{(2)},\dots, \mathfrak{y}_n^{(2)},\dots)$ are bases of H_1, H_2 respectively.

Since the direct product is continuous with respect to each factor, we obtain

$$
\begin{aligned}
&(\Phi_p \wedge \Psi_q)(x\colon X_1,\dots, X_{p+q})\\
&=\sum_\sigma \varepsilon(\sigma)\Phi_p(x\colon X_{\sigma(1)},\dots, X_{\sigma(p)})\otimes \Psi_q(x\colon X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\\
&=\sum_\sigma \varepsilon(\sigma)\{\sum_{i=1}^{\infty} \varphi_p^i(x\colon X_{\sigma(1)},\dots, X_{\sigma(p)})\mathfrak{y}_i^{(1)}\}\otimes\{\sum_{j=1}^{\infty} \psi_q^j(x\colon X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\mathfrak{y}_j^{(2)}\}\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \{\sum_\sigma \varepsilon(\sigma)\varphi_p^i(x\colon X_{\sigma(1)},\dots, X_{\sigma(p)})\psi_q^j(x\colon X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\}\mathfrak{y}_i^{(1)}\otimes\mathfrak{y}_j^{(2)}\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\varphi_p^i\wedge\psi_q^j)(x\colon X_1,\dots, X_{p+q})\mathfrak{y}_i^{(1)}\otimes\mathfrak{y}_j^{(2)}.
\end{aligned}
$$

Thus we get

Proposition.

For two differential forms $\Phi_p \in \Omega_p^{(n)}(M, H_1)$ and $\Psi_q \in \Omega_q^{(n)}(M, H_2)$ represented in the forms

$$\Phi_p(x)=\sum_{i=1}^{\infty} \varphi_p^i(x)\mathfrak{y}_i^{(1)},$$

$$\Psi_q(x)=\sum_{i=1}^{\infty} \psi_q^i(x)\mathfrak{y}_i^{(2)},$$

their exterior product $\Phi_p\wedge\Psi_q$ is represented by

$$(\Phi_p\wedge\Psi_q)(x)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\varphi_p^i\wedge\psi_q^j)(x)\mathfrak{y}_i^{(1)}\otimes\mathfrak{y}_j^{(2)}, \tag{5.2}$$

where $\mathfrak{y}^{(1)}=(\mathfrak{y}_1^{(1)}, \mathfrak{y}_2^{(1)},\dots, \mathfrak{y}_n^{(1)},\dots)$, $\mathfrak{y}^{(2)}=(\mathfrak{y}_1^{(2)}, \mathfrak{y}_2^{(2)},\dots, \mathfrak{y}_n^{(2)},\dots)$ are respectively the bases of H_1, H_2.

As a particular case of the above proposition, considering two differential forms $\varphi_p\in\Omega_p^{(n)}(M, R)$ and $\Psi_q\in\Omega_q^{(n)}(M, H)$, we have

$$(\varphi_p\wedge\Psi_q)(x)=\sum_{i=1}^{\infty} (\varphi_p\wedge\psi_q^i)(x)\mathfrak{y}_i,$$

where

$$\Psi_q(x)=\sum_{i=1}^{\infty} \psi_q^i(x)\mathfrak{y}_i$$

and $\mathfrak{y}=(\mathfrak{y}_1, \mathfrak{y}_2,\dots, \mathfrak{y}_n,\dots)$ is a base of H.

Now we shall define the exterior product of the differentials of coordinate functions. Let (U, φ) be a chart at a point x_0 of M and $dx^{i_1},\dots, dx^{i_n}$ be the differentials of coordinate functions $\varphi^{i_1},\dots, \varphi^{i_n}$ respectively. Then the exterior product $dx^{i_1}\wedge\cdots\wedge dx^{i_n}$ of the differentials $dx^{i_n},\dots, dx^{i_n}$ is defined by

$$(dx^{i_1}\wedge\cdots\wedge dx^{i_n})(X_1,\dots, X_n)=\sum_\sigma \varepsilon(\sigma)dx^{i_1}(X_{\sigma(1)})\cdots dx^{i_n}(X_{\sigma(n)})$$

where the sum $\sum_{\sigma}$ is to be extended over all permutations of $\{1, 2,\dots, n\}$ and $X_1,\dots, X_n$ are any n vector fields of class $C^{(n)}$ on M.

Since the differentials $d\mathfrak{x}^{i_1},\dots, d\mathfrak{x}^{i_n}$ are n linear continuous functionals on the tangent space and we have $d\mathfrak{x}^i\frac{\partial}{\partial\mathfrak{x}^j}=\delta^i_j$, we get

$$(d\mathfrak{x}^{i_1}\wedge\cdots\wedge d\mathfrak{x}^{i_n})(X_1,\dots, X_n)$$
$$=\sum_{\sigma}\varepsilon(\sigma)X^{i_1}_{\sigma(1)}\cdots\cdots X^{i_n}_{\sigma(n)}$$
$$=\begin{vmatrix} X^{i_1}_1 & X^{i_1}_2\cdots\cdots X^{i_1}_n \\ X^{i_2}_1 & X^{i_2}_2\cdots\cdots X^{i_2}_n \\ \vdots & \vdots \qquad\quad \vdots \\ X^{i_n}_1 & X^{i_n}_2\cdots\cdots X^{i_n}_n \end{vmatrix},$$

where

$$X_1=\sum_{i=1}^{\infty}X^i_1\frac{\partial}{\partial\mathfrak{x}^i},\dots, X_n=\sum_{i=1}^{\infty}X^i_n\frac{\partial}{\partial\mathfrak{x}^i}.$$

Finally we conclude this paragraph with remark on a property of exterior multiplication. Now let $\varphi_p\in\Omega^{(n)}_p(M, R)$, $\psi_q\in\Omega^{(n)}_q(M, R)$ be two real valued differential forms, then it is well known that we have

$$\varphi_p\wedge\psi_q=(-1)^{pq}\psi_q\wedge\varphi_p.$$

This property is called that the exterior multiplication of differential forms is anticommutative (see [13], Chapter 1, 1.5). However, the exterior multiplication of two differential forms is generally no anticommutative. To preserve this anticommutative property, we introduce the symmetric exterior multiplication. Let $\Phi_p\in\Omega^{(n)}_p(M, H_1)$ and $\Psi_q\in\Omega^{(n)}_q(M, H_2)$ be two differential forms on M. By the symmetric exterior product $\Phi_p\overset{s}{\wedge}\Psi_q$ of Φ_p, Ψ_q we mean a $H_1\otimes H_2$-valued differential form on M given by

$$(\Phi_p\overset{s}{\wedge}\Psi_q)(\mathfrak{x})=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\varphi^i_p\wedge\psi^j_q)(\mathfrak{x})\mathfrak{y}^{(1)}_i\overset{s}{\otimes}\mathfrak{y}^{(2)}_j,$$

where

$$\Phi_p(\mathfrak{x})=\sum_{i=1}^{\infty}\varphi^i_p(\mathfrak{x})\mathfrak{y}^{(1)}_i,\quad \Psi_q(\mathfrak{x})=\sum_{j=1}^{\infty}\psi^i_q(\mathfrak{x})\mathfrak{y}^{(2)}_i$$

and $\mathfrak{y}^{(1)}_i\overset{s}{\otimes}\mathfrak{y}^{(2)}_j=\frac{1}{2}(\mathfrak{y}^{(1)}_i\otimes\mathfrak{y}^{(2)}_j+\mathfrak{y}^{(2)}_j\otimes\mathfrak{y}^{(1)}_i)$.

Now we have

$$\Phi_p\overset{s}{\wedge}\Psi_q=(-1)^{pq}\Psi_q\overset{s}{\wedge}\Phi_p.$$

Indeed, by using that the exterior multiplication of real valued differential forms is anticommutative and $\mathfrak{y}^{(1)}_i\overset{s}{\otimes}\mathfrak{y}^{(2)}_j=\mathfrak{y}^{(2)}_j\overset{s}{\otimes}\mathfrak{y}^{(1)}_i$, we see

$$(\Phi_p \overset{s}{\wedge} \Psi_q)(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\varphi_p^i \wedge \psi_q^j)(x) \mathfrak{y}_i^{(1)} \overset{s}{\otimes} \mathfrak{y}_j^{(2)}$$

$$= (-1)^{pq} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (\psi_q^j \wedge \varphi_p^i)(x) \mathfrak{y}_j^{(2)} \overset{s}{\otimes} \mathfrak{y}_i^{(1)}$$

$$= (-1)^{pq} (\Psi_q \overset{s}{\wedge} \Phi_p)(x).$$

6. Exterior differentiation

At once we shall begin by defining the exterior differentiation.

Definition 1.

The exterior differential $d\Omega_p$ of a H-valued differential p--form Ω_p of class $C^{(n)}$ on M is defined by the following formula:

$$\begin{aligned} d\Omega_p(x: X_1,\dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \Omega_p'(x: X_1,\dots, \hat{X}_i,\dots, X_{p+1})(\overline{X}_i) \\ &+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega_p(x: [X_i, X_j], X_1,\dots, \hat{X}_i,\dots, \hat{X}_j,\dots, X_{p+1}), \end{aligned} \tag{6.1}$$

where $X_1,\dots, X_{p+1}$ are $p+1$ vector fields of class $C^{(n)}$ on M and the symbol ^ means that the term is omitted and further $\overline{X}_i$ means $\overline{X}_i(x) = \phi_{\alpha_0}(x)(X_i(x))$. Henceforth we shall omit the symbol $^{-}$.

We see immediately that the exterior differential $d\Omega_p$ is of class $C^{(n-1)}$ and that $d\Omega_p(x)$ is a $(p+1)$-linear mapping of $\mathfrak{X}^{(n)}(M) \times \cdots \times \mathfrak{X}^{(n)}(M)$ $(p+1$ times) into H.

To examine some properties of the exterior differential we introduce the Lie derivative and the interior product.

Definition 2.

Let Ω_p be a H-valued differential p-form of class $C^{(n)}$ on M and X be a vector field of class $C^{(n)}$ on M.

The Lie derivative $L_X\Omega_p$ of Ω_p with respect to X is defined by

$$\begin{aligned} L_X\Omega_p(x: X_1,\dots, X_p) &= \Omega_p'(x: X_1,\dots, X_p)(X) \\ &- \sum_{i=1}^{p} \Omega_p(x: X_1,\dots, [X, X_i],\dots, X_p). \end{aligned} \tag{6.2}$$

The interior product $i_X\Omega_p$ of X and Ω_p is defined by

$$i_X\Omega_p(x: X_1,\dots, X_{p-1}) = \Omega_p(x: X, X_1,\dots, X_{p-1}). \tag{6.3}$$

Here $X_1,\dots, X_p$ are any p vector fields of class $C^{(n)}$ on M.

We have the following proposition:

Proposition 1.

Let $\Phi_p \in \Omega_p^{(n)}(M, H_1)$, $\Psi_q \in \Omega_q^{(n)}(M, H_2)$ *be two differential forms on* M. *Then we have*

$$L_X(\Phi_p \wedge \Psi_q) = (L_X\Phi_p) \wedge \Psi_q + \Phi_p \wedge (L_X\Psi_q),$$
$$i_X(\Phi_p \wedge \Psi_q) = (i_X\Phi_p) \wedge \Psi_q + (-1)^p \Phi_p \wedge (i_X\Psi_q), \tag{6.4}$$

where H_1, H_2 are two Hilbert spaces.

Proof. Firstly, we prove the first formula of (6.4). By the definition of exterior product, we have

$$(\Phi_p \wedge \Psi_q)(x: X_1,\dots, X_{p+q})$$
$$= \sum_\sigma \varepsilon(\sigma)\Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)}) \otimes \Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)}),$$

where $X_1,\dots, X_{p+q} \in \mathfrak{X}^{(n)}(M)$.
Applying L_X to the above formula, we get

$$L_X(\Phi_p \wedge \Psi_q)(x: X_1,\dots, X_{p+q})$$
$$= \sum_\sigma \varepsilon(\sigma)\{\Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)}) \otimes \Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\}'(X)$$
$$- \sum_\sigma \varepsilon(\sigma)\{\sum_{i=1}^{p} \Phi_p(x: X_{\sigma(1)},\dots, [X, X_{\sigma(i)}],\dots, X_{\sigma(p)})$$
$$\otimes \Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})$$
$$+ \sum_{i=1}^{q} \Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)}) \otimes \Psi_q(x: X_{\sigma(p+1)},\dots, [X, X_{\sigma(p+i)}],\dots, X_{\sigma(p+q)})\}.$$

Moreover, by applying the law for the derivative of a product to the first term in the right-hand side of the above equality, we obtain

$$L_X(\Phi_p \wedge \Psi_q)(x: X_1,\dots, X_{p+q})$$
$$= \sum_\sigma \varepsilon(\sigma)\{\Phi'_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)})(X) - \sum_{i=1}^{p} \Phi_p(x: X_{\sigma(1)},\dots, [X, X_{\sigma(i)}],\dots, X_{\sigma(p)})\}$$
$$\otimes \Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})$$
$$+ \sum_\sigma \varepsilon(\sigma)\Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)})$$
$$\otimes \{\Psi'_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})(X) - \sum_{i=1}^{q} \Psi_q(x: X_{\sigma(p+1)},\dots, [X, X_{\sigma(p+i)}],\dots, X_{\sigma(p+q)})\},$$

where we used that the direct product is linear with respect to each factor.

By the definitions of Lie derivative and exterior product, consequently we get

$$L_X(\Phi_p \wedge \Psi_q)(x: X_1,\dots, X_{p+q})$$
$$= \sum_\sigma \varepsilon(\sigma)(L_X\Phi_p)(x: X_{\sigma(1)},\dots, X_{\sigma(p)}) \otimes \Psi_q(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})$$
$$+ \sum_\sigma \varepsilon(\sigma)\Phi_p(x: X_{\sigma(1)},\dots, X_{\sigma(p)}) \otimes (L_X\Psi_q)(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})$$
$$= \{(L_X\Phi_p) \wedge \Psi_q + \Phi_p \wedge (L_X\Psi_q)\}(x: X_1,\dots, X_{p+q}).$$

Next we proceed to the proof of the second formula of (6.4). From the definitions of the interior product and the exterior product, we have

$$
\begin{aligned}
&i_X(\Phi_p\wedge\Psi_q)(x\colon X_1,\dots,X_{p+q-1})\\
&=(\Phi_p\wedge\Psi_q)(x\colon X, X_1,\dots,X_{p+q-1})\\
&=\sum_\sigma \varepsilon(\sigma)\Phi_p(x\colon X, X_{\sigma(1)},\dots,X_{\sigma(p-1)})\otimes\Psi_q(x\colon X_{\sigma(p)},\dots,X_{\sigma(p+q-1)})\\
&\quad+(-1)^p\sum_\tau \varepsilon(\tau)\Phi_p(x\colon X_{\tau(1)},\dots,X_{\tau(p)})\otimes\Psi_q(x\colon X, X_{\tau(p+1)},\dots,X_{\tau(p+q-1)})\\
&=\sum_\sigma \varepsilon(\sigma)(i_X\Phi_p)(x\colon X_{\sigma(1)},\dots,X_{\sigma(p-1)})\otimes\Psi_q(x\colon X_{\sigma(p)},\dots,X_{\sigma(p+q-1)})\\
&\quad+(-1)^p\sum_\tau \varepsilon(\tau)\Phi_p(x\colon X_{\tau(1)},\dots,X_{\tau(p)})\otimes(i_X\Psi_q)(x\colon X_{\tau(p+1)},\dots,X_{\tau(p+q-1)})\\
&=\{(i_X\Phi_p)\wedge\Psi_q+(-1)^p\Phi_p\wedge(i_X\Psi_q)\}(x\colon X_1,\dots,X_{p+q-1}),
\end{aligned}
$$

where the sum $\sum_\sigma$ is to be extended over all permutations of $\{1, 2,\dots, p+q-1\}$ such that $\sigma(1)<\cdots<\sigma(p-1)$ and $\sigma(p)<\cdots<\sigma(p+q-1)$, while the sum $\sum_\tau$ is to be extended over those which satisfy $\tau(1)<\cdots<\tau(p)$ and $\tau(p+1)<\cdots<\tau(p+q-1)$.

Now, among the Lie derivative, the exterior differential and the interior product defined above, there exist the relations as follows:

Proposition 2.

Let Ω_p be an arbitrary H-valued differential p-form of class $C^{(n)}$ on M and X, Y be any two vector fields of class $C^{(n)}$ on M. Then we have

$$i_X d\Omega_p + d i_X\Omega_p = L_X\Omega_p, \tag{6.5}$$

$$\begin{cases} L_X i_Y\Omega_p - i_Y L_X\Omega_p = i_{[X,Y]}\Omega_p,\\ L_X L_Y\Omega_p - L_Y L_X\Omega_p = L_{[X,Y]}\Omega_p,\\ L_X d\Omega_p - dL_X\Omega_p = 0.\end{cases} \tag{6.6}$$

Proof. Let $X_1,\dots, X_{p+1}$ be any $p+1$ vector fields of class $C^{(n)}$ on M. From the definitions of the interior product and the exterior differential, we see

$$
\begin{aligned}
&(i_{X_1}d\Omega_p)(x\colon X_2,\dots,X_{p+1})\\
&=d\Omega_p(x\colon X_1, X_2,\dots,X_{p+1})\\
&=\sum_{i=1}^{p+1}(-1)^{i+1}\Omega'_p(x\colon X_1,\dots,\hat X_i,\dots,X_{p+1})(X_i)\\
&\quad+\sum_{1\le i<j\le p+1}(-1)^{i+j}\Omega_p(x\colon [X_i, X_j], X_1,\dots,\hat X_i,\dots,\hat X_j,\dots,X_{p+1}).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&d(i_{X_1}\Omega_p)(x\colon X_2,\dots,X_{p+1})\\
&=\sum_{i=2}^{p+1}(-1)^i(i_{X_1}\Omega_p)'(x\colon X_2,\dots,\hat X_i,\dots,X_{p+1})(X_i)
\end{aligned}
$$

$$+\sum_{2\leqq i<j\leqq p+1}(-1)^{i+j-2}(i_{X_1}\Omega_p)(x\colon [X_i, X_j], X_2,\dots, \hat{X}_i,\dots, \hat{X}_j,\dots, X_{p+1})$$

$$=\sum_{i=2}^{p+1}(-1)^i\Omega_p'(x\colon X_1, X_2,\dots, \hat{X}_i,\dots, X_{p+1})(X_i)$$

$$+\sum_{2\leqq i<j\leqq p+1}(-1)^{i+j-1}\Omega_p(x\colon [X_i, X_j], X_1, X_2,\dots, \hat{X}_i,\dots, \hat{X}_j,\dots, X_{p+1}).$$

Hence we get

$$\{i_{X_1}d\Omega_p+di_{X_1}\Omega_p\}(x\colon X_2,\dots, X_{p+1})$$

$$=\Omega_p'(x\colon X_2,\dots, X_{p+1})(X_1)-\sum_{j=2}^{p+1}(-1)^j\Omega_p(x\colon [X_1, X_j], X_2,\dots, \hat{X}_j,\dots, X_{p+1})$$

$$=\Omega_p'(x\colon X_2,\dots, X_{p+1})(X_1)-\sum_{j=2}^{p+1}\Omega_p(x\colon X_2,\dots, [X_1, X_j],\dots, X_{p+1}).$$

Therefore we have

$$(i_{X_1}d\Omega_p+di_{X_1}\Omega_p)(x\colon X_2,\dots, X_{p+1})$$
$$=(L_{X_1}\Omega_p)(x\colon X_2,\dots, X_{p+1}).$$

This completes our proof of (6.5).

We omit the proofs of (6.6), since those are merely straightforward calculations (see [12], Chapter III). We shall only give as a remark here that the justification of (6.6) depends on the definition of bracket [,] and Jacobi's identity.

From now on, we derive from the above propositions some results concerning the property of the exterior differential.

Proposition 3.

Let Ω_p be a H-valued differential p-form of class $C^{(n)}$ on M. Then its exterior differential $d\Omega_p$ is a H-valued differential $(p+1)$-form of class $C^{(n-1)}$ on M.

Proof. As we mentioned after the definition of exterior differential, $d\Omega_p(x)$ is a $(p+1)$-linear mapping of class $C^{(n-1)}$ from $\mathfrak{X}^{(n)}(M)\times\cdots\times\mathfrak{X}^{(n)}(M)$ $(p+1$ times) into H. Thus it is sufficient for our purpose to show that $d\Omega_p(x)$ is an alternating mapping, that is,

$$d\Omega_p(x\colon X_1,\dots, X_{p+1})=0$$

whenever $X_i=X_j$ for $i\neq j$ $(1\leqq i, j\leqq p+1)$, where $X_1,\dots, X_{p+1}\in\mathfrak{X}^{(n)}(M)$.

We prove by induction on p. In case $p=1$, from the definition of the exterior differential, we have

$$d\Omega_1(x\colon X_1, X_2)=\Omega_1'(x\colon X_2)(X_1)-\Omega_1'(x\colon X_1)(X_2)$$
$$-\Omega_1(x\colon [X_1, X_2]).$$

Clearly this show that $d\Omega_1$ is an alternating map.

Next, by (6.5) of Proposition 2, we have

$$d\Omega_p(\boldsymbol{x}: X_1,\dots, X_{p+1})=(i_{X_1}d\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})$$
$$=(L_{X_1}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})-d(i_{X_1}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1}),$$

where $X_1,\dots, X_{p+1}\in\mathfrak{X}^{(n)}(M)$. As can be seen immediately from the definition of the Lie derivative, $(L_{X_1}\Omega_p)(\boldsymbol{x})$ is an alternating map with respect to $X_2,\dots, X_{p+1}$. Moreover by hypothesis of induction, $d(i_{X_1}\Omega_p)(\boldsymbol{x})$ is also an alternating map with respect to $X_2,\dots, X_{p+1}$. Hence $d\Omega_p(\boldsymbol{x})$ is an alternating map with respect to $X_2,\dots, X_{p+1}$. Therefore it suffices to prove that, if $X_1=X_2$, we have

$$(L_{X_1}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})=d(i_{X_1}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1}).$$

Now, from the definition of the Lie derivative, we get

$$(L_{X_2}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})=\Omega_p'(\boldsymbol{x}: X_2,\dots, X_{p+1})(X_2)$$
$$-\sum_{i=3}^{p+1}\Omega_p(\boldsymbol{x}: X_2,\dots, [X_2, X_i],\dots, X_{p+1})$$
$$=(i_{X_2}\Omega_p)'(\boldsymbol{x}: X_3,\dots, X_{p+1})(X_2)-\sum_{i=3}^{p+1}(i_{X_2}\Omega_p)(\boldsymbol{x}: X_3,\dots, [X_2, X_i],\dots, X_{p+1})$$
$$=(L_{X_2}i_{X_2}\Omega_p)(\boldsymbol{x}: X_3,\dots, X_{p+1}).$$

Applying (6.5) of Proposition 2 to the right-hand side of the above equality, we see

$$(L_{X_2}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})=\{d(i_{X_2}i_{X_2}\Omega_p)+i_{X_2}d(i_{X_2}\Omega_p)\}(\boldsymbol{x}: X_3,\dots, X_{p+1}).$$

Since $i_{X_2}i_{X_2}\Omega_p=0$ by the alternating property of Ω_p, consequently we get

$$(L_{X_2}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1})=d(i_{X_2}\Omega_p)(\boldsymbol{x}: X_2,\dots, X_{p+1}).$$

This completes our proof of Proposition 3.

Proposition 4.
Let $\Phi_p\in\Omega_p^{(n)}(M, H_1)$ and $\Psi_q\in\Omega_q^{(n)}(M, H_2)$ be two differential forms of class $C^{(n)}$ on M. Then we have

$$d(\Phi_p\wedge\Psi_q)=(d\Phi_p)\wedge\Psi_q+(-1)^p\Phi_p\wedge(d\Psi_q), \tag{6.7}$$

where H_1, H_2 are Hilbert spaces.

Proof. We prove this proposition by induction on $p+q$. In case $p+q=0$, the above formula is none other than the law for the derivative of the direct product.

Suppose that Proposition 4 has already been proved for $p+q\leqq r-1$. We begin with an equality

$$d(\Phi_p\wedge\Psi_q)(\boldsymbol{x}: X_1,\dots, X_{p+q+1})=\{i_{X_1}d(\Phi_p\wedge\Psi_q)\}(\boldsymbol{x}: X_2,\dots, X_{p+q+1}).$$

Using Proposition 1 and (6.5) of Proposition 2, we get

$$i_{X_1}d(\Phi_p\wedge\Psi_q)=L_{X_1}(\Phi_p\wedge\Psi_q)-d\{i_{X_1}(\Phi_p\wedge\Psi_q)\}$$

$$=(L_{X_1}\Phi_p)\wedge\Psi_q+\Phi_p\wedge(L_{X_1}\Psi_q)$$
$$-d\{(i_{X_1}\Phi_p)\wedge\Psi_q\}-(-1)^p d\{\Phi_p\wedge(i_{X_1}\Psi_q)\}.$$

Here, by hypothesis of induction, we have

$$\begin{aligned} d\{(i_{X_1}\Phi_p)\wedge\Psi_q\}&=d(i_{X_1}\Phi_p)\wedge\Psi_q+(-1)^{p-1}(i_{X_1}\Phi_p)\wedge d\Psi_q,\\ d\{\Phi_p\wedge(i_{X_1}\Psi_q)\}&=d\Phi_p\wedge(i_{X_1}\Psi_q)+(-1)^p\Phi_p\wedge d(i_{X_1}\Psi_q). \end{aligned} \tag{6.8}$$

Inserting (6.8) in the right-hand side of the above equality, we obtain

$$\begin{aligned} i_{X_1}d(\Phi_p\wedge\Psi_q)=&\{L_{X_1}\Phi_p-d(i_{X_1}\Phi_p)\}\wedge\Psi_q\\ &+\Phi_p\wedge\{L_{X_1}\Psi_q-d(i_{X_1}\Psi_q)\}\\ &+(-1)^{p+1}(d\Phi_p)\wedge(i_{X_1}\Psi_q)+(-1)^p(i_{X_1}\Phi_p)\wedge(d\Psi_q). \end{aligned}$$

Moreover, by (6.5) and the second formula of (6.4), consequently we get

$$\begin{aligned} i_{X_1}d(\Phi_p\wedge\Psi_q)&=(i_{X_1}d\Phi_p)\wedge\Psi_q+(-1)^{p+1}(d\Phi_p)\wedge(i_{X_1}\Psi_q)\\ &\quad+\Phi_p\wedge(i_{X_1}d\Psi_q)+(-1)^p(i_{X_1}\Phi_p)\wedge(d\Psi_q)\\ &=i_{X_1}\{d\Phi_p\wedge\Psi_q\}+(-1)^p i_{X_1}\{\Phi_p\wedge(d\Psi_q)\}\\ &=i_{X_1}\{d\Phi_p\wedge\Psi_q+(-1)^p\Phi_p\wedge(d\Psi_q)\}. \end{aligned}$$

Therefore we have (6.7). As the particular case of Proposition 4 we have

Proposition 5.

Let f be a function of class $C^{(n)}$ and Ω_p be a H-valued differential p-form. Then we have

$$d(f\Omega_p)=df\wedge\Omega_p+fd\Omega_p.$$

Proposition 6.

Let Ω_p be an arbitrary H-valued differential p-form of class $C^{(n)}$ $(n\geqq 2)$ on M. Then we have

$$dd\Omega_p=0.$$

Proof. We prove by induction on p. In case $p=0$, for an arbitrary H-valued function $\Omega_0\in C^{(n)}(M, H)$, we see by the definition of the exterior differential the following formula:

$$\begin{aligned} \{dd\Omega_0(x)\}(X_1, X_2)&=\{d\Omega_0(x)X_2\}'(X_1)-\{d\Omega_0(x)X_1\}'(X_2)\\ &\quad-d\Omega_0(x)([X_1, X_2])\\ &=\{\Omega_0'(x)(X_2)\}'(X_1)-\{\Omega_0'(x)(X_1)\}'(X_2)\\ &\quad-\Omega_0'(x)([X_1, X_2]), \end{aligned}$$

where $X_1, X_2 \in \mathfrak{X}^{(n)}(M)$. By the definition of bracket $[X_1, X_2]$, we have

$$\{dd\Omega_0(x)\}(X_1, X_2)=0.$$

Next, suppose that Proposition 6 has already been proved for $p=r-1$. We start with the following formula:

$$(dd\Omega_p)(x: X_1,\dots, X_{p+2})=(i_{X_1}dd\Omega_p)(x: X_2,\dots, X_{p+2}),$$

where $X_1,\dots, X_{p+2} \in \mathfrak{X}^{(n)}(M)$.
Using (6.5) of Proposition 2 successively, we get

$$\begin{aligned} i_{X_1}dd\Omega_p &= L_{X_1}d\Omega_p - di_{X_1}d\Omega_p \\ &= L_{X_1}d\Omega_p - dL_{X_1}\Omega_p + ddi_{X_1}\Omega_p. \end{aligned}$$

In view of the last formula of (6.6) of Proposition 2 and by hypothesis of induction, we have

$$i_{X_1}dd\Omega_p=0,$$

and therefore

$$(dd\Omega_p)(x: X_1,\dots, X_{p+2})=0.$$

Finally we are led to the following proposition, as can be seen immediately by using the definition of exterior differentiation and Proposition 2 of § 2.1.

Proposition 7.

Suppose that Ω_p is a H-valued differential p-form of class $C^{(n)}$ on M expressed in the form

$$\Omega_p(x)=\sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{y}_i.$$

Then the exterior differential $d\Omega_p$ of Ω_p is given by

$$d\Omega_p(x)=\sum_{i=1}^{\infty} d\omega_p^i(x)\mathfrak{y}_i.$$

7. Canonical forms

Let M be a m dimensional differential manifold of class $C^{(n)}$, (U, φ) be a chart at an arbitrary point x_0 of M and $(x^1, x^2,\dots, x^m)$ be the coordinates of $x \in U$. Then it is well known that an arbitrary real valued differential p-form Ω_p of class $C^{(n)}$ on M may be written in the form

$$\Omega_p(x)=\sum_{1\leqq j_1<\cdots<j_p\leqq m} \omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

and its exterior differential $d\Omega_p$ is expressed by

$$d\Omega_p(x)=\sum_{1\leqq j_1<\cdots<j_p\leqq m} d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

Now the question arises: in case M is a Hilbert manifold of class $C^{(n)}$, under what conditions do we see the same situation as the above?

As the answer we have firstly the following:

Proposition 1.

Let Ω_p be a H-valued differential p-form of class $C^{(n)}$ on M and (U, φ) be a chart at an arbitrary point x_0 of M. Then this differential p-form Ω_p may be represented in the form

$$\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leqq j_1<\cdots<j_p<\infty}\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{y}_i,$$

where $dx^1,\ldots, dx^n,\ldots$ are the differentials of coordinate functions $\varphi^1,\ldots,\varphi^n,\ldots$ respectively and $\mathfrak{y}=(\mathfrak{y}_1,\ldots,\mathfrak{y}_n,\ldots)$ is a base of H, and furthermore the sum $\sum_{1\leqq j_1<\cdots<j_p<\infty}$ means $\lim_{N\to\infty}\sum_{1\leqq j_1<\cdots<j_p\leqq N}$.

Proof. By Proposition of § 2.4, we have

$$\Omega_p(x: X_1,\ldots, X_p)=\sum_{i=1}^{\infty}\omega^i_p(x: X_1,\ldots, X_p)\mathfrak{y}_i, \quad \text{for} \quad X_1,\ldots, X_p\in\mathfrak{X}^{(n)}(M)$$

where $\omega^i_p\in\Omega^{(n)}_p(M, R)$ for $i=1, 2, 3,\ldots$.
Inserting

$$X_1=\sum_{j_1=1}^{\infty}X_1^{j_1}\frac{\partial}{\partial x^{j_1}},\ldots, X_p=\sum_{j_p=1}^{\infty}X_p^{j_p}\frac{\partial}{\partial x^{j_p}}$$

in $\omega^i_p(x: X_1,\ldots, X_p)$, we get

$$\omega^i_p(x: X_1,\ldots, X_p)=\omega^i_p(x: \lim_{N\to\infty}\sum_{j_1=1}^{N}X_1^{j_1}\frac{\partial}{\partial x^{j_1}},\ldots,$$
$$\lim_{N\to\infty}\sum_{j_p=1}^{N}X_p^{j_p}\frac{\partial}{\partial x^{j_p}})$$

and furthermore, in view of $\omega^i_p(x)\in\mathscr{A}_p(T_x(M), R)$ and by the definition of the exterior product of the differentials of coordinate functions, we obtain

$$\omega^i_p(x: X_1,\ldots, X_p)=\lim_{N\to\infty}\sum_{j_1,\ldots,j_p=1}^{N}X_1^{j_1}\cdots X_p^{j_p}\omega^i_p\left(x: \frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_p}}\right)$$
$$=\lim_{N\to\infty}\sum_{1\leqq j_1<\cdots<j_p\leqq N}\omega^i_p(x: \partial_{j_1},\ldots,\partial_{j_p})$$
$$(dx^{j_1}\wedge\cdots\wedge dx^{j_p})(X_1,\ldots, X_p)$$
$$=\sum_{1\leqq j_1<\cdots<j_p<\infty}\{\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\}(X_1,\ldots, X_p)$$

where $\omega^i_{j_1\ldots j_p}(x)=\omega^i_p(x: \partial_{j_1},\ldots,\partial_{j_p})$.

Thus we see

$$\Omega_p(x: X_1,\dots, X_p)=\sum_{i=1}^{\infty}\sum_{1\leqq j_1<\cdots<j_p<\infty}\{\omega^i_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\}(X_1,\dots, X_p)\mathfrak{y}_i.$$

For the sake of simplicity we write formally

$$\Omega_p(x)=\sum_{1\leqq j_1<\cdots<j_p<\infty}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p} \tag{7.1}$$

for

$$\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leqq j_1<\cdots<j_p<\infty}\omega^i_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{y}_i,$$

where we put $\omega_{j_1\dots j_p}(x)=\sum_{i=1}^{\infty}\omega^i_{j_1\dots j_p}(x)\mathfrak{y}_i$ formally. The right-hand side in (7.1) is said to be the canonical form of a H-valued differential p-form Ω_p with reference to a chart (U, φ).

To obtain

$$d\Omega_p(x)=\sum_{1\leqq j_1<\cdots<j_p<\infty}d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

we propose the following definition:

Definition.

Let $\Omega_{p,1}, \Omega_{p,2},\dots, \Omega_{p,m},\dots, \Omega_p$ be H-valued differential p-forms of class $C^{(n)}$ on M and U be an open set of M. Then we say that *a sequence* $\{\Omega_{p,m}\}$ *of differential p-forms* $\Omega_{p,1}, \Omega_{p,2},\dots, \Omega_{p,m},\dots$ *converges uniformly to* Ω_p *on* U if, given $\varepsilon>0$, there exists a sufficiently large integer $N>0$ such that

$$\|\Omega_{p,m}(x: X_1,\dots, X_p)-\Omega_p(x: X_1,\dots, X_p)\|<\varepsilon \quad \text{for} \quad m>N,\ x\in U$$

where the symbol $\| \ \|$ means the norm in a Hilbert space H and $X_1,\dots, X_p\in\mathfrak{X}^{(n)}(M)$.

Consider a canonical form of Ω_p

$$\Omega_p(x)=\sum_{1\leqq j_1<\cdots<j_p<\infty}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

and let $\Omega^{(k)}_{p,p}, \Omega^{(k)}_{p,p+1},\dots, \Omega^{(k)}_{p,m},\dots$ be the derivatives of order k of the H-valued differential p-forms $\Omega_{p,p}, \Omega_{p,p+1},\dots, \Omega_{p,m},\dots$ of class $C^{(n)}$ given by

$$\Omega_{p,m}(x)=\sum_{1\leqq j_1<\cdots<j_p\leqq m}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}, \quad (m\geqq p).$$

A canonical form given above is said to be *a uniformly convergent canonical form of degree l* if each sequence $\{\Omega^{(k)}_{p,p}, \Omega^{(k)}_{p,p+1},\dots, \Omega^{(k)}_{p,m},\dots\}$ converges uniformly to $\Omega^{(k)}_p$ on U for $k=0, 1,\dots, l-1$ $(l\leqq n)$, and a sequence $\{\Omega^{(l)}_{p,p}, \Omega^{(l)}_{p,p+1},\dots, \Omega^{(l)}_{p,m},\dots\}$ converges uniformly to an element of $\mathscr{L}_{p+l}(E, H)$ on U where $\Omega^{(k)}_p$ is the derivative of order k of Ω_p. Here, by a uniformly convergent sequence $\{\Omega^{(k)}_{p,p}, \Omega^{(k)}_{p,p+1},\dots, \Omega^{(k)}_{p,m},\dots\}$ we mean that a sequence $\{\Omega^{(k)}_{p,p}(y_1,\dots, y_k), \Omega^{(k)}_{p,p+1}(y_1,\dots, y_k),\dots, \Omega^{(k)}_{p,m}(y_1,\dots, y_k),\dots\}$ of H-valued differential p-forms converges uniformly on U where $y_1,\dots, y_k\in E$.

Lemma 1.

Let $f_1(x), f_2(x),\dots, f_m(x),\dots$ be H-valued functions of class $C^{(n)}$ defined on U. Suppose that a sequence $\{f_m(x)\}$ converges uniformly to a H-valued function $f(x)$

on U and a sequence $\{f'_m(x)\}$ of the derivatives of $f_1(x), f_2(x), \ldots, f_m(x), \ldots$ converges uniformly to $g(x) \in \mathscr{L}(E, H)$ on U. Then $f(x)$ is differentiable on U and we have

$$f'(x) = \lim_{m\to\infty} f'_m(x) \qquad \text{for} \quad x \in U,$$

namely, for an arbitrary vector $X \in E$,

$$\lim_{m\to\infty} \|f'(x)X - f'_m(x)X\| = 0.$$

Proof. Let $(U_\alpha, \varphi_\alpha)$ be a chart at an arbitrary point x of U. We consider

$$f_\alpha(x_\alpha + \Delta x_\alpha) - f_\alpha(x_\alpha) - g_\alpha(x_\alpha)\Delta x_\alpha, \quad \text{for} \quad \Delta x_\alpha \in E,$$

where $f_\alpha = f \circ \varphi_\alpha^{-1}$, $x_\alpha = \varphi_\alpha(x)$ and $\varphi_\alpha^{-1}(x_\alpha + \Delta x_\alpha) \in U \cap U_\alpha$. Omitting the symbol α for simplicity, we have

$$\begin{aligned} \|f(x+\Delta x) - f(x) - g(x)\Delta x\| &\leqq \|f(x+\Delta x) - f_m(x+\Delta x)\| \\ &+ \|f_m(x) - f(x)\| + \|f'_m(x)\Delta x - g(x)\Delta x\| \\ &+ \|f_m(x+\Delta x) - f_m(x) - f'_m(x)\Delta x\|. \end{aligned}$$

For the sake of uniformly convergence on U of sequences $\{f_m(x)\}$ and $\{f'_m(x)\}$, given $\varepsilon > 0$, there exists a sufficiently large integer $N > 0$ such that

$$\begin{aligned} &\|f(x+\Delta x) - f_m(x+\Delta x)\| < \frac{\varepsilon}{4}, \\ &\|f_m(x) - f(x)\| < \frac{\varepsilon}{4}, \\ &\|f'_m(x)\Delta x - g(x)\Delta x\| < \frac{\varepsilon}{4}, \\ &\text{for} \quad m > N, \ x \in U. \end{aligned}$$

Moreover, since $f'_m(x)$ is the derivative of $f_m(x)$, there exists a sufficiently small number $\delta > 0$ for $m > N$ such that

$$\|f_m(x+\Delta x) - f_m(x) - f'_m(x)\Delta x\| < \frac{\varepsilon}{4}, \quad \text{for} \quad \|\Delta x\| < \delta.$$

Consequently, we have

$$\|f(x+\Delta x) - f(x) - g(x)\Delta x\| < \varepsilon, \quad \text{for} \quad \|\Delta x\| < \delta$$

This completes our proof of Lemma 1.

Lemma 2.

Let $\Omega_{p,1}, \Omega_{p,2}, \ldots, \Omega_{p,m}, \ldots, \Omega_p$ be H-valued differential p-forms. Suppose that a sequence $\{\Omega_{p,m}\}$ converges uniformly to Ω_p on an open set U of M and that a sequence $\{\Omega'_{p,m}\}$ converges uniformly to an element of $\mathscr{L}_{p+1}(E, H)$ on U where $\Omega'_{p,m}$ is the derivative of $\Omega_{p,m}$ for $m = 1, 2, \ldots$. Then we have

$$d\Omega_p(x)=\lim_{m\to\infty} d\Omega_{p,m}(x),$$

namely, for any $p+1$ vector fields $X_1,\dots, X_{p+1}\in\mathfrak{X}^{(n)}(M)$,

$$\lim_{m\to\infty}\|\{d\Omega_p(x)-d\Omega_{p,m}(x)\}(X_1,\dots, X_{p+1})\|=0.$$

Proof. From the definition of the exterior differential, we have

$$d\Omega_{p,m}(x\colon X_1,\dots, X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}\Omega'_{p,m}(x\colon X_1,\dots,\hat{X}_i,\dots, X_{p+1})(X_i)$$
$$+\sum_{1\leq i<j\leq p+1}(-1)^{i+j}\Omega_{p,m}(x\colon [X_i, X_j], X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots, X_{p+1}).$$

By the above Lemma 1, we get

$$\lim_{m\to\infty} d\Omega_{p,m}(x\colon X_1,\dots, X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}\lim_{m\to\infty}\Omega'_{p,m}(x\colon X_1,\dots,\hat{X}_i,\dots, X_{p+1})(X_i)$$
$$+\sum_{1\leq i<j\leq p+1}(-1)^{i+j}\lim_{m\to\infty}\Omega_{p,m}(x\colon [X_i, X_j], X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots, X_{p+1})$$
$$=\sum_{i=1}^{p+1}(-1)^{i+1}\Omega'_p(x\colon X_1,\dots,\hat{X}_i,\dots, X_{p+1})(X_i)$$
$$+\sum_{1\leq i<j\leq p+1}(-1)^{i+j}\Omega_p(x\colon [X_i, X_j], X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots, X_{p+1})$$
$$=d\Omega_p(x\colon X_1,\dots, X_{p+1}).$$

Thus Lemma 2 has been proved.

Now, consider a uniformly convergent canonical form of one degree

$$\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

A sequence $\{\Omega_{p,m}\}$ $(m=p,\ p+1,\dots)$ given by

$$\Omega_{p,m}(x)=\sum_{1\leq j_1<\cdots<j_p\leq m}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},\quad \text{for}\quad m=p,\ p+1,\dots,$$

converges uniformly to Ω_p on U and a sequence $\{\Omega'_{p,m}\}$ converges uniformly to an element of $\mathscr{L}_{p+1}(E, H)$ on U. Thus, using Proposition 4, 5, 6, 7 of § 2.6, we get

$$d\Omega_{p,m}(x)=\sum_{1\leq j_1<\cdots<j_p\leq m} d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

By Lemma 2, we see

$$d\Omega_p(x)=\lim_{m\to\infty} d\Omega_{p,m}(x)=\sum_{1\leq j_1<\cdots<j_p<\infty} d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

Thus we are derived to the following conclusion:

Proposition 2.

Consider a uniformly convergent canonical form of one degree of a H-valued differential form Ω_p:

$$\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

Then we have

$$d\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

i.e.,

$$d\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}d\omega^i_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{y}_i.$$

§3. Connection form and curvature form

1. $H(M)$-valued vector fields

In this paragraph, we shall construct "$H(M)$-valued vector fields" in the similar manner to the construction of vector fields on a Hilbert manifold.

For this purpose, we must start by considering a transformation group of bases of a Hilbert space. Let H be a Hilbert space and M be a Hilbert manifold of class $C^{(n)}$. By an isomorphism on H we mean a one-to-one linear continuous mapping of H onto itself. We denote by $GL(H)$ a set of all isomorphisms on H. Since the inverse mapping A^{-1} of an isomorphism A on H is also an isomorphism on H, it follows that a set $GL(H)$ is a group under the familiar multiplication of mappings and becomes a Banach space by introducing the norm in a well-known way: for an arbitrary isomorphism $A\in GL(H)$,

$$\|A\|=\sup\left\{\frac{\|Ah\|}{\|h\|}\ ;\ h\in H,\ h\neq 0\right\}.$$

Next we shall attach the group $GL(H)$ to each point on M. Let $(U_\alpha, \varphi_\alpha)$ be a chart at an arbitrary point x_0 of M. We consider a set $\mathfrak{U}_\alpha$ of all mappings A_α of U_α into a Banach space $GL(H)$ such that two mappings A_α, A_α^{-1} are of class $C^{(n)}$ in U_α where A_α^{-1} is a mapping which corresponds to each point x of U_α the inverse isomorphism $A_\alpha^{-1}(x)$ of isomorphism $A_\alpha(x)\in GL(H)$. By $\mathfrak{F}_{x_0}$ we denote a family of all pairs (U_α, A_α) such that $A_\alpha\in\mathfrak{U}_\alpha$, $\alpha\in I_{x_0}$ where I_{x_0} is the index set of all charts at x_0. Since a pair (U_α, A) belongs to the above family $\mathfrak{F}_{x_0}$ for an arbitrary isomorphism $A\in GL(H)$, we see that the group $GL(H)$ is attached to a point x_0 of M where we denoted by the same notation A a mapping of U_α into $GL(H)$ such that $A(x)=A$ for $x\in U_\alpha$.

Now, in order to construct "spaces" corresponding to tangent vector spaces, we begin by discussing the transformation of "moving frames" of a Hilbert space H. Let $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be two charts at a point x_0 of M. For two mappings $A_\alpha\in\mathfrak{U}_\alpha$ and $A_\beta\in\mathfrak{U}_\beta$, we define their multiplication as a mapping of $U_\alpha\cap U_\beta$ into $GL(H)$ given by

$$(A_\alpha A_\beta)(x)=A_\alpha(x)A_\beta(x),\quad \text{for}\quad x\in U_\alpha\cap U_\beta.$$

We may infer in the same manner as the proof for the derivative of the usual product of two functions that a product $A_\alpha A_\beta$ is also of class $C^{(n)}$ in $U_\alpha\cap U_\beta$ and that the following Leibniz's formula is held:

$$\{(A_\alpha A_\beta)(x)\}^{(m)}=\sum_{i=0}^{m}\binom{m}{i}A_\alpha^{(m-i)}(x)A_\beta^{(i)}(x),$$

where $A_\alpha^{(m-i)}(x)A_\beta^{(i)}(x)$ $(i=0, 1,\dots, m)$ are elements of $\mathscr{L}_m(E, GL(H))$ defined by

$$\{A_\alpha^{(m-i)}(x)A_\beta^{(i)}(x)\}(y_1,\dots, y_m)=A_\alpha^{(m-i)}(x)(y_1,\dots, y_{m-i})A_\beta^{(i)}(x)(y_{m-i+1},\dots, y_m),$$
$$\text{for}\quad y_1,\dots, y_m\in E.$$

Hereafter, we denote by $A_{\alpha\beta}$ the product $A_\alpha A_\beta^{-1}$ of $A_\alpha\in\mathfrak{U}_\alpha$ and $A_\beta^{-1}\in\mathfrak{U}_\beta$. Obviously $A_{\alpha\beta}$ is a mapping of $U_\alpha\cap U_\beta$ into $GL(H)$ of class $C^{(n)}$ and we have

$$\begin{aligned}&A_{\beta\alpha}(x)=A_{\alpha\beta}^{-1}(x),\quad\text{for}\quad x\in U_\alpha\cap U_\beta,\\&A_{\gamma\alpha}(x)=A_{\gamma\beta}(x)A_{\beta\alpha}(x),\quad\text{for}\quad x\in U_\alpha\cap U_\beta\cap U_\gamma.\end{aligned}\tag{1.1}$$

Let $\mathfrak{y}=(\mathfrak{y}_1, \mathfrak{y}_2,\dots, \mathfrak{y}_n,\dots)$ be a base of H and x be an arbitrary point of M, and further let $(U_{\alpha_0}, A_{\alpha_0})$, (U_α, A_α) be two elements of a family $\mathfrak{F}_x$ where α_0 is an arbitrary fixed index belonging to I_x. We consider a base $(A_{\alpha_0\alpha}(x)\mathfrak{y}_1,\dots, A_{\alpha_0\alpha}(x)\mathfrak{y}_n,\dots)$ of H and write $\mathfrak{y}A_{\alpha_0\alpha}(x)=(A_{\alpha_0\alpha}(x)\mathfrak{y}_1,\dots, A_{\alpha_0\alpha}(x)\mathfrak{y}_n,\dots)$. Moreover we define

$$\{\mathfrak{y}A_{\alpha_0\alpha}(x)\}A_{\alpha\beta}(x)=(A_{\alpha_0\alpha}(x)\{A_{\alpha\beta}(x)\mathfrak{y}_1\},\dots, A_{\alpha_0\alpha}(x)\{A_{\alpha\beta}(x)\mathfrak{y}_n\},\dots)\tag{1.2}$$

and for brevity we write

$$\mathfrak{b}_\alpha(x)=(\mathfrak{b}_1^\alpha(x),\dots, \mathfrak{b}_n^\alpha(x),\dots)$$

where $\mathfrak{b}_i^\alpha(x)=A_{\alpha_0\alpha}(x)\mathfrak{y}_i$ $(i=1, 2,\dots)$.
Then we have

$$\begin{aligned}&\mathfrak{b}_{\alpha_0}(x)=\mathfrak{y},\\&\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x).\end{aligned}\tag{1.3}$$

In fact, the first relation is clearly satisfied, since $A_{\alpha_0\alpha_0}(x)=A_{\alpha_0}(x)A_{\alpha_0}^{-1}(x)=1$ where 1 denote a unit operator on H. By using the second relation of (1.1) and by the Definition (1.2), the second formula of (1.3) is derived as follows:

$$\begin{aligned}\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)&=\{\mathfrak{b}_{\alpha_0}(x)A_{\alpha_0\alpha}(x)\}A_{\alpha\beta}(x)\\&=\mathfrak{b}_{\alpha_0}(x)\{A_{\alpha_0\alpha}(x)A_{\alpha\beta}(x)\}\\&=\mathfrak{b}_{\alpha_0}(x)A_{\alpha_0\beta}(x)\\&=\mathfrak{b}_\beta(x).\end{aligned}$$

Thus we are let to the following interpretation: a base $\mathfrak{b}_\alpha(x)$ of H is a moving frame of H-type on a Hilbert manifold M and the second relation of (1.3) show the transformation law of these moving frames on M.

From now on, we shall construct spaces corresponding to tangent spaces on M. Considering a product set $H\times I_x$, we denote by Z_α an element (Z, α) of $H\times I_x$ and moreover, for an isomorphism $A\in GL(H)$, we define AZ_α by $(AZ, \alpha)\in H\times I_x$. We say that two elements Z_α, Z_β of $H\times I_x$ are equivalent if and only if there exist two pairs (U_α, A_α), (U_β, A_β) of a family $\mathfrak{F}_x$ such that $Z_\beta = A_{\beta\alpha}(x)Z_\alpha$. For the sake of relations (1.1), we see readily that the equivalence thus defined satisfies the axiom of equivalence relation. By a fibre element $\mathfrak{z}$ at a point x of M we mean an equivalence class $\{Z_\alpha\}$ of elements of $H\times I_x$. Moreover, for two fibre elements $\mathfrak{z}_1 = \{(Z_1)_\alpha\}$, $\mathfrak{z}_2 = \{(Z_2)_\alpha\}$ at $x\in M$ and a scalar $\lambda\in R$, we introduce the addition and multiplication as follows:

$$\mathfrak{z}_1 + \mathfrak{z}_2 = \{(Z_1+Z_2)_\alpha\}, \quad \lambda\mathfrak{z}_1 = \{(\lambda Z_1)_\alpha\}.$$

Of course, as is easily seen, these addition and multiplication are independent to the representative of equivalence classes. Thus, a set of all fibre elements $\mathfrak{z}$ at a point x of M which has the addition and multiplication introduced above, is clearly a vector space over the real field R.

Definition 1.

A vector space constructed above is said to be *a fibre space of H-type* at a point x of M or *a H-fibre space* at $x\in M$ for simplicity. We denote by $H_x(M)$ this vector space.

Especially, in case a group $GL(H)$ is a group $GL(E)$ which consists of the derivatives $\varphi'_{\beta\alpha}(x)$ of all $C^{(n)}$-diffeomorphisms $\varphi_{\beta\alpha}(x)$ discussed in § 2.2, a E-fibre space at $x\in M$ seems to be a tangent space $T_x(M)$ at $x\in M$.

Since, for an index $\alpha\in I_x$ and a fibre element $\mathfrak{z}\in H_x(M)$, there exists a unique vector $Z\in H$ such that $Z_\alpha = (Z, \alpha)\in\mathfrak{z}$, we identify a representative Z_α of $\mathfrak{z}$ with this unique vector Z of H. Therefore we can consider a mapping of the fibre space $H_x(M)$ into the Hilbert space H given by

$$\Phi_\alpha(x)\colon \mathfrak{z} \longrightarrow Z_\alpha.$$

Moreover we define an inner product $\langle\mathfrak{z}_1, \mathfrak{z}_2\rangle$ of two fibre elements $\mathfrak{z}_1, \mathfrak{z}_2\in H_x(M)$ and a norm $\|\mathfrak{z}\|$ of a fibre element $\mathfrak{z}\in H_x(M)$ by

$$\langle\mathfrak{z}_1, \mathfrak{z}_2\rangle = \langle\Phi_{\alpha_0}(x)(\mathfrak{z}_1), \Phi_{\alpha_0}(x)(\mathfrak{z}_2)\rangle,$$
$$\|\mathfrak{z}\| = \sqrt{\langle\mathfrak{z}, \mathfrak{z}\rangle},$$

where α_0 is an arbitrary fixed index belonging I_x. We infer readily that the inner product thus defined satisfies the axiom of inner product and the fibre space $H_x(M)$ is complete with respect to the above norm. Consequently we obtain a fibre space $H_x(M)$ which has the same structure as the Hilbert space H. We have the following proposition which corresponds to Proposition of § 2.2.

Proposition 1.

Let M be a Hilbert manifold of class $C^{(n)}$ and $H_x(M)$ be the fibre space at a point x of M. Then a mapping $\Phi_{\alpha_0}(x)\colon H_x(M)\to H$ given by $Z_{\alpha_0} = \Phi_{\alpha_0}(x)(\mathfrak{z})$ for $\mathfrak{z}\in H_x(M)$, is an isomorphism of the fibre space $H_x(M)$ onto the Hilbert space H where Z_{α_0} is the representative of $\mathfrak{z}$ identified with a vector of H and α_0 is an arbitrary fixed index belonging to I_x.

Since this proposition can be verified in the same manner as in the proof of Proposition of § 2.2, we omit it.

In what follows, the fibre space is assumed to be the Hilbert space which has the same structure as the Hilbert space H, and we denote by Z or $Z_{\mathfrak{x}}$ the fibre element $\mathfrak{z}$ at $\mathfrak{x}\in M$.

Next, we shall go on to investigate the expression of a fibre element $Z\in H_{\mathfrak{x}}(M)$ in terms of a moving frame on M which corresponds to express the tangent vector $X\in T_{\mathfrak{x}}(M)$ as the derivation. For this aim, we search the matrix representation of the following relations with respect to a base $\mathfrak{b}_{\alpha_0}(\mathfrak{x})$ of H: the equivalence relation for two representatives Z_α, Z_β of a fibre element $Z\in H_{\mathfrak{x}}(M)$

$$Z_\beta=A_{\beta\alpha}(\mathfrak{x})Z_\alpha\,, \tag{1.4}$$

and the transformation law of moving frames

$$\mathfrak{b}_\beta(\mathfrak{x})=\mathfrak{b}_\alpha(\mathfrak{x})A_{\alpha\beta}(\mathfrak{x})\,. \tag{1.5}$$

We assume that two matrices

$$\begin{pmatrix} a^1_{\beta\alpha,1} & a^1_{\beta\alpha,2}\cdots a^1_{\beta\alpha,n}\cdots \\ a^2_{\beta\alpha,1} & a^2_{\beta\alpha,2}\cdots a^2_{\beta\alpha,n}\cdots \\ \vdots & \vdots \qquad \vdots \\ a^n_{\beta\alpha,1} & a^n_{\beta\alpha,2}\cdots a^n_{\beta\alpha,n}\cdots \\ \vdots & \vdots \qquad \vdots \end{pmatrix},\quad \begin{pmatrix} a^1_{\alpha\beta,1} & a^1_{\alpha\beta,2}\cdots a^1_{\alpha\beta,n}\cdots \\ a^2_{\alpha\beta,1} & a^2_{\alpha\beta,2}\cdots a^2_{\alpha\beta,n}\cdots \\ \vdots & \vdots \qquad \vdots \\ a^n_{\alpha\beta,1} & a^n_{\alpha\beta,2}\cdots a^n_{\alpha\beta,n}\cdots \\ \vdots & \vdots \qquad \vdots \end{pmatrix}$$

are the matrix representations of $A_{\beta\alpha}(\mathfrak{x})$, $A_{\alpha\beta}(\mathfrak{x})$ with respect to a base $\mathfrak{b}_{\alpha_0}(\mathfrak{x})=(\mathfrak{y}_1, \mathfrak{y}_2,\dots, \mathfrak{y}_n,\dots)$ of H where $A_{\beta\alpha}(\mathfrak{x})$, $A_{\alpha\beta}(\mathfrak{x})$ are the isomorphisms on H of class $C^{(n)}$ and we omit $\mathfrak{x}$ for matrix elements. As was mentioned in § 2.1, we obtain the matrix representations of (1.4), (1.5) in the form

$$\begin{pmatrix} Z^1_\beta \\ Z^2_\beta \\ \vdots \\ Z^i_\beta \\ \vdots \end{pmatrix} = \begin{pmatrix} a^1_{\beta\alpha,1} & a^1_{\beta\alpha,2}\cdots a^1_{\beta\alpha,j}\cdots \\ a^2_{\beta\alpha,1} & a^2_{\beta\alpha,2}\cdots a^2_{\beta\alpha,j}\cdots \\ \vdots & \vdots \qquad \vdots \\ a^i_{\beta\alpha,1} & a^i_{\beta\alpha,2}\cdots a^i_{\beta\alpha,j}\cdots \\ \vdots & \vdots \qquad \vdots \end{pmatrix} \begin{pmatrix} Z^1_\alpha \\ Z^2_\alpha \\ \vdots \\ Z^j_\alpha \\ \vdots \end{pmatrix}, \tag{1.6}$$

$$\begin{aligned}(\mathfrak{b}^\beta_1(\mathfrak{x}),\ &\mathfrak{b}^\beta_2(\mathfrak{x}),\dots,\ \mathfrak{b}^\beta_i(\mathfrak{x}),\dots) \\ &=(\mathfrak{b}^\alpha_1(\mathfrak{x}),\ \mathfrak{b}^\alpha_2(\mathfrak{x}),\dots,\ \mathfrak{b}^\alpha_j(\mathfrak{x}),\dots)\begin{pmatrix} a^1_{\alpha\beta,1} & a^1_{\alpha\beta,2}\cdots a^1_{\alpha\beta,i}\cdots \\ a^2_{\alpha\beta,1} & a^2_{\alpha\beta,2}\cdots a^2_{\alpha\beta,i}\cdots \\ \vdots & \vdots \qquad \vdots \\ a^j_{\alpha\beta,1} & a^j_{\alpha\beta,2}\cdots a^j_{\alpha\beta,i}\cdots \\ \vdots & \vdots \qquad \vdots \end{pmatrix},\end{aligned} \tag{1.7}$$

where $Z_\beta=\sum_{i=1}^{\infty} Z^i_\beta\mathfrak{y}_i$, $Z_\alpha=\sum_{i=1}^{\infty} Z^i_\alpha\mathfrak{y}_i$.

Moreover we get from a relation $A_{\alpha\beta}(\mathfrak{x})A_{\beta\alpha}(\mathfrak{x})=1$

$$\sum_{k=1}^{\infty} a^i_{\alpha\beta,k}\, a^k_{\beta\alpha,j} = \delta^i_j\,. \tag{1.8}$$

Here, for a representative Z_α of a fibre element $Z \in H_x(M)$ and a moving frame $\mathfrak{b}_\alpha(x)$ on M, we consider a vector of H which is denoted by $\mathfrak{b}_\alpha(x)Z_\alpha$:

$$\mathfrak{b}_\alpha(x)Z_\alpha = \sum_{i=1}^{\infty} Z^i_\alpha \mathfrak{b}^\alpha_i(x) \tag{1.9}$$

where $\mathfrak{b}_\alpha(x) = (\mathfrak{b}^\alpha_1(x), \mathfrak{b}^\alpha_2(x), \ldots, \mathfrak{b}^\alpha_n(x), \ldots)$.
Then we see

$$\mathfrak{b}_\beta(x)Z_\beta = \mathfrak{b}_\alpha(x)Z_\alpha \tag{1.10}$$

where Z_β is another representative of a fibre element Z and $\mathfrak{b}_\beta(x) = (\mathfrak{b}^\beta_1(x), \mathfrak{b}^\beta_2(x), \ldots, \mathfrak{b}^\beta_n(x), \ldots)$. In fact, by using (1.6), (1.7) and (1.8), we get

$$\mathfrak{b}_\beta(x)\, Z_\beta = (\mathfrak{b}^\beta_1(x), \mathfrak{b}^\beta_2(x), \ldots, \mathfrak{b}^\beta_i(x), \ldots) \begin{pmatrix} Z^1_\beta \\ Z^2_\beta \\ \vdots \\ Z^i_\beta \\ \vdots \end{pmatrix}$$

$$= (\mathfrak{b}^\alpha_1(x), \mathfrak{b}^\alpha_2(x), \ldots, \mathfrak{b}^\alpha_i(x), \ldots) \begin{pmatrix} a^1_{\alpha\beta,1} & a^1_{\alpha\beta,2} \cdots a^1_{\alpha\beta,i} \cdots \\ a^2_{\alpha\beta,1} & a^2_{\alpha\beta,2} \cdots a^2_{\alpha\beta,i} \cdots \\ \vdots & \vdots \quad\quad \vdots \\ a^i_{\alpha\beta,1} & a^i_{\alpha\beta,2} \cdots a^i_{\alpha\beta,i} \cdots \\ \vdots & \vdots \quad\quad \vdots \end{pmatrix}$$

$$\begin{pmatrix} a^1_{\beta\alpha,1} & a^1_{\beta\alpha,2} \cdots a^1_{\beta\alpha,i} \cdots \\ a^2_{\beta\alpha,1} & a^2_{\beta\alpha,2} \cdots a^2_{\beta\alpha,i} \cdots \\ \vdots & \vdots \quad\quad \vdots \\ a^i_{\beta\alpha,1} & a^i_{\beta\alpha,2} \cdots a^i_{\beta\alpha,i} \cdots \\ \vdots & \vdots \quad\quad \vdots \end{pmatrix} \begin{pmatrix} Z^1_\alpha \\ Z^2_\alpha \\ \vdots \\ Z^i_\alpha \\ \vdots \end{pmatrix}$$

$$= (\mathfrak{b}^\alpha_1(x), \mathfrak{b}^\alpha_2(x), \ldots, \mathfrak{b}^\alpha_i(x), \ldots) \begin{pmatrix} Z^i_\alpha \\ Z^2_\alpha \\ \vdots \\ Z^i_\alpha \\ \vdots \end{pmatrix}$$

$$= \mathfrak{b}_\alpha(x)\, Z_\alpha.$$

This formula (1.10) shows that a vector $\mathfrak{b}_\alpha(x)Z_\alpha$ of H is irrelevant to the representatives of a fibre element $Z \in H_x(M)$, and therefore that we may identify a fibre element Z with a vector $\mathfrak{b}_\alpha(x)Z_\alpha$ of H. Henceforth we write for a fibre element Z in the following form which corresponds to the expression [(2.12) of § 2.2] of a tangent vector $X \in T_x(M)$ by the derivation,

$$Z = \sum_{i=1}^{\infty} Z^i \mathfrak{b}_i(x). \tag{1.11}$$

Furthermore the above calculation derived (1.10), shows that it is allowed to operate formally in the following manner: by using (1.4), (1.5) and a relation $A_{\alpha\beta}(x)A_{\beta\alpha}(x)=1$,

$$\begin{aligned}\mathfrak{b}_\beta(x)Z_\beta &= \mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)A_{\beta\alpha}(x)Z_\alpha \\ &= \mathfrak{b}_\alpha(x)Z_\alpha.\end{aligned}$$

In particular, we have

$$\Phi_\alpha(x)(Z) = \sum_{i=1}^{\infty} Z_\alpha^i \mathfrak{y}_i, \quad \text{for} \quad Z \in H_x(M). \tag{1.12}$$

Finally, we shall give the definition of $H(M)$-valued vector fields. By the fibre bundle of H-type we mean a set $H(M) = \bigcup_{x \in M} H_x(M)$.

Definition 2.

Let U be an arbitrary open set of M. A mapping Z of U into $H(M)$ given by

$$Z(x) \in H_x(M), \quad \textit{for} \quad x \in U,$$

is said to be *a $H(M)$-valued vector field on M.*

A $H(M)$-valued vector field Z on M is said to be *of class $C^{(n)}$ on M* if and only if a mapping $\Phi_{\alpha_0} \circ Z$ of U into H given by

$$(\Phi_{\alpha_0} \circ Z)(x) = \Phi_{\alpha_0}(x)(Z(x)), \quad \textit{for} \quad x \in U,$$

is of class $C^{(n)}$ in U.

In what follows, we assume that the $H(M)$-valued vector field is of class $C^{(n)}$ on M, and we write

$$Z(x) = \sum_{i=1}^{\infty} Z^i(x)\mathfrak{b}_i(x). \tag{1.13}$$

We conclude this paragraph with remark on the vector field $\tilde{Z}(x)$ given by

$$\tilde{Z}(x) = \sum_{i=1}^{\infty} Z^i(x)\mathfrak{y}_i.$$

Since there exists a bounded linear operator $A(x)$ belonging to $GL(H)$ such that

$$\mathfrak{y}_i = A(x)\mathfrak{b}_i(x) \tag{1.14}$$

and hence we have

$$\tilde{Z}(x) = A(x)Z(x).$$

Obviously $\tilde{Z}(x)$ is a H-valued vector field on M. Moreover it is easy to show that $\tilde{Z}(x)$ is of class $C^{(n)}$ on M. In fact we have, from (1.10), (1.12) and (1.14),

$$(\Phi_{\alpha_0}\circ Z)(x)=\sum_{i=1}^{\infty} Z^i_{\alpha_0}(x)\mathfrak{y}_i$$

$$=\sum_{i=1}^{\infty} Z^i(x)\mathfrak{b}_i(x)$$

$$=\sum_{i=1}^{\infty} Z^i(x)A^{-1}(x)\mathfrak{y}_i$$

$$=A^{-1}(x)(\sum_{i=1}^{\infty} Z^i(x)\mathfrak{y}_i),$$

and hence we get

$$\tilde{Z}(x)=A(x)(\Phi_{\alpha_0}\circ Z)(x).$$

Thus, since $A(x)$ is of class $C^{(n)}$ on M, $\tilde{Z}(x)$ is also of class $C^{(n)}$ on M. We obtain the following conclusion:

Proposition 2.

Let $Z(x)$ be a $H(M)$-valued vector field of class $C^{(n)}$ on M given in the form

$$Z(x)=\sum_{i=1}^{\infty} Z^i(x)\mathfrak{b}_i(x).$$

Then the vector field $\tilde{Z}(x)$ defined by

$$\tilde{Z}(x)=\sum_{i=1}^{\infty} Z^i(x)\mathfrak{y}_i,$$

is the H-valued vector field of class $C^{(n)}$ on M.

2. $H(M)$-valued differential forms

We discussed about H-valued differential forms in § 2.4. In this paragraph, we shall attempt to construct the corresponding theory concerning $H(M)$-valued differential forms. First, we start by defining the $H(M)$-valued differential form. As one has been, by $\mathscr{A}_p(T_x(M), H_x(M))$ we mean a set of all p-linear alternating continuous mappings of the tangent space $T_x(M)$ into the fibre space $H_x(M)$. Moreover, by the differential p-form bundle of H-type on M we mean a set $\mathscr{A}_p(T(M), H(M))=\bigcup_{x\in M}\mathscr{A}_p\cdot(T_x(M), H_x(M))$.

Definition 1.

Let U be an arbitrary open set of M. A mapping Ω_p of U into $\mathscr{A}_p(T(M), H(M))$ given by

$$\Omega_p(x)\in\mathscr{A}_p(T_x(M), H_x(M)),\quad for\quad x\in U,$$

is said to be *a $H(M)$-valued differential p-form on M*. For brevity we denote $\Omega_p(x)\cdot(X_1,\dots, X_p)$ by $\Omega_p(x: X_1,\dots, X_p)$ where $X_1,\dots, X_p\in T_x(M)$.

A $H(M)$-valued differential p-form Ω_p on M is said to be *of class $C^{(n)}$ on M* if and only if a mapping $(\Phi_{\alpha_0}\circ\Omega_p)(X_1,\dots, X_p)$ of U into H given by

$$(\Phi_{\alpha_0}\circ\Omega_p)(X_1,\dots,X_p)(x)=\Phi_{\alpha_0}(x)(\Omega_p(x: X_1,\dots,X_p)), \quad \textit{for} \quad x\in U,$$

is of class $C^{(n)}$ in U, namely $\Phi_{\alpha_0}\circ\Omega_p\in\Omega_p^{(n)}(M, H)$, where $X_1,\dots, X_p$ are any p vector fields of class $C^{(n)}$ on M.

In what follows we assume Ω_p to be a $H(M)$-valued differential p-form of class $C^{(n)}$ on M and denote by $\Omega_p^{(n)}(M, H(M))$ a set of all $H(M)$-valued differential p-forms Ω_p on M. A set $\Omega_p^{(n)}(M, H(M))$ becomes a real vector space under the natural addition and scalar multiplication, and further is a $C^{(n)}(M, R)$-module under the following multiplication: for $f\in C^{(n)}(M, R)$ and $\Omega_p\in\Omega_p^{(n)}(M, H(M))$,

$$(f\Omega_p)(x)=f(x)\Omega_p(x), \quad \text{for} \quad x\in U.$$

In fact, we see evidently $\Phi_{\alpha_0}\circ(f\Omega_p)=f(\Phi_{\alpha_0}\circ\Omega_p)$ and hence we have $\Phi_{\alpha_0}\circ(f\Omega_p)\in\Omega_p^{(n)}(M, H)$.

Now let $\mathfrak{b}(x)=(\mathfrak{b}_1(x), \mathfrak{b}_2(x),\dots, \mathfrak{b}_n(x),\dots)$ *be a moving frame on* M. *Then a* $H(M)$-*valued differential p-form* Ω_p *is uniquely expressed in terms of* $\mathfrak{b}(x)$ *in the form*

$$\Omega_p(x: X_1,\dots, X_p)=\sum_{i=1}^{\infty}\omega_p^i(x: X_1,\dots, X_p)\mathfrak{b}_i(x), \tag{2.1}$$

where $X_1,\dots, X_p\in\mathfrak{X}^{(n)}(M)$. *It is obvious that each component* ω_p^i *is the real valued differential p-form of class* $C^{(n)}$ *on* M. *For simplicity we write* (2.1) *in the following form*:

$$\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x). \tag{2.2}$$

Next we proceed to the definition of exterior multiplication. Let H^1, H^2 be two Hilbert spaces and $H_x^1(M)$, $H_x^2(M)$ be fibre spaces of H^1-, H^2-type at a point x of M respectively. We shall denote by $H_x(M)=H_x^1(M)\otimes H_x^2(M)$ the direct product of Hilbert spaces $H_x^1(M)$, $H_x^2(M)$. This direct product $H_x(M)$ is a fibre space of $H^1\otimes H^2$-type at $x\in M$. An isomorphism $\Phi_{\alpha_0}(x)$ of $H_x(M)$ onto $H=H^1\otimes H^2$ is introduced as follows:

$$\Phi_{\alpha_0}(x)(Z)=\Phi_{\alpha_0}^1(x)(Z^1)\otimes\Phi_{\alpha_0}^2(x)(Z^2), \quad \text{for} \quad Z=Z^1\otimes Z^2\in H_x(M)$$

where $\Phi_{\alpha_0}^1(x)$, $\Phi_{\alpha_0}^2(x)$ are the isomorphisms of $H_x^1(M)$, $H_x^2(M)$ onto H^1, H^2 respectively.

Let $H^1(M)$, $H^2(M)$ and $H(M)$ be the fibre bundles of H^1-, H^2- and H-type on M respectively.

Definition 2.

Let Ω_p^1, Ω_q^2 be respectively a $H^1(M)$-valued differential p-form and a $H^2(M)$-valued differential q-form on M. By *the exterior product* of two differential forms Ω_p^1 and Ω_q^2, denoted by $\Omega_p^1\wedge\Omega_q^2$, we mean a $H(M)$-valued differential $(p+q)$-form on M such that, for an arbitrary point x of M and any $(p+q)$ vector fields $X_1,\dots, X_{p+q}\in\mathfrak{X}^{(n)}(M)$,

$$\begin{aligned}&(\Omega_p^1\wedge\Omega_q^2)(x: X_1,\dots, X_{p+q})\\ &=\sum_{\sigma}\varepsilon(\sigma)\Omega_p^1(x: X_{\sigma(1)},\dots, X_{\sigma(p)})\otimes\Omega_q^2(x: X_{\sigma(p+1)},\dots, X_{\sigma(p+q)})\end{aligned}$$

where the sum $\sum_{\sigma}$ is to be extended over all permutations of $\{1, 2,\dots, p+q\}$ satisfying $\sigma(1)<\cdots<\sigma(p)$ and $\sigma(p+1)<\cdots<\sigma(p+q)$, and

$$\varepsilon(\sigma)=\begin{cases} +1, & \text{for even permutation } \sigma, \\ -1, & \text{for odd permutation } \sigma. \end{cases}$$

A mapping $\Phi_{\alpha_0}\circ(\Omega_p^1\wedge\Omega_q^2)(X_1,\dots, X_{p+q})$ of U into H is defined by

$$\begin{aligned}
&\Phi_{\alpha_0}\circ(\Omega_p^1\wedge\Omega_q^2)(X_1,\dots, X_{p+q})(x) \\
&=\sum_{\sigma}\varepsilon(\sigma)(\Phi_{\alpha_0}^1\circ\Omega_p^1)(x\colon X_{\sigma(1)},\dots, X_{\sigma(p)}) \\
&\quad\otimes(\Phi_{\alpha_0}^2\circ\Omega_q^2)(x\colon X_{\sigma(p+1)},\dots, X_{\sigma(p+q)}), \quad \text{for} \quad x\in U,
\end{aligned}$$

where U is an arbitrary open set of M. By the introduction of this mapping Φ_{α_0}, the $H(M)$-valued differential $(p+q)$-form $\Omega_p^1\wedge\Omega_q^2$ becomes of class $C^{(n)}$ on M.

Now, let us consider two differential forms $\Omega_p^1\in\Omega_p^{(n)}(M, H^1(M))$, $\Omega_q^2\in\Omega_q^{(n)}(M, H^2(M))$ which are expressed in the forms

$$\begin{aligned}
\Omega_q^1(x)&=\sum_{i=1}^{\infty}\omega_p^{1^i}(x)\mathfrak{b}_i^1(x), \\
\Omega_q^2(x)&=\sum_{i=1}^{\infty}\omega_q^{2^i}(x)\mathfrak{b}_i^2(x),
\end{aligned}\tag{2.3}$$

where $(\mathfrak{b}_1^1(x), \mathfrak{b}_2^1(x),\dots, \mathfrak{b}_n^1(x),\dots)$, $(\mathfrak{b}_1^2(x), \mathfrak{b}_2^2(x),\dots, \mathfrak{b}_n^2(x),\dots)$ are the moving frames on M which are the bases of $H_x^1(M)$, $H_x^2(M)$ respectively. Then, as was mentioned in § 2.5, their exterior product $\Omega_p^1\wedge\Omega_q^2$ is expressed by

$$(\Omega_p^1\wedge\Omega_q^2)(x)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\omega_p^{1^i}\wedge\omega_q^{2^i})(x)\mathfrak{b}_i^1(x)\otimes\mathfrak{b}_j^2(x) \tag{2.4}$$

where $(\mathfrak{b}_i^1(x)\otimes\mathfrak{b}_j^2(x))$ $(i, j=1, 2,\dots)$ is a moving frame on M which is a base of $H_x(M)$.

As a particular case of the exterior product defined above, for $\omega_p\in\Omega_p^{(n)}(M, R)$ and $\Omega_q\in\Omega_q^{(n)}(M, H(M))$ expressed in the form

$$\Omega_q(x)=\sum_{i=1}^{\infty}\omega_q^i(x)\mathfrak{b}_i(x),$$

their exterior product $\omega_p\wedge\Omega_q$ is given by

$$(\omega_p\wedge\Omega_q)(x)=\sum_{i=1}^{\infty}(\omega_p\wedge\omega_q^i)(x)1\otimes\mathfrak{b}_i(x),$$

where 1 denotes a base of R and $\mathfrak{b}(x)=(\mathfrak{b}_1(x), \mathfrak{b}_2(x),\dots, \mathfrak{b}_n(x),\dots)$ is a moving frame on M which is a base of $H_x(M)$. By identifying $1\otimes\mathfrak{b}_i(x)$ with $\mathfrak{b}_i(x)$, we get

$$(\omega_p\wedge\Omega_q)(x)=\sum_{i=1}^{\infty}(\omega_p\wedge\omega_q^i)(x)\mathfrak{b}_i(x).$$

Thus $\omega_p\wedge\Omega_q$ is a $H(M)$-valued differential $(p+q)$-form.

From now, we shall discuss the exterior differential of $H(M)$-valued differential form with respect to a moving frame on M. Letting Ω_p be the $H(M)$-valued differential p-form on M written in the form

$$\Omega_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{b}_i(x),$$

we consider the differential form $\tilde{\Omega}_p(x)$ given by

$$\tilde{\Omega}_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{y}_i,$$

which is a H-valued differential p-form of class $C^{(n)}$ on M in virtue of Proposition 2 in § 3.1. By Proposition 7 in § 2.6, the exterior differential $d\tilde{\Omega}_p$ of $\tilde{\Omega}_p$ is given in the form

$$d\tilde{\Omega}_p(x) = \sum_{i=1}^{\infty} d\omega_p^i(x)\mathfrak{y}_i.$$

Since there exists a bounded linear operator $A(x)$ of class $C^{(n)}$ belonging to $GL(H)$ such that $\mathfrak{b}_i(x) = A(x)\mathfrak{y}_i$ $(i=1, 2,\ldots)$, we see

$$(A\, d\tilde{\Omega}_p)(x) = \sum_{i=1}^{\infty} d\omega_p^i(x)\mathfrak{b}_i(x),$$

where we mean by a notation $(A\, d\tilde{\Omega}_p)(x)$ that

$$(A\, d\tilde{\Omega}_p)(x)(X_1,\ldots, X_{p+1}) = A(x)d\tilde{\Omega}_p(x\colon X_1,\ldots, X_{p+1}),$$
$$\text{for} \quad X_1,\ldots, X_{p+1} \in \mathfrak{X}^{(n)}(M).$$

Obviously this differential form $A\, d\tilde{\Omega}_p$ is a $H(M)$-valued differential $(p+1)$-form of class $C^{(n-1)}$ on M. Thus we attain to the following definition:

Definition 3.

Let Ω_p be a $H(M)$-valued differential p-form expressed in terms of a moving frame $\mathfrak{b}(x) = (\mathfrak{b}_1(x), \mathfrak{b}_2(x),\ldots, \mathfrak{b}_n(x),\ldots)$ on M as follows:

$$\Omega_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{b}_i(x).$$

Then *the exterior differential* $d\Omega_p$ *of* Ω_p *with respect to* $\mathfrak{b}(x)$ is defined by

$$d\Omega_p(x) = \sum_{i=1}^{\infty} d\omega_p^i(x)\mathfrak{b}_i(x).$$

As has been defined above, the exterior differential of $H(M)$-valued differential form with respect to a moving frame on M is determined by the exterior differential of real valued differential form. Therefore the following proposition is derived immediately from Proposition 4, 5, 6 in § 2.6.

Proposition 1.

The exterior differential of $H(M)$-valued differential form with respect to a *moving frame on M has the following properties*:

$$d(f\Omega_p)=df\wedge\Omega_p+fd\Omega_p,$$

$$d(\Omega_p^1\wedge\Omega_q^2)=(d\Omega_p^1)\wedge\Omega_q^2+(-1)^p\Omega_p^1\wedge(d\Omega_q^2),$$

$$dd\Omega_p=0,$$

where $f\in C^{(n)}(M, R)$, $\Omega_p\in\Omega_p^{(n)}(M, H(M))$ *and*

$$\Omega_p^1\in\Omega_p^{(n)}(M, H^1(M)),\quad \Omega_q^2\in\Omega_q^{(n)}(M, H^2(M)).$$

Finally we shall mention to the canonical form of $H(M)$-valued differential form. For a $H(M)$-valued differential p-form Ω_p written in the form

$$\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x),$$

we consider the H-valued differential p-form $\tilde{\Omega}_p$ given by

$$\tilde{\Omega}_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{y}_i.$$

The canonical form of $\tilde{\Omega}_p$ with respect to a chart (U, φ) is the following:

$$\tilde{\Omega}_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

i.e.,

$$\tilde{\Omega}_p(x)=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\ldots j_p}^i(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{y}_i,$$

where $\omega_{j_1\ldots j_p}^i(x)=\omega_p^i(x\colon \partial_{j_1},\ldots,\partial_{j_p})$ and the sum $\sum_{1\leq j_1<\cdots<j_p<\infty}$ means $\lim_{N\to\infty}\sum_{1\leq j_1<\cdots<j_p\leq N}$, namely,

$$\sum_{1\leq j_1<\cdots<j_p<\infty}\{\omega_{j_1\ldots j_p}^i(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\}(X_1,\ldots,X_p)$$

$$=\lim_{N\to\infty}\sum_{1\leq j_1<\cdots<j_p\leq N}\{\omega_{j_1\ldots j_p}^i(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\}(X_1,\ldots,X_p),$$

for $X_1,\ldots,X_p\in\mathfrak{X}^{(n)}(M)$.
By the canonical form of Ω_p with respect to a chart (U, φ) we mean $A\tilde{\Omega}_p$, namely,

$$(A\tilde{\Omega}_p)(x)=A(x)\tilde{\Omega}_p(x)$$

$$=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\ldots j_p}^i(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{b}_i(x)$$

where $A(x)$ is a bounded linear operator of class $C^{(n)}$ belonging to $GL(H)$ such that $\mathfrak{b}_i(x)=A(x)\mathfrak{y}_i$ $(i=1, 2,\ldots)$. For simplicity we write for this canonical form formally, if there is no possibility of confusion,

$$\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

where we put $\omega_{j_1\dots j_p}(x)=\sum_{i=1}^{\infty}\omega^i_{j_1\dots j_p}(x)\mathfrak{b}_i(x)$ formally.

Here we give a remark on coefficients $\omega_{j_1\dots j_p}(x)$ of the canonical form: according to whether Ω_p is a H-valued or $H(M)$-valued differential p-form, the coefficients $\omega_{j_1\dots j_p}(x)$ of the canonical form are given in the form

$$\omega_{j_1\dots j_p}(x)=\begin{cases}\sum_{i=1}^{\infty}\omega^i_{j_1\dots j_p}(x)\mathfrak{y}_i, & \text{for } \Omega_p\in\Omega_p^{(n)}(M, H),\\ \sum_{i=1}^{\infty}\omega^i_{j_1\dots j_p}(x)\mathfrak{b}_i(x), & \text{for } \Omega_p\in\Omega_p^{(n)}(M, H(M)).\end{cases}$$

From now, we denote by $\tilde{\Omega}_p$ a H-valued differential p-form $A^{-1}\Omega_p$ for a $H(M)$-valued differential p-form Ω_p.

Let $\Omega_{p,1}, \Omega_{p,2},\dots, \Omega_{p,m},\dots, \Omega_p$ be $H(M)$-valued differential p-forms and U be an open set of M. *Then we say that a sequence* $\{\Omega_{p,m}\}$ *of* $\Omega_{p,1}, \Omega_{p,2},\dots, \Omega_{p,m},\dots$ *converges uniformly to* Ω_p *on* U *if a sequence* $\{\tilde{\Omega}_{p,m}\}$ *of* $\tilde{\Omega}_{p,1}, \tilde{\Omega}_{p,2},\dots, \tilde{\Omega}_{p,m},\dots$ *converges uniformly to* $\tilde{\Omega}_p$ *on* U.

Moreover a canonical form of Ω_p *is said to be a uniformly convergent canonical form of degree* l *if a canonical form of* $\tilde{\Omega}_p$ *is a uniformly convergent canonical form of degree* l.

Proposition 2.

Suppose that a canonical form of Ω_p *written in the form*

$$\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}\omega_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

i.e.,

$$\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}\omega^i_{j_1\dots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{b}_i(x),$$

is a uniformly convergent canonical form of degree l.

Then the exterior differential $d\Omega_p$ *of* Ω_p *with respect to a moving frame* $\mathfrak{b}(x)$ *is given by*

$$d\Omega_p(x)=\sum_{1\leq j_1<\cdots<j_p<\infty}d\omega_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p},$$

i.e.,

$$d\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}d\omega^i_{j_1\dots j_p}(x)\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{b}_i(x).$$

This proposition is readily proved by using the definition of the exterior differential $d\Omega_p$ of Ω_p with respect to a moving frame $\mathfrak{b}(x)$ and Proposition 2 in § 2.7.

3. Exterior covariant differentiation

Definition 1.

The exterior covariant differentiation of H-type is a linear mapping D_H over the real field R of the $C^{(n)}(M, R)$-module $\Omega_p^{(n)}(M, H(M))$ into the $C^{(n-1)}(M, R)$-module $\Omega_{p+1}^{(n-1)}(M, H(M))$ satisfying the following requirements: for each p,

(A_1) *a mapping* Ψ_H *of the fibre space* $H_x(M)$ *of H-type at* x *into itself given by*

$$\Psi_H\colon \Omega_p(x\colon X_1,\dots, X_p) \longrightarrow (D_H\Omega_p)(x\colon X_1,\dots, X_{p+1}),$$

$$\textit{for} \quad \Omega_p \in \Omega_p^{(n)}(M, H(M)),$$

is continuous,

(A_2) $D_H(\omega_q \wedge \Omega_p) = d\omega_q \wedge \Omega_p + (-1)^q \omega_q \wedge D_H\Omega_p$,

$$\textit{for} \quad \omega_q \in \Omega_q^{(n)}(M, R) \quad \textit{and} \quad \Omega_p \in \Omega_p^{(n)}(M, H(M)),$$

(A_3) $D_H C = 0$, *for constant* $C \in R$,

where x *is a point of* M *and* $X_1,\dots, X_{p+1}$ *are any* $(p+1)$ *vector fields belonging to* $\mathfrak{X}^{(n)}(M)$.

In what follows if there is no possibility of confusion, we omit the symbol of type of the exterior covariant differentiation and denote it by D merely.

Definition 2.

Let H^1, H^2 be two Hilbert spaces. *The exterior covariant differentiation* $D_{H^1\otimes H^2}$ *of* $H^1\otimes H^2$*-type* is defined by the following formula:

$$D_{H^1\otimes H^2}(\Omega_p^1 \wedge \Omega_q^2) = (D_{H^1}\Omega_p^1) \wedge \Omega_q^2 + (-1)^p \Omega_p^1 \wedge (D_{H^2}\Omega_q^2),$$

$$\textit{for} \quad \Omega_p^1 \in \Omega_p^{(n)}(M, H^1(M)) \quad \textit{and} \quad \Omega_q^2 \in \Omega_q^{(n)}(M, H^2(M)). \tag{3.1}$$

Proposition.

We see

$$D_R 1 = 0 \quad \textit{and} \quad D_R\omega_p = d\omega_p, \quad \textit{for} \quad 1 \in \Omega_0^{(n)}(M, R) \quad \textit{and} \quad \omega_p \in \Omega_p^{(n)}(M, R),$$

where 1 *denotes a base of* R.

Proof. By using (3.1), we have

$$D_{R\otimes R}(\omega_p \wedge 1) = (D_R\omega_p) \wedge 1 + (-1)^p \omega_p \wedge (D_R 1).$$

From $R\otimes R = R$, $\omega_p \wedge 1 = \omega_p$ and $(D_R\omega_p) \wedge 1 = D_R\omega_p$, we get

$$\omega_p \wedge (D_R 1) = 0.$$

Thus we see $D_R 1 = 0$.

Next, by using (A_2), we have

$$D_R(\omega_p \wedge 1) = d\omega_p \wedge 1 + (-1)^p \omega_p \wedge (D_R 1).$$

From $D_R 1 = 0$, we see $D_R\omega_p = d\omega_p$.

Finally we shall give a remark concerning to apply the exterior covariant differentiation to a $H(M)$-valued differential p-form Ω_p expressed in the form

$$\Omega_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{b}_i(x).$$

From $\mathfrak{b}_i(x)=1\otimes\mathfrak{b}_i(x)=1\wedge\mathfrak{b}_i(x)$, we see

$$\begin{aligned}\Omega_p(x)&=\sum_{i=1}^{\infty}\omega_p^i(x)1\wedge\mathfrak{b}_i(x)\\&=\sum_{i=1}^{\infty}\omega_p^i(x)\wedge\mathfrak{b}_i(x).\end{aligned}$$

Applying the exterior covariant differentiation D to the above formula, in virtue of (A_1) and (A_2), we have

$$(D\Omega_p)(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\wedge\mathfrak{b}_i(x)+(-1)^p\sum_{i=1}^{\infty}\omega_p^i(x)\wedge D\mathfrak{b}_i(x).$$

Consequently we get

$$(D\Omega_p)(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{b}_i(x)+(-1)^p\sum_{i=1}^{\infty}\omega_p^i(x)\wedge D\mathfrak{b}_i(x). \tag{3.2}$$

4. Connection forms

In this paragraph, we shall introduce the connection form and dual connection form by means of the exterior covariant differentiation. Moreover we shall investigate the transformation law of connection forms.

Let Ω_0 be a $H(M)$-valued differential 0-form expressed in the form

$$\Omega_0(x)=\sum_{i=1}^{\infty}\omega_0^i(x)\mathfrak{b}_i(x),$$

where $\omega_0^i\in\Omega_0^{(n)}(M,R)$. Applying D_H to the above formula, we get from (3.2)

$$(D_H\Omega_0)(x)=\sum_{i=1}^{\infty}d\omega_0^i(x)\mathfrak{b}_i(x)+\sum_{i=1}^{\infty}\omega_0^i(x)D_H\mathfrak{b}_i(x). \tag{4.1}$$

In virtue of (A_1), we have

$$D_H\mathfrak{b}_i(x)=\sum_{i=1}^{\infty}\omega_{H\,i}^j(x)\mathfrak{b}_j(x), \tag{4.2}$$

where $\omega_{H\,i}^j\in\Omega_1^{(n-1)}(M,R)$ and the matrix

$$\omega_H(x\colon X)=\begin{pmatrix}\omega_{H1}^1(x\colon X) & \omega_{H2}^1(x\colon X)\cdots\cdots\omega_{Hi}^1(x\colon X)\cdots\\ \omega_{H1}^2(x\colon X) & \omega_{H2}^2(x\colon X)\cdots\cdots\omega_{Hi}^2(x\colon X)\cdots\\ \vdots & \vdots \qquad\qquad \vdots\\ \omega_{H1}^j(x\colon X) & \omega_{H2}^j(x\colon X)\cdots\cdots\omega_{Hi}^j(x\colon X)\cdots\\ \vdots & \vdots \qquad\qquad \vdots\end{pmatrix},\quad\text{for } X\in\mathfrak{X}^{(n)}(M),$$

is a bounded linear operator on fibre space $H_x(M)$. Inserting (4.2) into the right-hand side of (4.1), we obtain

$$(D_H\Omega_0)(x)=\sum_{i=1}^{\infty}\{d\omega_0^i(x)+\sum_{j=1}^{\infty}\omega_{H\,j}^i(x)\omega_0^j(x)\}\mathfrak{b}_i(x). \tag{4.3}$$

We use for simplicity the matrix notations as follows:

$$d\omega_0=\begin{pmatrix} d\omega_0^1 \\ d\omega_0^2 \\ \vdots \\ d\omega_0^i \\ \vdots \end{pmatrix}, \quad D_H\mathfrak{b}=(D_H\mathfrak{b}_1,\ D_H\mathfrak{b}_2,\dots,\ D_H\mathfrak{b}_i,\dots)\,,$$

$$\omega_H=\begin{pmatrix} \omega_{H1}^1 & \omega_{H2}^1\cdots\cdots\omega_{Hi}^1\cdots \\ \omega_{H1}^2 & \omega_{H2}^2\cdots\cdots\omega_{Hi}^2\cdots \\ \vdots & \vdots \qquad\quad \vdots \\ \omega_{H1}^j & \omega_{H2}^j\cdots\cdots\omega_{Hi}^j\cdots \\ \vdots & \vdots \qquad\quad \vdots \end{pmatrix}.$$

Then the formulas (4.1), (4.2), (4.3) are written in the following forms:

$$D_H(\Omega_0)(x)=\mathfrak{b}(x)d\omega_0(x)+D_H\mathfrak{b}(x)\omega_0(x)\,, \tag{4.4}$$

$$D_H\mathfrak{b}(x)=\mathfrak{b}(x)\omega_H(x)\,, \tag{4.5}$$

$$(D_H\Omega_0)(x)=\mathfrak{b}(x)\{d\omega_0(x)+\omega_H(x)\omega_0(x)\}\,, \tag{4.6}$$

where $\Omega_0(x)=\mathfrak{b}(x)\omega_0(x)$.

Now, since $d\omega_0^i\in\Omega_1^{(n-1)}(M, R)$ and $\omega_{Hj}^i\in\Omega_1^{(n-1)}(M, R)$, these are represented in terms of the differentials dx^i of coordinate functions as follows:

$$d\omega_0^i(x)=\sum_{k=1}^{\infty}\partial_k\omega_0^i(x)dx^k\,,$$

$$\omega_{Hj}^i(x)=\sum_{k=1}^{\infty}\Gamma_{Hjk}^i(x)dx^k\,, \tag{4.7}$$

where $\Gamma_{Hjk}^i\in C^{(n-1)}(M, R)$. Inserting the above two formulas into (4.3), we obtain by a straightforward calculation

$$\begin{aligned}(D_H\Omega_0)(x\colon X)&=\sum_{i=1}^{\infty}\{d\omega_0^i(x\colon X)+\sum_{j=1}^{\infty}\omega_{Hj}^i(x\colon X)\omega_0^j(x)\}\mathfrak{b}_i(x)\\ &=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\overset{H}{\nabla}{}_{Xj}^i(x)\omega_0^j(x)\mathfrak{b}_i(x)\,,\end{aligned}$$

where

$$\overset{H}{\nabla}{}_{Xj}^i(x)=\sum_{k=1}^{\infty}dx^k(X)(\delta_j^i\partial_k+\Gamma_{Hjk}^i(x)), \qquad \text{for}\quad X\in\mathfrak{X}^{(n)}(M)\,. \tag{4.8}$$

By using the matrix notation

$$\overset{H}{\nabla}_X=\begin{pmatrix} \overset{H}{\nabla}{}_{X1}^1 & \overset{H}{\nabla}{}_{X2}^1\cdots\cdots\overset{H}{\nabla}{}_{Xj}^1\cdots \\ \overset{H}{\nabla}{}_{X1}^2 & \overset{H}{\nabla}{}_{X2}^2\cdots\cdots\overset{H}{\nabla}{}_{Xj}^2\cdots \\ \vdots & \vdots \qquad\quad \vdots \\ \overset{H}{\nabla}{}_{X1}^i & \overset{H}{\nabla}{}_{X2}^i\cdots\cdots\overset{H}{\nabla}{}_{Xj}^i\cdots \\ \vdots & \vdots \qquad\quad \vdots \end{pmatrix},$$

we have

$$(D_H\Omega_0)(\mathfrak{x}\colon X)=\mathfrak{b}(\mathfrak{x})\overset{H}{\nabla}_X(\mathfrak{x})\omega_0(\mathfrak{x})\,. \tag{4.9}$$

Definition 1.

We say that ω_H, Γ_H, $\overset{H}{\nabla}_X$ are respectively *the connection form, the connection, the covariant derivation in the direction of X of H-type with respect to a moving frame* $\mathfrak{b}(\mathfrak{x})$ *on* M where $\Gamma_H=(\Gamma^i_{Hjk})_{i,j,k=1,2,\dots}$.

Henceforth if there is no possibility of confusion, we shall omit the symbol of type.

As can be seen easily from (4.8) and the definition of the differential of functions, we have

Proposition 1.

Let X, $Y\in\mathfrak{X}^{(n)}(M)$ *and let* Ω_0^1, $\Omega_0^2\in\Omega_0^{(n)}(M, H(M))$ *which are expressed in the forms*

$$\Omega_0^1(\mathfrak{x})=\sum_{i=1}^{\infty}\overset{1}{\omega}{}_0^i(\mathfrak{x})\mathfrak{b}_i(\mathfrak{x})\,,$$

$$\Omega_0^2(\mathfrak{x})=\sum_{i=1}^{\infty}\overset{1}{\omega}{}_0^i(\mathfrak{x})\mathfrak{b}_i(\mathfrak{x})\,.$$

Then we have

$$\begin{aligned}
&\nabla_{X+Y}\overset{1}{\omega}_0=\nabla_X\overset{1}{\omega}_0+\nabla_Y\overset{1}{\omega}_0\,,\\
&\nabla_X(\overset{1}{\omega}_0+\overset{2}{\omega}_0)=\nabla_X\overset{1}{\omega}_0+\nabla_X\overset{2}{\omega}_0\,,\\
&\nabla_{\lambda X}\overset{1}{\omega}_0=\lambda\nabla_X\overset{1}{\omega}_0, \quad \textit{for} \quad \lambda\in R\,,\\
&\nabla_X(f\overset{1}{\omega}_0)=f\nabla_X\overset{1}{\omega}_0+X(f)\overset{1}{\omega}_0, \quad \textit{for} \quad f\in C^{(n)}(M, R)\,,
\end{aligned} \tag{4.10}$$

where

$$\overset{1}{\omega}_0=\begin{pmatrix}\overset{1}{\omega}{}_0^1\\ \overset{1}{\omega}{}_0^2\\ \vdots\\ \overset{1}{\omega}{}_0^i\\ \vdots\end{pmatrix},\quad \overset{2}{\omega}_0=\begin{pmatrix}\overset{2}{\omega}{}_0^1\\ \overset{2}{\omega}{}_0^2\\ \vdots\\ \overset{2}{\omega}{}_0^i\\ \vdots\end{pmatrix}.$$

In the above formulas, we omitted the symbol $\mathfrak{x}$.

Next we shall proceed to the discussion of dual connection forms. Since the fibre space $H_{\mathfrak{x}}(M)$ of H-type at $\mathfrak{x}\in M$ is a Hilbert space, we may consider the dual space $H^*_{\mathfrak{x}}(M)$ of $H_{\mathfrak{x}}(M)$ which is said to be the dual fibre space of $H_{\mathfrak{x}}(M)$ or the fibre space of H^*-type at $\mathfrak{x}$, and furthermore we have the dual fibre bundle $H^*(M)=\bigcup_{\mathfrak{x}\in X}H^*_{\mathfrak{x}}(M)$ of $H(M)$ (or the fibre bundle of H^*-type on M). By $\Omega_p^{(n)}(M, H^*(M))$ we mean a set of all $H^*(M)$-valued differential p-forms of class $C^{(n)}$ on M. For any two elements $X\in H_{\mathfrak{x}}(M)$ and $X^*\in H^*_{\mathfrak{x}}(M)$, we denote the value of the bounded linear functional X^*

at X by $\langle X^* | X\rangle$ according to the Dirac's notation. Moreover we denote by $\Omega_{(p,q)}(M, H(M))$ a set of all exterior products $\Omega_p^* \wedge \Omega_q$ of any two differential forms $\Omega_p^* \in \Omega_p^{(n)}(M, H^*(M))$ and $\Omega_q \in \Omega_q^{(n)}(M, H(M))$.

Definition 2.

The contraction C is a mapping of $\Omega_{(p,q)}(M, H(M))$ into $C^{(n)}(M, R)$ given by

$$C(\Omega_p^* \wedge \Omega_q)(x: X_1,\dots, X_{p+q})$$
$$=\langle \Omega_p^*(x: X_1,\dots, X_p) \,|\, \Omega_q(x: X_{p+1},\dots, X_{p+q})\rangle ,$$
$$\text{for} \quad x\in M \quad \text{and} \quad X_1,\dots, X_{p+q}\in \mathfrak{X}^{(n)}(M).$$

Now, we set up the following requirement:

(R_1) *the exterior covariant differentiation commute with the contraction.*

The above requirement leads us to the following conclusion:

Proposition 2.

Let $\mathfrak{b}^*(x)=(\mathfrak{b}^1(x), \mathfrak{b}^2(x),\dots, \mathfrak{b}^n(x),\dots)$ be the dual moving frame of a moving frame $\mathfrak{b}(x)=(\mathfrak{b}_1(x), \mathfrak{b}_2(x),\dots, \mathfrak{b}_n(x),\dots)$, namely

$$\langle \mathfrak{b}^i(x) \,|\, \mathfrak{b}_j(x)\rangle =\delta^i_j .$$

Assume that

$$D_H\mathfrak{b}(x)=\mathfrak{b}(x)\omega_H(x) \quad \text{and} \quad D_{H^*}\mathfrak{b}^*(x)=\mathfrak{b}^*(x)\omega_{H^*}(x) .$$

Then we have

$$\omega_H(x)+\omega_{H^*}(x)=0 . \tag{4.11}$$

Proof. From the definition of the contraction, we have immediately

$$C(\mathfrak{b}^i(x)\wedge \mathfrak{b}_j(x))=\langle \mathfrak{b}^i(x) \,|\, \mathfrak{b}_j(x)\rangle =\delta^i_j .$$

In virtue of the requirements (R_1) and (A_3), we get

$$CD_{H^*\otimes H}(\mathfrak{b}^i(x)\wedge \mathfrak{b}_j(x))=D_{H^*\otimes H}C(\mathfrak{b}^i(x)\wedge \mathfrak{b}_j(x))$$
$$=D_{H^*\otimes H}\delta^i_j$$
$$=0 .$$

On the other hand, for the sake of (3.1), we see

$$CD_{H^*\otimes H}(\mathfrak{b}^i(x)\wedge \mathfrak{b}_j(x))=C(D_{H^*}\mathfrak{b}^i(x)\wedge \mathfrak{b}_j(x)+\mathfrak{b}^i(x)\wedge D_H\mathfrak{b}_j(x))$$
$$=C(\{\sum_{k=1}^{\infty} \omega_{H^*}{}^i_k(x)\mathfrak{b}^k(x)\}\wedge \mathfrak{b}_j(x)+\mathfrak{b}^i(x)\wedge \{\sum_{k=1}^{\infty} \omega^k_{H\,j}(x)\mathfrak{b}_k(x)\})$$
$$=\sum_{k=1}^{\infty} \omega_{H^*}{}^i_k(x)\langle \mathfrak{b}^k(x) \,|\, \mathfrak{b}_j(x)\rangle + \sum_{k=1}^{\infty} \omega^k_{H\,j}(x)\langle \mathfrak{b}^i(x) \,|\, \mathfrak{b}_k(x)\rangle$$
$$=\omega_{H^*}{}^i_j(x)+\omega^i_{H\,j}(x) .$$

Consequently, we have

$$\omega_{H^*}{}^i_j(x)+\omega^i_{H\,j}(x)=0\,.$$

Definition 3.

Let ω_H be a connection form of H-type. Then the connection form ω_{H^*} of H^*-type satisfying the relation

$$\omega_{H^*}+\omega_H=0\,, \tag{4.12}$$

is said to be *the dual connection form of* ω_H.

Before we proceed to the investigation of the transformation law of connection forms, we shall give the formal discussion about "matrix differential forms". By a matrix differential p-form we shall mean a matrix such that its all elements are real valued differential p-forms, and by the exterior differential of the matrix differential form we shall mean the exterior differential of all elements of this matrix differential form. Moreover the exterior multiplication of two matrix differential forms is defined in the same manner as in the familiar matrix multiplication, but the product of matrix elements must be replaced by the exterior product. In particular, the exterior multiplication of two matrix differential o-forms and the exterior multiplication of a matrix differential o-form with a matrix differential 1-form, is the usual matrix multiplication.

Let A_p be a matrix differential p-form and A_q be a matrix differential q-form. Then it is easy to see that

$$d(A_p\wedge A_q)=(dA_p)\wedge A_q+(-1)^pA_p\wedge(dA_q)\,. \tag{4.13}$$

In fact, we have formally

$$d\Big(\sum_{k=1}^{\infty}\omega^i_{p,k}(x)\wedge\omega^k_{q,j}(x)\Big)=\sum_{k=1}^{\infty}d(\omega^i_{p,k}(x)\wedge\omega^k_{q,j}(x))$$

$$=\sum_{k=1}^{\infty}d\omega^i_{p,k}(x)\wedge\omega^k_{q,j}(x)+(-1)^p\sum_{k=1}^{\infty}\omega^i_{p,k}(x)\wedge d\omega^k_{q,j}(x)\,,$$

where $\omega^i_{p,j}$, $\omega^i_{q,j}$ are (i,j) elements of A_p, A_q respectively.

From now on, we begin to discuss the transformation law of connection forms. As was mentioned in § 3.1, there exists the following relation between two moving frames $\mathfrak{b}_\alpha(x)$, $\mathfrak{b}_\beta(x)$ of the same type:

$$\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)\,, \tag{4.14}$$

where $A_{\alpha\beta}(x)$ is an isomorphism on H of class $C^{(n)}$. We get immediately from (4.5) that

$$D\mathfrak{b}_\beta(x)=\mathfrak{b}_\beta(x)\omega^\beta(x)\,,$$

$$D\mathfrak{b}_\alpha(x)=\mathfrak{b}_\alpha(x)\omega^\alpha(x)\,, \tag{4.15}$$

where ω^α, ω^β are connection forms of H-type with respect to the moving frames $\mathfrak{b}_\alpha(x)$, $\mathfrak{b}_\beta(x)$ respectively. On the other hand, applying D to both sides of (4.14), we get

$$D\mathfrak{b}_\beta(x)=D(\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)),$$

i.e., for each component,

$$D\mathfrak{b}^\beta_i(x)=D(A_{\alpha\beta}(x)\mathfrak{b}^\alpha_i(x)),$$

and further by using the matrix representation of $A_{\alpha\beta}(x)$, we obtain from (1.7), (3.2), (4.2),

$$\begin{aligned}D\mathfrak{b}^\beta_i(x)&=D(\sum_{j=1}^{\infty}a^j_{\alpha\beta,i}(x)\mathfrak{b}^\alpha_j(x))\\&=\sum_{j=1}^{\infty}da^j_{\alpha\beta,i}(x)\mathfrak{b}^\alpha_j(x)+\sum_{j=1}^{\infty}a^j_{\alpha\beta,i}(x)D\mathfrak{b}^\alpha_j(x)\\&=\sum_{j=1}^{\infty}da^j_{\alpha\beta,i}(x)\mathfrak{b}^\alpha_j(x)+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\omega^{\alpha j}_{k}(x)a^k_{\alpha\beta,i}(x)\mathfrak{b}^\alpha_j(x),\end{aligned}$$

where we used that $\omega^\alpha(x: X)$ is a bounded linear operator on fibre space $H_x(M)$ for $X\in\mathfrak{X}^{(n)}(M)$. Consequently, in terms of matrix notation, we have

$$D\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)dA_{\alpha\beta}(x)+\mathfrak{b}_\alpha(x)\omega^\alpha(x)A_{\alpha\beta}(x). \tag{4.16}$$

Form (4.14), the first formula of (4.15) and (4.16), we get

$$\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)\omega^\beta(x)=\mathfrak{b}_\alpha(x)dA_{\alpha\beta}(x)+\mathfrak{b}_\alpha(x)\omega^\alpha(x)A_{\alpha\beta}(x),$$

and hence

$$\omega^\beta(x)=A_{\beta\alpha}(x)dA_{\alpha\beta}(x)+A_{\beta\alpha}(x)\omega^\alpha(x)A_{\alpha\beta}(x). \tag{4.17}$$

Thus we come to the following conclusion:

Proposition 3.

Suppose that $\mathfrak{b}_\alpha(x)$, $\mathfrak{b}_\beta(x)$ are two moving frames of H-type which satisfy a relation

$$\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)$$

and that ω^α, ω^β are the connection forms of H-type with respect to $\mathfrak{b}_\alpha(x)$, $\mathfrak{b}_\beta(x)$ respectively.

Then the transformation law of connection forms is given by (4.17) *where $A_{\alpha\beta}(x)$ is an isomorphism on H of class $C^{(n)}$.*

In a particular case of

$$A_{\beta\alpha}(x)=\begin{pmatrix}\dfrac{\partial x^1_\beta}{\partial x^1_\alpha} & \dfrac{\partial x^1_\beta}{\partial x^2_\alpha}\cdots\cdots & \dfrac{\partial x^1_\beta}{\partial x^i_\alpha}\cdots\\ \dfrac{\partial x^2_\beta}{\partial x^1_\alpha} & \dfrac{\partial x^2_\beta}{\partial x^2_\alpha}\cdots\cdots & \dfrac{\partial x^2_\beta}{\partial x^i_\alpha}\cdots\\ \vdots & \vdots & \vdots\\ \dfrac{\partial x^i_\beta}{\partial x^1_\alpha} & \dfrac{\partial x^i_\beta}{\partial x^2_\alpha}\cdots\cdots & \dfrac{\partial x^i_\beta}{\partial x^i_\alpha}\cdots\\ \vdots & \vdots & \vdots\end{pmatrix},$$

by using $\omega^i_j(x\colon \partial/\partial x^k)=\Gamma^i_{jk}(x)$ and $\frac{\partial}{\partial x^i_\beta}=\sum_{j=1}^{\infty}\frac{\partial x^j_\alpha}{\partial x^i_\beta}\frac{\partial}{\partial x^j_\alpha}$, we obtain from (4.17) by a straightforward calculation

$$\overset{\beta}{\Gamma}{}^i_{jk}(x)=\sum_{l,m,n=1}^{\infty}\frac{\partial x^m_\alpha}{\partial x^j_\beta}\frac{\partial x^n_\alpha}{\partial x^k_\beta}\frac{\partial x^i_\beta}{\partial x^l_\alpha}\overset{\alpha}{\Gamma}{}^l_{mn}(x)+\sum_{l=1}^{\infty}\frac{\partial^2 x^l_\alpha}{\partial x^j_\beta\partial x^k_\beta}\frac{\partial x^i_\beta}{\partial x^l_\alpha},$$

where $\overset{\alpha}{\Gamma}=(\overset{\alpha}{\Gamma}{}^i_{jk})$, $\overset{\beta}{\Gamma}=(\overset{\beta}{\Gamma}{}^i_{jk})$ are connections with respect to $\mathfrak{b}_\alpha(x)=(\partial/\partial x^1_\alpha, \partial/\partial x^2_\alpha,\dots)$, $\mathfrak{b}_\beta(x)=(\partial/\partial x^1_\beta, \partial/\partial x^2_\beta,\dots)$ respectively. This is the generalization of the transformation law of affine connections.

We conclude this paragraph with a slight mention on the exterior covariant differentiation of $(H\otimes H^*)$-type. Let Ω_p be a $(H\otimes H^*)(M)$-valued differential p-form expressed in the form

$$\Omega_p(x)=\sum_{i,j=1}^{\infty}\omega^i_{p,j}(x)\mathfrak{b}_i(x)\otimes\mathfrak{b}^j(x)$$

where $\mathfrak{b}^*(x)=(\mathfrak{b}^1(x), \mathfrak{b}^2(x),\dots, \mathfrak{b}^n(x),\dots)$ is the dual moving frame of $\mathfrak{b}(x)=(\mathfrak{b}_1(x), \mathfrak{b}_2(x),\dots, \mathfrak{b}_n(x),\dots)$. Applying $D_{H\otimes H^*}$ to both sides of the above formula, we get easily in virtue of (3.1), (3.2), (4.11),

$$\begin{aligned}D_{H\otimes H^*}\Omega_p(x)=\sum_{i,j=1}^{\infty}\{d\omega^i_{p,j}(x)+\sum_{k=1}^{\infty}\omega^i_{Hk}(x)\wedge\omega^k_{p,j}(x)\\ -\sum_{k=1}^{\infty}\omega^k_{Hj}(x)\wedge\omega^i_{p,k}(x)\}\mathfrak{b}_i(x)\otimes\mathfrak{b}^j(x).\end{aligned}\tag{4.18}$$

5. Parallelism

By a differentiable curve of class $C^{(n)}$ (or a smooth curve) on M we shall mean a differentiable mapping $\gamma=x(t)$ of class $C^{(n)}$ of a closed interval $[a, b]$ of R into M: for an arbitrary point $t\in(a, b)$ and for each chart (U, φ) at $x(t)\in M$, a mapping $\varphi\circ\gamma$ of an open interval $I_\delta=(t-\delta, t+\delta)$ into E is a differentiable mapping of class $C^{(n)}$ on I_δ where δ is a positive number. We shall denote the derivative $(\varphi\circ\gamma)'\in\mathscr{L}(I_\delta, E)$ by a notation $\dot{x}$. The vector $\dot{x}(t)$ belongs to $T_{x(t)}(M)$ and hence can be expressed in the form

$$\dot{x}(t)=\sum_{i=1}^{\infty}\dot{x}^i(t)\left(\frac{\partial}{\partial x^i}\right)_{x(t)}.\tag{5.1}$$

Definition.

A $H(M)$-valued differential p-form Ω_p is said to *be parallel along a smooth curve* $\gamma=x(t)$ *on* M, if the equation

$$(D\Omega_p)(x(t)\colon \underbrace{\dot{x}(t),\dots, \dot{x}(t)}_{(p+1)-\text{factors}})=0$$

is satified everywhere on the curve $\gamma=x(t)$.

Now we consider the uniformly convergent canonical form of degree 1 of Ω_p with respect to a chart (U, φ) as follows:

$$\Omega_p(x)=\sum_{i=1}^{\infty}\sum_{1\leq j_1<\cdots<j_p<\infty}\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{b}_i(x).$$

In virtue of (A_1) and (A_2), we get

$$(D\Omega_p)(x)=\sum_{i=1}^{\infty}\{d(\sum_{1\leq j_1<\cdots<j_p<\infty}\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}\mathfrak{b}_i(x)$$
$$+(-1)^p(\sum_{1\leq j_1<\cdots<j_p<\infty}\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p})\wedge D\mathfrak{b}_i(x).$$

Since the real valued p-form

$$\sum_{1\leq j_1<\cdots<j_p<\infty}\omega^i_{j_1\ldots j_p}(x)dx^{j_1}\wedge\cdots\wedge dx^{j_p}$$

is a canonical form of degree 1, we apply Proposition 2 of §2.7 to the first term in the right-hand side of the above formula and we use (4.2) for $D\mathfrak{b}_i(x)$ of the second term. Hence we have

$$(D\Omega_p)(x)=\sum_{i=1}^{\infty}\{\sum_{1\leq j_1<\cdots<j_p<\infty}(d\omega^i_{j_1\ldots j_p}(x)+(-1)^p\sum_{k=1}^{\infty}\omega^i_k(x)\cdot$$
$$\omega^k_{j_1\ldots j_p}(x))\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}\}\mathfrak{b}_i(x).$$

Thus we obtain by using (5.1)

$$(D\Omega_p)(x(t):\dot{x}(t),\ldots,\dot{x}(t))$$
$$=\sum_{i=1}^{\infty}\{\sum_{1\leq j_1<\cdots<j_p<\infty}d(\omega^i_{j_1\ldots j_p}(x(t):\dot{x}(t)) \tag{5.2}$$
$$+(-1)^p\sum_{k=1}^{\infty}\omega^i_k(x(t):\dot{x}(t))\omega^k_{j_1\ldots j_p}(x(t)))\dot{x}^{j_1}(t)\cdots\dot{x}^{j_p}(t)\}\mathfrak{b}_i(x(t)).$$

In case $p=0$: $\Omega_0(x)=\sum_{i=1}^{\infty}\omega^i_0(x)\mathfrak{b}_i(x)$, the formula (5.2) becomes

$$(D\Omega_0)(x(t):\dot{x}(t))=\sum_{i=1}^{\infty}\{d\omega^i_0(x(t):\dot{x}(t))+\sum_{k=1}^{\infty}\omega^i_k(x(t):\dot{x}(t))\omega^k_0(x(t))\}\mathfrak{b}_i(x(t)).$$

Moreover we see by using (3.2) of §2.3 and (4.7)

$$(D\Omega_0)(x(t):\dot{x}(t))=\sum_{i=1}^{\infty}\left\{\frac{d\omega^i_0(x(t))}{dt}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Gamma^i_{jk}(x(t))\dot{x}^k(t)\omega^j_0(x(t))\right\}\mathfrak{b}_i(x(t)).$$

In the special case where Ω_0 is a $T(M)$-valued differential o-form:

$$\Omega_0(x)=\sum_{i=1}^{\infty}\dot{x}^i(t)\left(\frac{\partial}{\partial x^i}\right)_{x(t)},$$

we obtain

$$(D\Omega_0)(x(t:\dot{x}(t))=\sum_{i=1}^{\infty}\left\{\frac{d^2x^i(t)}{dt^2}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Gamma^i_{jk}(x(t))\dot{x}^k(t)\dot{x}^j(t)\right\}\left(\frac{\partial}{\partial x^i}\right)_{x(t)}.$$

Consequently we have the equation of the parallel translation of $T(M)$-valued differential o-form along a smooth curve $\gamma = x(t)$ as follows:

$$\frac{d^2 x^i(t)}{dt^2} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Gamma^i_{jk}(x(t)) \dot{x}^k(t) \dot{x}^j(t) = 0 .$$

This is the generalization of the equation which determine a geodesic in the usual differential geometry.

6. Curvature forms and structure equation

We shall now define the curvature form by means of the exterior covariant differentiation.

Definition.

By *the curvature form with respect to a moving frame* $\mathfrak{b}(x)$ we shall mean the matrix differential 2-form R such that

$$D(D\mathfrak{b}(x)) = \mathfrak{b}(x) R(x) .$$

In case $D\mathfrak{b}(x) = \mathfrak{b}(x)\omega(x)$, the curvature form R is said to be *the curvature form of the connection form* ω.

It goes without saying in the above definition that the curvature form has the same type as the exterior covariant differentiation.

Theorem 1 (*Structure equation*).
Let ω *be a connection form and* R *its curvature form. Then we have*

$$R = d\omega + \omega \wedge \omega .$$

Proof. In virtue of (A_1) and (A_2), we get

$$\begin{aligned} D(D\mathfrak{b}(x)) &= D(\mathfrak{b}(x)\omega(x)) \\ &= \mathfrak{b}(x) d\omega(x) + (D\mathfrak{b}(x)) \wedge \omega(x) \\ &= \mathfrak{b}(x) d\omega(x) + \mathfrak{b}(x)\omega(x) \wedge \omega(x) \\ &= \mathfrak{b}(x)(d\omega(x) + \omega(x) \wedge \omega(x)) . \end{aligned}$$

Therefore we have

$$R(x) = d\omega(x) + \omega(x) \wedge \omega(x) .$$

Lemma 1.
Let ω^i_j $(i, j = 1, 2, \ldots)$ be real valued differential 1-forms of class $C^{(n)}$ such that

$$\sum_{i=1}^{\infty} |\omega^i_j(x : X)|^2 < \infty, \ \sum_{j=1}^{\infty} |\omega^i_j(x : X)|^2 < \infty, \quad \text{for} \quad x \in U \quad \text{and} \quad X \in \mathfrak{X}^{(n)}(M),$$

where U is an open set of M. Then we have

$$(\sum_{k=1}^{\infty} \omega_k^i(x) \wedge \omega_j^k(x))' = \sum_{k=1}^{\infty} (\omega_k^{i\prime}(x) \wedge \omega_j^k(x) + \omega_k^i(x) \wedge \omega_j^{k\prime}(x)).$$

Lemma 2.

Under the same conditions as in Lemma 1, we have

$$d(\sum_{k=1}^{\infty} \omega_k^i(x) \wedge \omega_j^k(x)) = \sum_{k=1}^{\infty} d(\omega_k^i(x) \wedge \omega_j^k(x)).$$

The proof of Lemma 1, 2 is a straightforward calculation. It will therefore be omitted.

Now, let $\mathfrak{b}^*(x) = (\mathfrak{b}^1(x), \mathfrak{b}^2(x), \ldots, \mathfrak{b}^n(x), \ldots)$ be the dual moving frame of $\mathfrak{b}(x)$ and R_j^i be the (i, j)-element of curvature form R with respect to $\mathfrak{b}(x)$. Here we set a condition for R such that

$$\sum_{i,j=1} |R_j^i(x : X_1, X_2)|^2 < \infty, \quad \text{for} \quad x \in U \quad \text{and} \quad X_1, X_2 \in \mathfrak{X}^{(n)}(M).$$

Then the differential form $\Omega(x)$ given by

$$\Omega(x) = \sum_{i,j=1}^{\infty} R_j^i(x) \mathfrak{b}_i(x) \otimes \mathfrak{b}^j(x),$$

is a $(H \otimes H^*)(M)$-valued differential 2-form of class $C^{(n-2)}$.

Theorem 2 (*Bianchi's identity*).

Under the above condition, we have

$$D\Omega = 0.$$

Proof. By using (4.18), we see readily

$$D\Omega(x) = \sum_{i,j=1}^{\infty} \{dR_j^i(x) + \sum_{k=1}^{\infty} \omega_k^i(x) \wedge R_j^k(x)$$
$$- \sum_{k=1}^{\infty} \omega_j^k(x) \wedge R_k^i(x)\} \mathfrak{b}_i(x) \otimes \mathfrak{b}^j(x).$$

In virtue of structure equation and Lemma 2, we get

$$dR_j^i(x) + \sum_{k=1}^{\infty} \omega_k^i(x) \wedge R_j^k(x) - \sum_{k=1}^{\infty} \omega_j^k(x) \wedge R_k^i(x)$$
$$= d\{d\omega_j^i(x) + \sum_{k=1}^{\infty} \omega_k^i(x) \wedge \omega_j^k(x)\}$$
$$+ \sum_{k=1}^{\infty} \omega_k^i(x) \wedge \{d\omega_j^k(x) + \sum_{l=1}^{\infty} \omega_l^k(x) \wedge \omega_j^l(x)\}$$
$$- \sum_{k=1}^{\infty} \omega_j^k(x) \wedge \{d\omega_k^i(x) + \sum_{l=1}^{\infty} \omega_l^i(x) \wedge \omega_k^l(x)\}$$
$$= \sum_{k=1}^{\infty} d\omega_k^i(x) \wedge \omega_j^k(x) - \sum_{k=1}^{\infty} \omega_k^i(x) \wedge d\omega_j^k(x)$$

$$+\sum_{k=1}^{\infty}\omega^i_k(x)\wedge d\omega^k_j(x)-\sum_{k=1}^{\infty}\omega^k_j(x)\wedge d\omega^i_k(x)$$

$$+\sum_{k,l=1}^{\infty}\omega^i_k(x)\wedge\omega^k_l(x)\wedge\omega^l_j(x)$$

$$-\sum_{k,l=1}^{\infty}\omega^k_j(x)\wedge\omega^i_l(x)\omega\wedge{}^l_k(x).$$

Now, 1st and 4th terms, 2nd and 3rd terms, 5th and 6th terms in the right-hand side of the above formula, cancel respectively where we used that $\omega(x: X)$ is a bounded linear operator on $H_x(M)$. Thus we have

$$dR^i_j(x)+\sum_{k=1}^{\infty}\omega^i_k(x)\wedge R^k_j(x)-\sum_{k=1}^{\infty}\omega^k_j(x)\wedge R^i_k(x)=0. \tag{6.1}$$

This completes our proof of Theorem 2.

7. Examples

Let M be the 4-dimensional differentiable manifold which has the Minkowski space as the base space and H be the isotopic spin space (the 2-dimensional Euclidean space):

$$\psi=\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\text{proton}\\ \text{neutron}\end{pmatrix},\qquad \text{for}\quad \psi\in H.$$

Furthermore, let be $GL(H)=SU(2)$. Throughout this paragraph we use the convention that the sum is to be performed with respect to this index, whenever the same index appears twice in a term of a formula.

Now, we consider an infinitesimal transformation of $A(x)\in SU(2)$:

$$A(x)=I+\varepsilon^a(x)\tau_a,\quad (I\text{ is the unit matrix}),$$

where τ_a are the generators of $SU(2)$ (the isotopic spin matrices):

$$\tau_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \tau_3=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

with the commutation relations

$$[\tau_a, \tau_b]=f^c_{ab}\tau_c, \tag{7.1}$$

where

$$f^c_{ab}=\begin{cases}-2, & \text{for even permutation } (a, b, c) \text{ of } \{1, 2, 3\},\\ 2, & \text{for odd permutation } (a, b, c) \text{ of } \{1, 2, 3\},\\ 0, & \text{for else}.\end{cases}$$

Then the transformation law (4.17) of connection forms reduces to

$$\begin{cases} \omega'(x)=\omega(x)+\delta\omega(x), \\ \delta\omega(x)=\varepsilon^a(x)[\omega(x), \tau_a]+d\varepsilon^a(x)\tau_a. \end{cases} \tag{7.2}$$

This result shows that $\omega(x)$ may be chosen to be linear combinations of the generators τ_a:

$$\omega(x)=B^a(x)\tau_a=B^a_j(x)dx^j\tau_a. \tag{7.3}$$

The fields $B^a_j(x)$ $(a=1, 2, 3)$ are called the Yang-Mills potential and it satisfies the following transformation law obtained from (7.1), (7.2), (7.3):

$$\begin{cases} B'^c_j(x)=B^c_j(x)+\delta B^c_j(x), \\ \delta B^c_j(x)=-f^c_{ab}\varepsilon^a(x)B^b_j(x)+\partial_j\varepsilon^c(x). \end{cases}$$

From (4.6) and (4.9), we get

$$\nabla_X\psi(x)=d\psi(x)(X)+\omega(x: X)\psi(x). \tag{7.4}$$

By setting $X=\partial_j$ in the above formula and by using $\omega(x: \partial_j)=B^a_j(x)\tau_a$, we have

$$\nabla_{\partial_j}\psi(x)=\partial_j\psi(x)+B^a_j(x)\tau_a\psi(x). \tag{7.5}$$

This gives the covariant derivative of $\psi(x)$.

Next we see immediately from (7.3)

$$d\omega(x)=\partial_k B^a_j(x)dx^k\wedge dx^j\tau_a,$$

$$\omega(x)\wedge\omega(x)=B^a_j(x)B^b_k(x)dx^j\wedge dx^k\tau_a\tau_b,$$

and hence

$$d\omega(x: \partial_j, \partial_k)=(\partial_j B^a_k(x)-\partial_k B^a_j(x))\tau_a,$$

$$(\omega\wedge\omega)(x: \partial_j, \partial_k)=B^a_j(x)B^b_k(x)[\tau_a, \tau_b]$$

$$=\frac{1}{2}f^a_{bc}(B^b_j(x)B^c_k(x)-B^b_k(x)B^c_j(x))\tau_a.$$

Thus we get from the structure equation

$$R(x: \partial_j, \partial_k)=R^a_{jk}(x)\tau_a \tag{7.6}$$

with

$$R^a_{jk}(x)=\partial_j B^a_k(x)-\partial_k B^a_j(x)+\frac{1}{2}f^a_{bc}(B^b_j(x)B^c_k(x)-B^b_k(x)B^c_j(x)). \tag{7.7}$$

The fields $R^a_{jk}(x)$ $(a=1, 2, 3)$ are called the Yang-Milles field. Next it is seen easily from (6.1) that the Bianchi's identity is written by the matrix differential forms in the following form:

$$dR(x)+\omega(x)\wedge R(x)-R(x)\wedge\omega(x)=0. \tag{7.8}$$

From (7.3) and (7.6), we obtain

$$dR(\mathscr{x}\colon \partial_j, \partial_k, \partial_l)=2(\partial_j R^a_{kl}(\mathscr{x})+\partial_k R^a_{lj}(\mathscr{x})+\partial_l R^a_{jk}(\mathscr{x}))\tau_a,$$

$$\omega(\mathscr{x})\wedge R(\mathscr{x})(\partial_j, \partial_k, \partial_l)=2(B^a_j(\mathscr{x})R^b_{kl}(\mathscr{x})+B^a_k(\mathscr{x})R^b_{lj}(\mathscr{x})+B^a_l(\mathscr{x})R^b_{jk}(\mathscr{x}))\tau_a\tau_b,$$

$$R(\mathscr{x})\wedge\omega(\mathscr{x})(\partial_j, \partial_k, \partial_l)=2(R^a_{jk}(\mathscr{x})B^b_l(\mathscr{x})+R^a_{kl}(\mathscr{x})B^b_j(\mathscr{x})+R^a_{lj}(\mathscr{x})B^b_k(\mathscr{x}))\tau_a\tau_b,$$

where we used $R^a_{jk}(\mathscr{x})=-R^a_{kj}(\mathscr{x})$. Thus we have from (7.8) the following equation:

$$\partial_j R^a_{kl}(\mathscr{x})+\partial_k R^a_{lj}(\mathscr{x})+\partial_l R^a_{jk}(\mathscr{x})+f^a_{bc}(B^b_j(\mathscr{x})R^c_{kl}(\mathscr{x})+B^b_k(\mathscr{x})R^c_{lj}(\mathscr{x})+B^b_l(\mathscr{x})R^c_{jk}(\mathscr{x}))=0,$$

where we used (7.1). This equation is called the Yang-Milles equation.

As the simplest example, we consider the case of $GL(H)=SU(1)$ where H is the state space of the charged particle, for instance, the spinor space describing the state of electrons. Since the generator of this group $SU(1)$ is obviously the unit matrix, henceforth we omit it.

For $A(\mathscr{x})=e^{i\lambda(\mathscr{x})}\in SU(1)$, the transformation law (4.17) of connection forms reduces to

$$\omega'(\mathscr{x})=\omega(\mathscr{x})+id\lambda(\mathscr{x}).$$

This result shows that $\omega(\mathscr{x})$ may be chosen as follows:

$$\omega(\mathscr{x})=iA_j(\mathscr{x})d\mathscr{x}^j.$$

Moreover, the fields $A_j(\mathscr{x})$ $(j=1, 2, 3, 4)$ satisfy the transformation law

$$A'_j(\mathscr{x})=A_j(\mathscr{x})+\partial_j\lambda(\mathscr{x}). \tag{7.9}$$

These facts suggest that the fields $A_j(\mathscr{x})$ are the electromagnetic potential and that the transformation law (7.9) gives the gauge transformation. The covariant derivatives of $\psi(\mathscr{x})$ become to

$$\nabla_{\partial_j}\psi(\mathscr{x})=\partial_j\psi(\mathscr{x})+iA_i(\mathscr{x})\psi(\mathscr{x}).$$

This gives the electromagnetic interaction of the electron in quantum mechanics.

Next, we have immediately from the structure equation

$$R_{jk}(\mathscr{x})=\partial_j A_k(\mathscr{x})-\partial_k A_j(\mathscr{x}).$$

The fields $R_{jk}(\mathscr{x})$ are the electromagnetic field. Since the generator of $SU(1)$ is the unit matrix and $R(\mathscr{x})$ is the differential 2-form, we have

$$\omega(\mathscr{x})\wedge R(\mathscr{x})-R(\mathscr{x})\wedge\omega(\mathscr{x})=0.$$

Thus we obtain from the Bianchi's identity

$$\partial_j R_{kl}(\mathscr{x})+\partial_k R_{lj}(\mathscr{x})+\partial_l R_{jk}(\mathscr{x})=0.$$

This equation gives the second group of Maxwell equations.

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