ELEMENTARY PROOF OF CLARKSON'S INEQUALITIES AND THEIR GENERALIZATION

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Clarkson's inequalities, which are related to functions in L^p-spaces, are generalized. At the same time, we propose a direct and elementary way of proofs of these inequalities. It is also shown that estimates in these inequalities are best possible.

Key Words:Clarkson's inequalities, L^p-space

1. INTRODUCTION

Clarkson showed the following inequalities for functions in L^p $(1 < p < \infty)$ in order to prove that L^p-spaces are uniformly convex [1, 2].

Theorem (Clarkson's inequalities [1])

Let (Ω, μ) be a measure space, $1 < p < \infty$ and $p' = p/(p-1)$. Let u and v be in $L^p(\Omega)$. If $1 < p \leq 2$, then

(1.1)
$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \leq \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{\frac{p'}{p}},$$

(1.2)
$$\left\|\frac{u+v}{2}\right\|_p^{p} + \left\|\frac{u-v}{2}\right\|_p^{p} \geq \frac{1}{2}(\|u\|_p^p + \|v\|_p^p).$$

If $2 \leq p < \infty$, then

(1.3)
$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \geq \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{\frac{p'}{p}},$$

(1.4)
$$\left\|\frac{u+v}{2}\right\|_p^{p} + \left\|\frac{u-v}{2}\right\|_p^{p} \leq \frac{1}{2}(\|u\|_p^p + \|v\|_p^p).$$

The key of proof by Clarkson is the inequality below,

$$(1.5)\quad \left|\frac{1+t}{2}\right|^{p'} + \left|\frac{1-t}{2}\right|^{p'} \leq \left(\frac{1}{2} + \frac{1}{2}t^p\right)^{\frac{p'}{p}},$$

where $1 < p \leq 2$ and $0 \leq t \leq 1$.

In order to prove this, he uses the generalized binomial theorem. However, this part of the proof is not necessarily easy to understand. In authors' opinion it is better to use differentiation, since in doing so we can obtain more general results. Also, the proof of inequalities (1.1) - (1.4) is not so straightforward [1]. In this paper we shall give straightforward and elementary proofs of a generalized version of inequalities (1.1) - (1.5).

In §2 equalities (2.2) and (2.3) related to inequality (1.5) are proved. In §3 generalized Clarkson's inequalities (3.1) in the field $\mathbf{C}$ of complex numbers are given by using equalities (2.2) and (2.3). Theorem 3.4, in which the Clarkson's inequalities are extended to the case of any $1 < p, q < \infty$, is the main theorem of this paper. By using inequalities (3.1), we find the maximum value of k and minimum value of K such as

$$k\,(\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}}$$

$$
\begin{aligned}
&\le \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \\
&\le K(\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}}
\end{aligned}
$$

for u, v in $L^p(\Omega, \mu)$.

As an example of another proof of Clarkson's inequalities (1.2) and (1.4), there is a result of Hanner[3] in interval Ω=[0,1]. Hanner uses convexity of a function in Ω to prove these. Moreover, at the regular meeting of the Japanese Mathematical Society in the spring of 1993, Chinami WATARI (Touhokugakuin Univ.) proved the inequality (1.5) by using Riesz-Thorin interpolation theorem, and pointed out that the inequality (1.5) coincides with Housdorff-Young's inequality in the case of the group {0,1}.

2. MAXIMUM OF A KEY FUNCTION

If $1 < p \le 2$ and $p' = p/(p-1)$, then the following inequality is obtained from inequality (1.5),

$$\frac{\{(1+t)^q + (1-t)^q\}^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}} \le 2^{1-\frac{1}{p}}$$

for $0 \le t \le 1$, where we set $q = p'$. Let us introduce a function $f(t)$ as follows:

$$f(t) = \frac{\{(1+t)^q + (1-t)^q\}^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}} \tag{2.1}$$

for $0 \le t \le 1$, where $1 < p \le 2$ and $1 < q < \infty$. To estimate the function $f(t)$, we provide some lemmas.

Lemma 2.1 *Let* $0 < \alpha, \beta < 1$ *and*

$$g_1(t) = 1 - t^{2\alpha} - \alpha\beta t^{\alpha-1} + \alpha\beta t^{\alpha+1}$$

for $0 < t < 1$. *Then there exists* t_1 *in the interval* $(0,1)$ *such that* $g_1(t_1) = 0$, $g_1(t) < 0$ *for* $0 < t < t_1$, *and* $g_1(t) > 0$ *for* $t_1 < t < 1$.

PROOF. Differentiating $g_1(t)$,

$$g_1'(t) = \alpha t^{\alpha-2}\{(\alpha+1)\beta t^2 - 2t^{\alpha+1} + (1-\alpha)\beta\}.$$

We set

$$h(t) = (\alpha+1)\beta t^2 - 2t^{\alpha+1} + (1-\alpha)\beta.$$

Since $h(t) \to (1-\alpha)\beta > 0$ (as $t \to +0$), $h(t) \to 2(\beta-1) < 0$ (as $t \to 1-0$), and $h'(t) = 2(\alpha+1)(\beta t - t^\alpha) < 0$ in the interval $(0,1)$, there exists $0 < t_0 < 1$ such that $h(t_0) = 0, h(t) > 0$ for $0 < t < t_0$, and $h(t) < 0$ for $t_0 < t < 1$. Therefore $g_1'(t) > 0$ for $0 < t < t_0$ and $g_1'(t) < 0$ for $t_0 < t < 1$. We have $g_1(t) > 0$ in $[t_0, 1)$ because $g_1(t) \to 0$ (as $t \to 1-0$) and $g_1(t)$ is strictly decreasing in $[t_0, 1)$. From the fact $g_1(t_0) > 0$ and $g_1(t) \to -\infty$ (as $t \to +0$), it follows that there exists $0 < t_1 < t_0$ such that $g_1(t) = 0$. Since $g_1(t)$ is strictly increasing in $(0, t_0]$, we have $g_1(t) < 0$ for $0 < t < t_1$ and $g_1(t) > 0$ for $t_1 < t \le t_0$. Therefore we find the desired t_1. This completes the proof. □

Lemma 2.2 *Let* $0 < \alpha, \beta < 1$ *and*

$$
\begin{aligned}
&g_2(t) \\
&\quad = \log(1+t) + \beta\log(1-t^\alpha) - \log(1-t) \\
&\qquad -\beta\log(1+t^\alpha)
\end{aligned}
$$

for $0 < t < 1$. *Then the following hold:*
(1) $g_2(t) \to 0$ *(as* $t \to +0$*) and* $g_2(t) \to \infty$ *(as* $t \to 1-0$*),*
(2) *there existes* $0 < t_2 < 1$ *such that* $g_2(t)$ *is strictly decreasing in the interval* $(0, t_2)$ *and strictly increasing in the interval* $(t_2, 1)$.

PROOF. (1) From the fact that

$$\frac{(1-t^\alpha)^\beta}{1-t} = \left(\frac{1-t^\alpha}{1-t}\right)^\beta \frac{1}{(1-t)^{1-\beta}} \to \infty$$

as $t \to 1-0$, we can obtain $g_2(t) \to \infty$ (as $t \to 1-0$) easily. It is obvious that $g_2(t) \to 0$ (as $t \to +0$).
(2) Using the function $g_1(t)$ in Lemma 2.1, we have

$$g_2'(t) = \frac{2g_1(t)}{(1-t^2)(1-t^{2\alpha})} \quad .$$

By Lemma 2.1, there is $0 < t_2 < 1$ such that $g_1(t) < 0$ for $0 < t < t_2$, and $g_1(t) > 0$ for $t_2 < t < 1$. Since $(1-t^2)(1-t^{2\alpha})$ is positive for $0 < t < 1$, we have $g_2'(t) < 0$ for $0 < t < t_2$, and $g_2'(t) > 0$ for $t_2 < t < 1$. This completes the proof. □

Lemma 2.3 *Let* $1 < p < 2$, $q > 2$ *and*

$$g_3(t) = (1+t)^{q-1}(1-t^{p-1}) - (1-t)^{q-1}(1+t^{p-1})$$

for $0 < t < 1$. *Then there exists* $0 < t_3 < 1$ *such that* $g_3(t_3) = 0$, $g_3(t) < 0$ *for* $0 < t < t_3$, *and* $g_3(t) > 0$ *for* $t_3 < t < 1$.

PROOF. We remark that

$$(1+t)^{q-1}(1-t^{p-1}) < (1-t)^{q-1}(1+t^{p-1})$$

if and only if

$$
\begin{aligned}
&\log\{(1+t)^{q-1}(1-t^{p-1})\} \\
&\quad < \log\{(1-t)^{q-1}(1+t^{p-1})\} .
\end{aligned}
$$

So we set

$$\begin{aligned} g_4(t) & \\ = & (q-1)\log(1+t)+\log(1-t^{p-1}) \\ & -(q-1)\log(1-t)-\log(1+t^{p-1}) , \end{aligned}$$

and determine the sign of the function $g_4(t)$.

If we put $\alpha = p-1$ and $\beta = 1/(q-1)$ in Lemma 2.2, we have $g_4(t) = (1/\beta)g_2(t)$. By using Lemma 2.2, there is $0 < t_2 < 1$ such that $g_4(t)$ is strictly decreasing in the interval $(0, t_2)$ and strictly increasing in the interval $(t_2, 1)$. As $g_4(t) \to 0$ $(t \to +0)$ from (1) of Lemma 2.2 and $g_4(t)$ is strictly decreasing in the interval $(0, t_2)$, we get $g_4(t_2) < 0$. Since $g_4(t) \to \infty$ $(t \to 1-0)$ from (1) of Lemma 2.2 and $g_4(t_2) < 0$, there is $0 < t_3 < 1$ such that $g_4(t_3) = 0$. By considering the behavior of $g_4(t)$, it is evident that $g_4(t) < 0$ for $0 < t < t_3$ and $g_4(t) > 0$ for $t_3 < t < 1$. This completes the proof. □

Remark. $g_3(t) \to 0$ (as $t \to 1-0$), although $g_4(t) \to \infty$ (as $t \to 1-0$).

Lemma 2.4 *Let* $1 < p \leq 2$ *and* $1 < q \leq 2$ *except for* $p = q = 2$. *Let*

$$g_3(t) = (1+t)^{q-1}(1-t^{p-1}) - (1-t)^{q-1}(1+t^{p-1})$$

for $0 < t < 1$. *Then* $g_3(t) < 0$ *for* $0 < t < 1$.

PROOF. Assume that $1 < p < 2$ and $1 < q < 2$. From an inequality

$$\frac{1+a}{1-a}\,\frac{1-b}{1+b} < 1$$

for $0 < a < b$, we have

$$\frac{1+t}{1-t}\,\frac{1-t^{p-1}}{1+t^{p-1}} < 1 .$$

Therefore

$$\begin{aligned} & \frac{1+t}{1-t}\left(\frac{1-t^{p-1}}{1+t^{p-1}}\right)^{\frac{1}{q-1}} \\ = & \frac{1+t}{1-t}\,\frac{1-t^{p-1}}{1+t^{p-1}}\left(\frac{1-t^{p-1}}{1+t^{p-1}}\right)^{\frac{1}{q-1}-1} < 1 . \end{aligned}$$

So we have

$$\frac{(1+t)^{q-1}(1-t^{p-1})}{(1-t)^{q-1}(1+t^{p-1})} < 1$$

for $0 < t < 1$. From this inequality we get this lemma. For other cases of p and q, it is easily shown that $g_3(t) < 0$ for $0 < t < 1$. □

Under the knowledges about $g_3(t)$, we can achieve the maximum value of $f(t)$.

Theorem 2.5 *Let* $1 < p \leq 2$, $q > 1$ *and*

$$f(t) = \frac{\{(1+t)^q + (1-t)^q\}^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}$$

for $0 \leq t \leq 1$. *Let* $C(p,q)$ *be the maximum value of* $f(t)$. *Then*

(2.2)

$$\begin{aligned} C(p,q) &= \max\{f(0), f(1)\} \\ &= \begin{cases} 2^{1-\frac{1}{p}} & \text{if } q \geq p' \\ 2^{\frac{1}{q}} & \text{if } 1 < q < p' \end{cases} \end{aligned}$$

PROOF. The derivative $f'(t)$ is as follows:

$$f'(t) = \frac{\{(1+t)^q + (1-t)^q\}^{\frac{1}{q}-1}}{(1+t^p)^{1+\frac{1}{p}}} g_3(t) ,$$

where $g_3(t)$ is the function in Lemma 2.3 and Lemma 2.4. Noting that the sign of $f'(t)$ accords with the sign of $g_3(t)$, the following are easily obtained:

(1) if $1 < p < 2$ and $q > 2$,

$$C(p,q) = \max\{f(0), f(1)\}$$

from Lemma 2.3,

(2) if $1 < p \leq 2$ and $1 < q \leq 2$ except for $p = q = 2$, $C(p,q) = f(0)$ from Lemma 2.4,

(3) if $p = 2$ and $q > 2$, $C(2,q) = f(1)$ from $g_3(t) > 0$,

(4) noting that $f(t)$ is identical with constant $2^{\frac{1}{2}}$ for $p = q = 2$, it is trivial.

This completes the proof. □

Remark. If $p > 2$, we have opposite inequalities corresponding to those in Lemma 2.1–Lemma 2.4 in the similar fashion.

The following are proved by the same method used to prove Theorem 2.5.

Theorem 2.6 *Let* $p > 2$ *and* $f(t)$ *be the same function as in Theorem* 2.5. *Let* $c(p,q)$ *be the minimum value of* $f(t)$ *in* $[0,1]$. *Then*

(2.3)

$$\begin{aligned} c(p,q) &= \min\{f(0), f(1)\} \\ &= \begin{cases} 2^{1-\frac{1}{p}} & \text{if } 1 < q < p' \\ 2^{\frac{1}{q}} & \text{if } q \geq p' \end{cases} \end{aligned}$$

3. GENERALIZATION OF CLARKSON'S INEQUALITIES

In this section, we generalize Clarkson's inequalities.

The following lemma gives more detailed results than those obtained by setting $q = 2$ in Theorem 2.5 and Theorem 2.6.

Lemma 3.1 *Let $p > 1$ and*

$$h(t) = \frac{(1+t^2)^{\frac{1}{2}}}{(1+t^p)^{\frac{1}{p}}}$$

for $0 \leq t \leq 1$. Then the following hold:
(1) *if $1 < p < 2$, $h(t)$ is decreasing,*
(2) *if $p = 2$, $h(t) = 1$,*
(3) *if $p > 2$, $h(t)$ is increasing.*

PROOF. Differentiating $h(t)$,

$$h'(t) = \frac{(1+t^p)^{\frac{1}{p}-1}(1+t^2)^{-\frac{1}{2}}}{(1+t^p)^{\frac{2}{p}}}(t - t^{p-1}).$$

Let $1 < p < 2$. Then $h'(t) < 0$, because $t - t^{p-1} < 0$ for $0 < t < 1$. If $p > 2$, similarly we have $h'(t) > 0$. This completes the proof. □

Remark. Lemma 3.1 is easily generalized as follows:
Let $1 \leq p$, $1 \leq q$ and

$$h(t) = \frac{(1+t^q)^{\frac{1}{q}}}{(1+t^p)^{\frac{1}{p}}}$$

for $0 \leq t \leq 1$. Then $h(t)$ is decreasing if $p < q$, and $h(t)$ is increasing if $p > q$. From this fact, we have the following relations among l^p-norms in $\mathbf{C}^2$. If $1 \leq p \leq q$, then

$$\|x\|_q \leq \|x\|_p \leq 2^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$$

for x in $\mathbf{C}^2$.

Using Theorem 2.5, Theorem 2.6 and Lemma 3.1, the following theorem is shown in the similar fashion of Adams [1, Lemma 2.27].

Theorem 3.2 *Let $p, q > 1$,*

$$M(p,q) = \max\{2^{1-\frac{q}{2}-\frac{q}{p}}, 2^{1-q}, 2^{-\frac{q}{p}}\}$$

and

$$m(p,q) = \min\{2^{1-\frac{q}{2}-\frac{q}{p}}, 2^{1-q}, 2^{-\frac{q}{p}}\} .$$

Then

$$\begin{aligned} (3.1)\qquad & m(p,q)\,(|z|^p + |w|^p)^{\frac{q}{p}} \\ \leq\ & \left|\frac{z+w}{2}\right|^q + \left|\frac{z-w}{2}\right|^q \\ \leq\ & M(p,q)\,(|z|^p + |w|^p)^{\frac{q}{p}} \end{aligned}$$

for z, w in $\mathbf{C}$.

PROOF. If $z = 0$ or $w = 0$, the inequalities evidently hold. Let $z \neq 0$ and $w \neq 0$. We may assume $|z| \geq |w| > 0$ without loss of generality. If we set $w/z = re^{i\theta}$, then

$$\begin{aligned} & \left|\frac{z+w}{2}\right|^q + \left|\frac{z-w}{2}\right|^q \\ &= |z|^q \left(\left|\frac{1+re^{i\theta}}{2}\right|^q + \left|\frac{1-re^{i\theta}}{2}\right|^q\right), \end{aligned}$$

$$(|z|^p + |w|^p)^{\frac{q}{p}} = |z|^q (1+r^p)^{\frac{q}{p}}.$$

Let $g(\theta)$ be as follows:

$$\begin{aligned} g(\theta) &= \left|1+re^{i\theta}\right|^q + \left|1-re^{i\theta}\right|^q \\ &= (1+r^2+2r\cos\theta)^{\frac{q}{2}} \\ &\quad +(1+r^2-2r\cos\theta)^{\frac{q}{2}}. \end{aligned}$$

Since $g(\theta+\pi) = g(\theta) = g(\pi-\theta)$, we consider only in the interval $[0, \frac{\pi}{2}]$. Since

$$\begin{aligned} g'(\theta) &= -qr\sin\theta\,\left\{(1+r^2+2r\cos\theta)^{\frac{q}{2}-1}\right. \\ &\qquad \left. -(1+r^2-2r\cos\theta)^{\frac{q}{2}-1}\right\}, \end{aligned}$$

we have $g'(\theta) = 0$ if and only if $\theta = 0, \frac{\pi}{2}$.

We shall divide the proof of the inequalities in some cases.

(1) Let $1 < p \leq 2$ and $1 < q < 2$. Since $q/2 - 1 < 0$, then $g'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$. Thus $g(0) \leq g(\theta) \leq g(\frac{\pi}{2})$. Therefore

$$\begin{aligned} (3.2)\qquad & \left(\frac{1+r}{2}\right)^q + \left(\frac{1-r}{2}\right)^q \\ \leq\ & \left|\frac{1+re^{i\theta}}{2}\right|^q + \left|\frac{1-re^{i\theta}}{2}\right|^q \\ \leq\ & 2^{1-q}(1+r^2)^{\frac{q}{2}}. \end{aligned}$$

From Lemma 3.1, we have

$$(3.3)\qquad (1+r^2)^{\frac{q}{2}} \leq (1+r^p)^{\frac{q}{p}}.$$

Since the function f in Theorem 2.5 is decreasing from the proof of Theorem 2.5,

$$\frac{\{(1+r)^q + (1-r)^q\}^{\frac{1}{q}}}{(1+r^p)^{\frac{1}{p}}} = f(r) \geq f(1) = 2^{1-\frac{1}{p}}.$$

Hence

$$(3.4)\qquad \left(\frac{1+r}{2}\right)^q + \left(\frac{1-r}{2}\right)^q \geq 2^{-\frac{q}{p}}(1+r^p)^{\frac{q}{p}}.$$

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} 2^{-\frac{q}{p}}(1+r^p)^{\frac{q}{p}} &\leq \left|\frac{1+re^{i\theta}}{2}\right|^q + \left|\frac{1-re^{i\theta}}{2}\right|^q \\ &\leq 2^{1-q}(1+r^p)^{\frac{q}{p}}. \end{aligned}$$

We have $m(p,q) = 2^{-\frac{q}{p}}$ and $M(p,q) = 2^{1-q}$ by simple calculation. Therefore inequalities (3.1) are proved in this case.
(2) Let $1 < p \le 2$ and $2 < q$. Since $q/2 - 1 > 0$, then $g'(\theta) < 0$ for $0 < \theta < \frac{\pi}{2}$. Thus $g(\frac{\pi}{2}) \le g(\theta) \le g(0)$. Therefore

(3.5)
$$\begin{aligned} 2^{1-q}(1+r^2)^{\frac{q}{2}} &\le \left|\frac{1+re^{i\theta}}{2}\right|^q + \left|\frac{1-re^{i\theta}}{2}\right|^q \\ &\le \left(\frac{1+r}{2}\right)^q + \left(\frac{1-r}{2}\right)^q. \end{aligned}$$

From Lemma 3.1,

$$(1+r^2)^{\frac{q}{2}} \ge 2^{\frac{q}{2}-\frac{q}{p}}(1+r^p)^{\frac{q}{p}}. \tag{3.6}$$

Let us put $A(p,q)$ as

$$A(p,q) = \begin{cases} 2^{1-\frac{1}{p}} & \text{if} \quad q \ge p' \\ 2^{\frac{1}{q}} & \text{if} \quad 2 < q < p' \end{cases}$$

From Theorem 2.5, we have

$$\frac{\{(1+r)^q + (1-r)^q\}^{\frac{1}{q}}}{(1+r^p)^{\frac{1}{p}}} \le A(p,q).$$

Therefore

(3.7)
$$\left(\frac{1+r}{2}\right)^q + \left(\frac{1-r}{2}\right)^q \le \left\{\frac{A(p,q)}{2}\right\}^q (1+r^p)^{\frac{q}{p}}.$$

Calculating $m(p,q)$ and $M(p,q)$ in this case, it follows from (3.5), (3.6) and (3.7) that inequalities (3.1) hold.

Since inequlities (3.1) are proved similarly for other cases, we omit the proof. □

Remark. It is easily shown that $m(p,q)$ and $M(p,q)$ in inequalities (3.1) are the best possible estimations.

We note the following inequalities which are well known and used in the proof of the next theorem.

Lemma 3.3 *Let $p > 0$. Then*

$$c(a+b)^p \le a^p + b^p \le C(a+b)^p$$

for $a, b > 0$, where $c = \min\{1, 2^{1-p}\}$ and $C = \max\{1, 2^{1-p}\}$.

Now we can get generalized Clarkson's inequalities as follows.

Theorem 3.4 *Let (Ω, μ) be a measure space and $p, q > 1$. Let*

$$K(p,q) = \max\{2^{1-2\frac{q}{p}}, 2^{1-q}, 2^{-\frac{q}{p}}\}$$

and

$$k(p,q) = \min\{2^{1-2\frac{q}{p}}, 2^{1-q}, 2^{-\frac{q}{p}}\}\ .$$

Then

(3.8)
$$\begin{aligned} & k(p,q)\ (\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}} \\ \le\ & \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \\ \le\ & K(p,q)\ (\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}} \end{aligned}$$

for u, v in $L^p(\Omega)$.

Proof. Inequalities (3.8) are proved similarly to those in Adams [1.Theorem 2.28].

We remark that

(3.9)
$$\begin{aligned} & \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \\ =\ & \left\| \left|\frac{u+v}{2}\right|^q \right\|_{\frac{p}{q}} + \left\| \left|\frac{u-v}{2}\right|^q \right\|_{\frac{p}{q}} \end{aligned}$$

for u, v in $L^p(\Omega)$.
(1) Let $1 < p \le 2$.
(i) Let $1 < q \le p$. Combining equality (3.9), Minkowski's inequality with $p/q \ge 1$ and Theorem 3.2, we have

$$\begin{aligned} & \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \\ \ge\ & \left\| \left|\frac{u+v}{2}\right|^q + \left|\frac{u-v}{2}\right|^q \right\|_{\frac{p}{q}} \\ \ge\ & \left\{\int_\Omega m(p,q)^{\frac{p}{q}}(|u|^p + |v|^p)d\mu(x)\right\}^{\frac{q}{p}} \\ \ge\ & 2^{-\frac{q}{p}}\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}}, \end{aligned}$$

where we used $m(p,q) = 2^{-\frac{q}{p}}$ for $1 < p < 2$, $1 < q \le p$. Therefore the left-hand side of inequalities (3.8) is proved. From Theorem 3.2 and Lemma 3.3, we have

$$\begin{aligned} & \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \\ =\ & \left(\left\|\frac{u+v}{2}\right\|_p^p\right)^{\frac{q}{p}} + \left(\left\|\frac{u-v}{2}\right\|_p^p\right)^{\frac{q}{p}} \\ \le\ & 2^{1-\frac{q}{p}}\left(\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p\right)^{\frac{q}{p}} \\ \le\ & 2^{1-\frac{q}{p}}M(p,p)^{\frac{q}{p}}\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}} \\ =\ & 2^{1-q}\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}}. \end{aligned}$$

Therefore the right-hand side of inequalities (3.8) is proved.

(ii) Let $p < q$. Combining equality (3.9), reverse Minkowski's inequality with $p/q < 1$ and Theorem 3.2, we have

$$\left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q \le \left\| \left|\frac{u+v}{2}\right|^q + \left|\frac{u-v}{2}\right|^q \right\|_{\frac{p}{q}} \le M(p,q)(\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}}.$$

It is obvious $M(p,q) = 2^{1-q} = K(p,q)$ for $p < q \le p'$, $1 < p \le 2$ and $M(p,q) = 2^{-\frac{q}{p}} = K(p,q)$ for $p' < q$, $1 < p \le 2$. Therefore the right-hand side of inequalities (3.8) is proved. From Theorem 3.2 and Lemma 3.3, we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q &= \left(\left\|\frac{u+v}{2}\right\|_p^p\right)^{\frac{q}{p}} + \left(\left\|\frac{u-v}{2}\right\|_p^p\right)^{\frac{q}{p}} \\ &\ge 2^{1-\frac{q}{p}} \left(\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p\right)^{\frac{q}{p}} \\ &\ge 2^{1-\frac{q}{p}} m(p,p)^{\frac{q}{p}} \left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}} \\ &= 2^{1-2\frac{q}{p}} \left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}}.\end{aligned}$$

Since $c(p,q) = 2^{1-2\frac{q}{p}}$ for $p < q$ and $1 < p \le 2$, the left-hand side of inequalities (3.8) is proved.
(2) Inequalities (3.8) can also be proved similarly in the case $2 < p$. Hence we omit the proof. □

Corollary 3.5 *Let (Ω, μ) be a measure space, $1 < p < \infty$ and $p' = p/(p-1)$. Let $A(p) = \max\{2^{-1}, 2^{1-p}\}$, $a(p) = \min\{2^{-1}, 2^{1-p}\}$, $B(p) = \max\{2^{1-2\frac{p'}{p}}, 2^{-\frac{p'}{p}}\}$, and $b(p) = \min\{2^{1-2\frac{p'}{p}}, 2^{-\frac{p'}{p}}\}$. Then*

$$a(p)\left(\|u\|_p^p + \|v\|_p^p\right) \le \left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le A(p)\left(\|u\|_p^p + \|v\|_p^p\right),$$

$$b(p)\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{p'}{p}} \le \left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \le B(p)\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{p'}{p}}$$

for u, v in $L^p(\Omega)$.

Remark. If a measure space (Ω, μ) is trivial, that is, Ω is composed of one element and $\mu(\Omega) = 1$, then $L^p(\Omega)$ is isomorphic to the field $\mathbf{C}$ of complex numbers as Banach space. In this case, inequalities (3.8) are not necessarily the best possible estimation from Theorem 3.1. But if (Ω, μ) is a "usual" measure space, that is, there are measurable sets A, B of Ω satisfying $A \cap B = \phi$ and $0 < \mu(A), \mu(B) < \infty$, then inequalities (3.8) are the best possible estimations. To show this, noting $k(p,q) = 2^{1-2\frac{q}{p}}$ for $1 < p < 2$ and $2 < q$, we set

$$u(x) = \begin{cases} \mu(A)^{-\frac{1}{p}} & \text{if } x \in A \\ \mu(B)^{-\frac{1}{p}} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = \begin{cases} \mu(A)^{-\frac{1}{p}} & \text{if } x \in A \\ -\mu(B)^{-\frac{1}{p}} & \text{if } x \in B \\ 0 & \text{otherwise .} \end{cases}$$

Then the equality below holds:

$$2^{1-\frac{2q}{p}}\left(\|u\|_p^p + \|v\|_p^p\right)^{\frac{q}{p}} = \left\|\frac{u+v}{2}\right\|_p^q + \left\|\frac{u-v}{2}\right\|_p^q .$$

We can also show similarly that inequalities (3.8) are the best possible estimations in other caces.

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クラークソンの不等式の初等的な証明とその拡張

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関数空間 L^p に属する関数に対するクラークソンの不等式を拡張する。同時に、この不等式を初等的に証明し、この不等式の評価が最善であることを示す。