A Generalized Skew Information and Uncertainty Relation

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***Abstract*—A generalized skew information is defined and a generalized uncertainty relation is established with the help of a trace inequality which was recently proven by Fujii. In addition, we prove the trace inequality conjectured by Luo and Zhang. Finally, we point out that Theorem 1 in S. Luo and Q. Zhang, *IEEE Trans. Inf. Theory*, vol. 50, pp. 1778–1782, no. 8, Aug. 2004 is incorrect in general, by giving a simple counter-example.**

***Index Terms*—Skew information, trace inequalities and uncertainty relation.**

I. Introduction

As one of the mathematical studies on entropy, the skew entropy [14], [15] and the problem of its concavity are famous. The concavity problem for the skew entropy generalized by Dyson, was solved by Lieb in [9]. It is also known that the skew entropy represents the degree of noncommutativity between a certain quantum state represented by the density matrix ρ (which is a positive semidefinite matrix with unit trace) and an observable represented by the selfadjoint matrix X. Quite recently, S. Luo and Q. Zhang studied the relation between skew information (which is equal to the opposite signed skew entropy) and the uncertainty relation in [10]. Inspired by their interesting work, we define a generalized skew information and then study the relationship between it and the uncertainty relation. In addition, we prove the trace inequality conjectured in [11].

II. Preliminaries

Let f and g be functions on the domain $D \subset \boldsymbol{R}$. (f,g) is called a monotonic pair if $(f(a)-f(b))(g(a)-g(b)) \geq 0$ for all $a, b \in D$. (f,g) is also called an antimonotonic pair if $(f(a)-f(b))(g(a)-g(b)) \leq 0$ for all $a, b \in D$.

In what follows we consider selfadjoint matrices whose spectra are included in D so that functional calculus makes sense.

Lemma II.1 ([1], [2]): For any selfadjoint matrices A and X, we have the following trace inequalities.

1) If (f,g) is a monotonic pair, then

$$\mathrm{Tr}(f(A)Xg(A)X) \leq \mathrm{Tr}(f(A)g(A)X^2).$$

2) If (f,g) is an antimonotonic pair, then

$$\mathrm{Tr}(f(A)Xg(A)X) \geq \mathrm{Tr}(f(A)g(A)X^2).$$

From this lemma, we can obtain the following lemma.

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Lemma II.2: For any selfadjoint matrices A and B, and any matrix X, we have the following trace inequalities.

1) If (f, g) is a monotonic pair, then

$$\mathrm{Tr}(f(A)X^*g(B)X + f(B)Xg(A)X^*) \leq \mathrm{Tr}(f(A)g(A)X^*X + f(B)g(B)XX^*).$$

2) If (f, g) is an antimonotonic pair, then

$$\mathrm{Tr}(f(A)X^*g(B)X + f(B)Xg(A)X^*) \geq \mathrm{Tr}(f(A)g(A)X^*X + f(B)g(B)XX^*).$$

Proof: Define on $\mathcal{H} \oplus \mathcal{H}$

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$$

where A, B and X act on a finite-dimensional Hilbert space $\mathcal{H}$. Then $\hat{A}$ and $\hat{X}$ are selfadjoint. Therefore, one may apply Lemma II.1 to get

$$\begin{aligned}
&\mathrm{Tr}(f(A)X^*g(B)X + f(B)Xg(A)X^*) \\
&= \mathrm{Tr}\left(\begin{pmatrix} f(A) & 0 \\ 0 & f(B) \end{pmatrix}\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}\begin{pmatrix} g(A) & 0 \\ 0 & g(B) \end{pmatrix}\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}\right) \\
&= \mathrm{Tr}(f(\hat{A})\hat{X}g(\hat{A})\hat{X}) \\
&\leq \mathrm{Tr}(f(\hat{A})g(\hat{A})\hat{X}^2) \\
&= \mathrm{Tr}\left(\begin{pmatrix} f(A) & 0 \\ 0 & f(B) \end{pmatrix}\begin{pmatrix} g(A) & 0 \\ 0 & g(B) \end{pmatrix}\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}\right) \\
&= \mathrm{Tr}(f(A)g(A)X^*X + f(B)g(B)XX^*)
\end{aligned}$$

which is (1). Inequality (2) is proven in a similar way. □

III. Generalized Uncertainty Relation

For a density matrix (quantum state) ρ and arbitrary matrices X and Y acting on $\mathcal{H}$, we denote $\tilde{X} \equiv X - \mathrm{Tr}(\rho X)I$ and $\tilde{Y} \equiv Y - \mathrm{Tr}(\rho Y)I$, where I represents the identity matrix. Then we define the covariance by $\mathrm{Cov}_\rho(X, Y) = \mathrm{Tr}(\rho\tilde{X}\tilde{Y})$. Each variance is defined by $V_\rho(X) \equiv \mathrm{Cov}_\rho(X, X)$ and $V_\rho(Y) \equiv \mathrm{Cov}_\rho(Y, Y)$.

The famous Heisenberg's uncertainty relation [6], [12] can be easily proven by the application of the Schwarz inequality and it was generalized by Schrödinger as follows:

Propostion III.1 (Schrödinger [13]): For any density matrix ρ and any two selfadjoint matrices A and B, we have the uncertainty relation

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A, B))|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2 \quad (1)$$

where $[X, Y] \equiv XY - YX$.

Definition III.2: For arbitrary matrices X and Y, we define

$$I_p(\rho; X, Y) \equiv \mathrm{Tr}(\rho XY) - \mathrm{Tr}\left(\rho^{\frac{1}{p}} X \rho^{\frac{1}{p^*}} Y\right)$$

where $p \in [1, +\infty]$ and with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$. If A is selfadjoint, the Wigner-Yanase-Dyson information is defined by

$$\begin{aligned}
I_p(\rho; A) \equiv I_p(\rho; A, A) &= \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{\frac{1}{p}} A \rho^{\frac{1}{p^*}} A) \\
&= -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{p}}, A][\rho^{\frac{1}{p^*}}, A]).
\end{aligned}$$

We use the parameters p and p^*, since many papers [3]–[5], [7] in this field use such notations. The Wigner–Yanase skew information is

$$\begin{aligned}
I(\rho; A) \equiv I_2(\rho; A) &= \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} A) \\
&= -\frac{1}{2}\mathrm{Tr}([\rho^{\frac{1}{2}}, A]^2).
\end{aligned}$$

An interpretation of skew information as a measure of quantum uncertainty is given in [10]. They claimed the following uncertainty relation

$$I(\rho, A)I(\rho, B) - |\mathrm{Re}(\mathrm{Corr}_\rho(A, B))|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2 \quad (2)$$

for two selfadjoint matrices A and B, and density matrix ρ, where their correlation measure was defined by

$$\mathrm{Corr}_\rho(A, B) \equiv \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^{1/2} A \rho^{1/2} B).$$

However, we show (2) does not hold in general. We give a counterexample for (2) in Section IV.

We define the generalized skew correlation and the generalized skew information as follows.

Definition III.3: For arbitrary X and Y, $p \in [1, +\infty]$ with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\varepsilon \geq 0$, set

$$\phi_{p,\varepsilon}(\rho; X, Y) \equiv \varepsilon \mathrm{Cov}_\rho(X^*, Y) + \frac{1}{2}I_p(\rho; \tilde{X^*}, \tilde{Y}) + \frac{1}{2}I_p(\rho; \tilde{Y}, \tilde{X^*}).$$

If A and B are selfadjoint, the generalized skew correlation is defined by

$$\mathrm{Corr}_{p,\varepsilon}(\rho; A, B) \equiv \phi_{p,\varepsilon}(\rho; A, B).$$

The generalized skew information is defined by

$$I_{p,\varepsilon}(\rho; A) \equiv \mathrm{Corr}_{p,\varepsilon}(\rho; A, A) = \varepsilon V_\rho(A) + I_p(\rho; \tilde{A})$$

so that

$$I_{p,0}(\rho; A) = I_p(\rho; \tilde{A}) = V_\rho(A) - \mathrm{Tr}(\rho^{\frac{1}{p}} \tilde{A} \rho^{\frac{1}{p^*}} \tilde{A}).$$

Then we have the following theorem.

Theorem III.4: For any two selfadjoint matrices A and B, any density matrix ρ, any $p \in [1, +\infty]$ with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\varepsilon \geq 0$, we have a generalized uncertainty relation

$$I_{p,\varepsilon}(\rho; A)I_{p,\varepsilon}(\rho; B) - |\mathrm{Re}(\mathrm{Corr}_{p,\varepsilon}(\rho; A, B))|^2 \geq \frac{\varepsilon^2}{4}|\mathrm{Tr}(\rho[A, B])|^2.$$

Proof: By Lemma II.2, $\phi_{p,\varepsilon}(\rho; X, X) \geq 0$. Furthermore it is clear that $\phi_{p,\varepsilon}(\rho; X, Y)$ is sesquilinear and Hermitian. Then we have

$$|\phi_{p,\varepsilon}(\rho; X, Y)^2 \leq \phi_{p,\varepsilon}(\rho; X, X)\phi_{p,\varepsilon}(\rho; Y, Y)$$

by the Schwarz inequality. It follows that

$$|\mathrm{Corr}_{p,\varepsilon}(\rho; A, B)|^2 \leq \mathrm{Corr}_{p,\varepsilon}(\rho; A, A)\mathrm{Corr}_{p,\varepsilon}(\rho; B, B)$$

for any two selfadjoint matrices A and B. Then

$$|\mathrm{Corr}_{p,\varepsilon}(\rho; A, B)|^2 \leq I_{p,\varepsilon}(\rho; A) I_{p,\varepsilon}(\rho; B). \tag{3}$$

Simple calculations imply

$$\mathrm{Corr}_{p,\varepsilon}(\rho; A, B) - \mathrm{Corr}_{p,\varepsilon}(\rho; B, A) = \varepsilon \mathrm{Tr}(\rho[\tilde{A}, \tilde{B}]) = \varepsilon \mathrm{Tr}(\rho[A, B]) \tag{4}$$

$$\mathrm{Corr}_{p,\varepsilon}(\rho; A, B) + \mathrm{Corr}_{p,\varepsilon}(\rho; B, A) = 2\mathrm{Re}(\mathrm{Corr}_{p,\varepsilon}(\rho; A, B)). \tag{5}$$

Summing both sides in the above two equalities, we have

$$2\mathrm{Corr}_{p,\varepsilon}(\rho; A, B) = \varepsilon \mathrm{Tr}(\rho[A, B]) + 2\mathrm{Re}(\mathrm{Corr}_{p,\varepsilon}(\rho; A, B)). \tag{6}$$

Since $[A, B]$ is skew-adjoint, $\mathrm{Tr}(\rho[A, B])$ is a purely imaginary number, we have

$$|\mathrm{Corr}_{p,\varepsilon}(\rho; A, B)|^2 = \frac{\varepsilon^2}{4}|\mathrm{Tr}(\rho[A, B])|^2 + |\mathrm{Re}(\mathrm{Corr}_{p,\varepsilon}(\rho; A, B))|^2. \tag{7}$$

Thus the proof of the theorem is completed by the use of (3) and (7). □

We are interested in the relationship between the left-hand sides in Proposition III.1 and Theorem III.4. The following proposition gives the relationship.

Proposition III.5: For any two selfadjoint matrices A and B, any density matrix ρ, any $p \in [1, +\infty]$ with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\varepsilon \geq 0$, we have

$$I_{p,\varepsilon}(\rho; A) I_{p,\varepsilon}(\rho; B) - |\mathrm{Re}(\mathrm{Corr}_{p,\varepsilon}(\rho; A, B))|^2 \geq \varepsilon^2 V_\rho(A) V_\rho(B) - \varepsilon^2 |\mathrm{Re}(\mathrm{Cov}_\rho(A, B))|^2.$$

Proof: From Proposition III.1, we have

$$V_\rho(A) V_\rho(B) \geq |\mathrm{Re}(\mathrm{Cov}_\rho(A, B))|^2$$

that is,

$$|\mathrm{Re}(\mathrm{Tr}(\rho \tilde{A}\tilde{B}))|^2 \leq \mathrm{Tr}(\rho \tilde{A}^2)\mathrm{Tr}(\rho \tilde{B}^2). \tag{8}$$

By putting $\varepsilon = 0$ in (3), we have

$$|\mathrm{Corr}_{p,0}(\rho; A, B)|^2 \leq I_{p,0}(\rho; A) I_{p,0}(\rho; B).$$

It follows from (4) and (5) that

$$\mathrm{Corr}_{p,0}(\rho; A, B) = \mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B)).$$

Thus,

$$|\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B))|^2 \leq I_{p,0}(\rho; A) I_{p,0}(\rho; B). \tag{9}$$

Using (8), (9) and direct calculations, we get

$$\begin{aligned}
&\mathrm{L.H.S.} - \mathrm{R.H.S.} \\
&\quad = \varepsilon \mathrm{Tr}(\rho \tilde{A}^2) I_{p,0}(\rho; B) + \varepsilon \mathrm{Tr}(\rho \tilde{B}^2) I_{p,0}(\rho; A) \\
&\qquad - 2\varepsilon \mathrm{Re}(\mathrm{Tr}(\rho \tilde{A}\tilde{B}))\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B)) \\
&\qquad + I_{p,0}(\rho; A) I_{p,0}(\rho; B) - \{\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B))\}^2 \\
&\quad \geq \varepsilon \mathrm{Tr}(\rho \tilde{A}^2) I_{p,0}(\rho; B) + \varepsilon \mathrm{Tr}(\rho \tilde{B}^2) I_{p,0}(\rho; A) \\
&\qquad - 2\varepsilon \mathrm{Re}(\mathrm{Tr}(\rho \tilde{A}\tilde{B}))\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B)) \\
&\quad \geq \varepsilon \mathrm{Tr}(\rho \tilde{A}^2) I_{p,0}(\rho; B) + \varepsilon \mathrm{Tr}(\rho \tilde{B}^2) I_{p,0}(\rho; A) \\
&\qquad - 2\varepsilon \sqrt{\mathrm{Tr}(\rho \tilde{A}^2)\mathrm{Tr}(\rho \tilde{B}^2)}\sqrt{I_{p,0}(\rho; A) I_{p,0}(\rho; B)} \\
&\quad = \varepsilon \{\sqrt{\mathrm{Tr}(\rho \tilde{A}^2) I_{p,0}(\rho; B)} - \sqrt{\mathrm{Tr}(\rho \tilde{B}^2) I_{p,0}(\rho; A)}\}^2 \\
&\quad \geq 0.
\end{aligned}$$

□

Remark III.6: Theorem III.4 can be also proven by Proposition III.1 and Proposition III.5.

IV. An Inequality Related to the Uncertainty Relation

The trace inequality

$$V_\rho(A) V_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A, B))|^2 \geq I_{2,0}(\rho; A) I_{2,0}(\rho; B) - |\mathrm{Re}(\mathrm{Corr}_{2,0}(\rho; A, B))|^2.$$

was conjectured in [11] and proven in [10]. As a generalization of [10, Theorem 2], we prove a one-parameter extention of the above inequality.

Proposition IV.1: For any two selfadjoint matrices A and B, any density matrix ρ and any $p \in [1, +\infty]$ with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$, we have

$$V_\rho(A) V_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A, B))|^2 \geq I_{p,0}(\rho; A) I_{p,0}(\rho; B) - |\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B))|^2. \tag{10}$$

Proof: Let $\{\varphi_i\}$ be a complete orthonormal basis composed by eigenvectors of ρ. Then we calculate

$$\mathrm{Tr}(\rho^{\frac{1}{p}} \tilde{A} \rho^{\frac{1}{p^*}} \tilde{A}) = \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} a_{ij} a_{ji}$$

where $a_{ij} \equiv \langle \tilde{A}\varphi_i | \varphi_j \rangle$ and $a_{ji} \equiv \overline{a_{ij}}$. Thus, we get

$$I_{p,0}(\rho; A) = V_\rho(A) - \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} a_{ij} a_{ji}$$

$$I_{p,0}(\rho; B) = V_\rho(B) - \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} b_{ij} b_{ji}$$

where $b_{ij} \equiv \langle \tilde{B}\varphi_i | \varphi_j \rangle$ and $b_{ji} \equiv \overline{b_{ij}}$. In a similar way, we obtain

$$\begin{aligned}\mathrm{Re}(\mathrm{Corr}_{p,0}(\rho; A, B)) &= \mathrm{Re}(\mathrm{Cov}_\rho(A, B)) \\ &- \frac{1}{2}\sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(a_{ij} b_{ji}) \\ &- \frac{1}{2}\sum_{j,i} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(b_{ij} a_{ji}).\end{aligned}$$

In order to prove the present proposition, we have only to show the inequality $\xi \geq \eta$, where

$$\begin{aligned}\xi \equiv{}& V_\rho(A) \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} b_{ij} b_{ji} + V_\rho(B) \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} a_{ij} a_{ji} \\ &- \left(\sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} a_{ij} a_{ji}\right)\left(\sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} b_{ij} b_{ji}\right), \\ \eta \equiv{}& \mathrm{Re}(\mathrm{Cov}_\rho(A, B)) \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(a_{ij} b_{ji}) \\ &+ \mathrm{Re}(\mathrm{Cov}_\rho(A, B)) \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(b_{ij} a_{ji}) \\ &- \frac{1}{4}\left(\sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(a_{ij} b_{ji}) + \sum_{i,j} \lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \mathrm{Re}(b_{ij} a_{ji})\right)^2.\end{aligned}$$

Since

$$\begin{aligned}V_\rho(A) &= \mathrm{Tr}(\rho \tilde{A}^2) = \frac{1}{2}\sum_{i,j} (\lambda_i + \lambda_j) a_{ij} a_{ji} \\ V_\rho(B) &= \mathrm{Tr}(\rho \tilde{B}^2) = \frac{1}{2}\sum_{i,j} (\lambda_i + \lambda_j) b_{ij} b_{ji}\end{aligned}$$

and

$$(\lambda_i + \lambda_j)\lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}} + (\lambda_k + \lambda_l)\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} - 2\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}} \geq 0$$

we calculate

$$\begin{aligned}\xi ={}& \frac{1}{4}\sum_{i,j,k,l} \Big\{(\lambda_i + \lambda_j)\lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}} + (\lambda_k + \lambda_l)\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \\ &\qquad -2\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}}\Big\}(a_{ij} a_{ji} b_{kl} b_{lk} + b_{ij} b_{ji} a_{kl} a_{lk}) \\ \geq{}& \frac{1}{2}\sum_{i,j,k,l} \Big\{(\lambda_i + \lambda_j)\lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}} + (\lambda_k + \lambda_l)\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \\ &\qquad -2\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}}\Big\} |a_{ij} b_{ji}||a_{kl} b_{lk}|. \end{aligned} \tag{11}$$

Since $\mathrm{Re}(b_{kl} a_{lk}) = \mathrm{Re}(\overline{b_{lk}}\,\overline{a_{kl}}) = \mathrm{Re}(b_{lk} a_{kl}) = \mathrm{Re}(a_{kl} b_{lk})$, $\mathrm{Re}(b_{ij} a_{ji}) = \mathrm{Re}(a_{ij} b_{ji})$, we calculate

$$\begin{aligned}\eta ={}& \frac{1}{2}\sum_{i,j,k,l} \Big\{(\lambda_i + \lambda_j)\lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}} + (\lambda_k + \lambda_l)\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \\ &\qquad -2\lambda_i^{\frac{1}{p}} \lambda_j^{\frac{1}{p^*}} \lambda_k^{\frac{1}{p}} \lambda_l^{\frac{1}{p^*}}\Big\} \mathrm{Re}(a_{ij} b_{ji}) \mathrm{Re}(a_{kl} b_{lk}).\end{aligned}$$

Thus, we conclude $\xi \geq \eta$, since

$$|a_{ij} b_{ji}||a_{kl} b_{lk}| \geq |\mathrm{Re}(a_{ij} b_{ji}) \mathrm{Re}(a_{kl} b_{lk})|. \qquad \square$$

Inequality (10) was independently proven in [8]. Our proof is simpler than Kosaki's one.

As a concluding remark, we point out that [10, Theorem 1] is incorrect in general.

Remark IV.2: Reference [10, Theorem 1] is not true in general. A counterexample is given as follows. Let

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have, $I(\rho, A) I(\rho, B) - |\mathrm{Re}(\mathrm{Corr}_\rho(A, B))|^2 = \frac{7 - 4\sqrt{3}}{4}$ and $|\mathrm{Tr}(\rho[A, B])|^2 = 1$. These imply

$$I(\rho, A) I(\rho, B) - |\mathrm{Re}(\mathrm{Corr}_\rho(A, B))|^2 < \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2.$$

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