On the connection mapping of the induced (Cl)-pair connections on homogeneous spaces

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Summary: Connection mappings of induced (Cl)-pair connections are studied to get $\tilde{\Gamma}$-distribution of horizontal spaces of the (Cl)-connection.

1. Introduction

To formulate the Finsler geometry without Finsler metrics, M. Matsumoto and T. Okada have developed the theory of pair connections[3)4)], and they get several results about linear[2)] and affine[1)] transformations on Finsler manifolds. The present author has studied this connection on homogeneous spaces, especially under the condition that it is invariant under tangential transformation groups induced from transformation groups on the manifolds.

In this paper we want to study an induced (Cl)-connection on a homogeneous space P/G, where G has a closed subgroup H.

The author wishes to express his hearty thanks to Prof. M. Matsumoto for his encouragements.

2. Preliminaries[7)]

We use notations as in the previous paper. Let P be a Lie group, and G its closed subgroup, and H, closed subgroup of G. Following K. Nomizu[6)] a homogeneous space P/G is called reductive if the following relations hold about Lie algebras: $\mathfrak{P}=\mathfrak{G}+\mathfrak{F}$, $\mathrm{ad}\,(G)\mathfrak{F}=\mathfrak{F}$ where Germann letters represent corresponding Lie algebras. In this case, we have $[\mathfrak{G}, \mathfrak{G}]\subset\mathfrak{G}$, $[\mathfrak{G}, \mathfrak{F}]\subset\mathfrak{F}$. Now from relations $P\supset G\supset H$ we have principal bundles $P(P/G, G)$ $P(P/H, H)$ $G(G/H, H)$

Let $T(P)$, $T(G)$ etc be tangent bundle spaces, P_e be the tangent space of P at $e\in P$, then $P_e\cong\mathfrak{P}$ and we can identify $T(P)$ with $P\times\mathfrak{P}$. The projections $T(P)\to P$, and $T(P)\to\mathfrak{P}$ are denoted with τ, σ respectively. Following S. Kobayashi we can introduce in $P\times\mathfrak{P}$ the product as follows; $(p_1, x_1)\cdot(p_2, x_2)=(p_1 p_2, \mathrm{ad}(p_2^{-1})x_1+x_2)$ and $P\times\mathfrak{P}$ is made into a Lie group.

Left and right translations with respect to this product are denoted with $L_{(p,x)}$ and $R_{(g,U)}$ respectively.

The bundle structure $P(P/G, G)$ induces on its tangent bundle spaces, the bundle structure with differentials of their mappings, called tangential bundle structure. We shall denote differential mappings of mappings with the same symbols, if there seems to be no confusions.

The Lie algebra of $T(P)$ is identified with $\mathfrak{P}+\mathfrak{P}$, and adjoint transformation of a group $T(G)$ is given by $\mathrm{ad}\,(g, U)(A, B)=(\mathrm{ad}\,(g)A, \mathrm{ad}\,(g)\,B-\mathrm{ad}\,(g)\cdot[A, U])$ for $(g, U)\in G\times\mathfrak{G}$ $(A, B)\in\mathfrak{P}+\mathfrak{P}$.

Pair connections on $T(P)$ $(T(P/G), T(G))$ are defined as follows:

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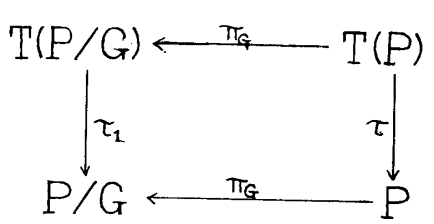
(a) $T(P)_{(p.X)}=T(P)^{v}_{(p.X)}+\Gamma_{(p.x)}$ (direct sum)
(b) $R_{(g.U)}\,\Gamma_{(p.X)}=\Gamma_{(p.X)(g.U)}$
(c) $\Gamma_{(p.X)}$ depends C^{∞}-ly on (p, X)
(d) $\Gamma_{(p.X)}=\Gamma^{v}_{(p.X)}+\Gamma^{h}_{(p.X)}$ here $\Gamma^{v}_{(p.X)}$ covers vertical spaces of $T(P/G)$
(e) $R_{(g.U)}\Gamma^{h}_{(p.X)}=\Gamma^{h}_{(p.X)(g.U)}$
(f) $\Gamma^{h}_{(p.X)}$ depends C^{∞}-ly on (p, X).

Projections of distribution $\Gamma^{h}_{(p.X)}$ on $T(P/G)$ evidently give non-linear connection on $T(P/G)$ and there arises C^{∞}-ditsributions Λ of horizontal subspaces. Conversely, given non-linear connection Λ and linear connection Γ on $T(P/G)$ and $T(P)$ respectively, we have a pair connection.

3. (Cl)-connection on $T(P)$ $(T(P/G), T(G))$



Suppose pair connection Γ, Λ is given on $T(P)$, $T(P/G)$ we say (Cl)-connection is defined in the diagram if the distribution Γ satisfies the following condition.

(Cl) $$L_{(p.X)}\Gamma_{(q.Y)}=\Gamma_{(p.X)(q.Y)}$$

The necessary condition that (Cl)-connection exists is reductivity of $T(P)/T(G)$ that is

(Dl) $$\mathfrak{P}^{(1)}+\mathfrak{P}^{(2)}=\mathfrak{G}^{(1)}+\mathfrak{G}^{(2)}+\mathfrak{M}.\quad \mathfrak{P}^{(i)}\supset\mathfrak{G}^{(i)}\ \mathrm{ad}\,(T(G))\,\mathfrak{M}=\mathfrak{M}$$

Proposition 1. If $T(P)/T(G)$ is reductive, then $\mathrm{ad}(G)\ (\tau\mathfrak{M})=\tau\mathfrak{M}$ Proof Take any $(A, B)\in\mathfrak{M}$, where $A=\tau(A, B)\in\tau\mathfrak{M}$. From $\mathrm{ad}\,(g, U)\,(A, B)=(\mathrm{ad}\,(g)A,\ \mathrm{ad}\,(g)B-\mathrm{ad}\,(g)\cdot[A, U])\in\mathfrak{M}$. $\tau\cdot\mathrm{ad}\,(g, U)\,(A, B)=\mathrm{ad}\,(g)A$, so ad (g) $A\in\tau\mathfrak{M}$ and $\mathrm{ad}\,(G)\,(\tau\mathfrak{M})\subset\tau\mathfrak{M}$.

Remark If P/G is reductive then $T(P)/T(G)$ is reductive, also from reductive decompositions of $T(P)/T(G)$ we can get any reductive decomposition of P/G.

Proposition 2. We fix decomposition (Dl) on reductive $T(P)/T(G)$, then left invariant distribution Γ corresponding to it, is uniquely determined.

The all other invariantdistribution $\tilde{\Gamma}$ is given from the pair of R-linear mappings

$$\rho_1:\mathfrak{M}\to\mathfrak{G};\ \rho_1\circ\ \mathrm{ad}\,(g, U)\,(A, B)=\mathrm{ad}\,(g)\circ\rho_1(A, B)$$
$$\rho_2:\mathfrak{M}\to\mathfrak{G};\ \rho_2\circ\ \mathrm{ad}\,(g, U)\,(A, B)=\mathrm{ad}\,(g)\circ\rho_2(A, B)\text{-}\mathrm{ad}\,(g)\cdot[\rho_1(\mathrm{A, B}), U]$$
for any $(g, U)\in G\times\mathfrak{G}$ and $(A, B)\in\mathfrak{M}$.

Proof $T(P/G)\cong T(P)/T(G)$, and $\pi_G:\mathfrak{M}\cong T(P/G)_{(e.0)}$

Take any $(A, B)\in\mathfrak{M}$, and $\tilde{l}$ denotes the lifting operator with respect to $\tilde{\Gamma}$,

$$\tilde{l}_{(e.0)}\,\pi_G(A, B)=\{(e, 0), (A, B)\}+L_{(e.0)}(\rho(A, B))$$

where $\rho:\mathfrak{M}\to\mathfrak{G}^{(1)}+\mathfrak{G}^{(2)}$ is a R-linear mapping.

In $\tilde{\Gamma}_{(g.U)}$ we have

$$\tilde{l}_{(g.U)}\,\pi_G(A, B)=R_{(g.U)}\{(e, 0), (A, B)\}+L_{(g.U)}\,(\mathrm{ad}\,(g, U)^{-1}\cdot\rho(A, B))$$

On the other hand the invariance of Γ assures the existence $(C, D)\in\mathfrak{M}$ such that

$$\tilde{l}_{(g.U)}\,\pi_G(A,\,B)=L_{(g.U)}\{(e,\,0),\,(C,\,D)\}+L_{(g.U)}(\rho(C,\,D))$$

The above two relations give $(C,\,D)=\mathrm{ad}\,(g,\,U)^{-1}\cdot(A,\,B)$ and $\rho(C,\,D)=\mathrm{ad}\,(g,\,U)^{-1}\cdot\rho(A,\,B)$ and thus

$$\rho\circ\mathrm{ad}\,(g,\,U)\,(A,\,B)=\mathrm{ad}\,(g,\,U)\circ\rho(A,\,B) \text{ holds for any } (A,\,B)\in\mathfrak{M}$$

If we put $\rho_1\equiv\tau\circ\rho,\ \rho_2\equiv\sigma\circ\rho,\ \rho_1,\ \rho_2:\ \mathfrak{M}\to\mathfrak{G}$ then they enjoy the following relations

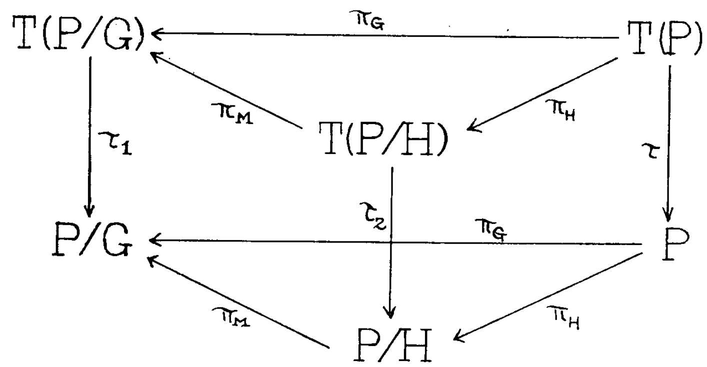
$$\rho\circ\mathrm{ad}\,(g,\,U)\,(A,\,B)=\mathrm{ad}\,(g)\circ\rho_1(A,\,B)$$
$$\rho_2\circ\mathrm{ad}\,(g,\,U)\,(A,\,B)=\mathrm{ad}\,(g)\circ\rho_2(A,\,B)-\mathrm{ad}\,(g)\circ[\rho_1(A,\,B),\,U]$$
for any $(g,\,U)\in G\times\mathfrak{G},\ (A,\,B)\in\mathfrak{M}$

Conversely, these $\rho_1,\ \rho_2$ determines $\tilde{\Gamma}$ and $\tilde{\Gamma}$ has property (Cl) and this correspondence is 1 to 1. We shall call these ρ or $\rho_1,\ \rho_2$ the connection mapping of $\tilde{\Gamma}$ with respect to (Dl).

Remark The connection form $\tilde{\omega}$ of $\tilde{\Gamma}$ has the value

$$\tilde{\omega}\{(e,\,0),\,(A,\,B)\}=-(\rho_1(A,\,B),\,\rho_2(A,\,B))$$

4. Induced (Cl)-connection on $T(P)$ $(T(P/H),\ T(H))$



In this section, we shall study the case where $T(P)/T(G)$, $T(P)/T(H)$ are reductive. Make the diagram, there $T(P)$ $(T(P/G),\ T(G))$, $T(P)$ $(T(P/H),\ T(H))$ are principal bundles, and our purpose in this section is to induce (Cl)-connection on $T(P)$ $(T(P/H),\ T(H))$ from the (Cl)-connection on $T(P)$ $(T(P/G),\ T(G))$

Proposition 3 If $T(P)/T(G)$, and $T(G)/T(H)$ are reductive then $T(P)/T(H)$ is reductive.

Proof From the reductivity of $T(P)/T(G)$ and $T(G)/T(H)$ we have

$$(D_G)\qquad \mathfrak{P}^{(1)}+\mathfrak{P}^{(2)}=\mathfrak{G}^{(1)}+\mathfrak{G}^{(2)}+\mathfrak{M}\qquad \mathrm{ad}\,(T(G))\mathfrak{M}=\mathfrak{M}\qquad \mathfrak{P}^{(i)}\supset\mathfrak{G}^{(i)}$$
$$(D')\qquad \mathfrak{G}^{(1)}+\mathfrak{G}^{(2)}=\mathfrak{H}^{(1)}+\mathfrak{H}^{(2)}+\mathfrak{N}\qquad \mathrm{ad}\,(T(H))\mathfrak{N}=\mathfrak{N}\qquad \mathfrak{G}^{(i)}\supset\mathfrak{H}^{(i)}$$

thus

$$(D_H)\qquad \mathfrak{P}^{(1)}+\mathfrak{P}^{(2)}=\mathfrak{H}^{(1)}+\mathfrak{H}^{(2)}+\widetilde{\mathfrak{M}}\qquad \widetilde{\mathfrak{M}}=\mathfrak{M}+\mathfrak{N}$$

The adjoint invariance $\mathrm{ad}\,(T(H))\widetilde{\mathfrak{M}}=\widetilde{\mathfrak{M}}$ are easily proved.

Proposition 4 Fix the decomposition (D_H), let $\rho'_i:\ \mathfrak{N}\to\mathfrak{H}$ be mappings such that

$$\rho_1'\cdot \text{ad}\,(h, V)\,(C, D) = \text{ad}\,(h)\circ\rho_1'(C, D)$$
$$\rho_2'\cdot \text{ad}\,(h, V)\,(C, D) = \text{ad}\,(h)\circ\rho_2'(C, D) - \text{ad}\,(h)\cdot[\rho_1'(C, D), V]$$
$$\text{for any } (h, V) \in \mathfrak{H}\ (C, D) \in \mathfrak{N}$$

then the mappings defined by $\tilde{\rho}_i : \widetilde{\mathfrak{M}} \to \mathfrak{H}$.

$$\tilde{\rho}_i(A+X, B+Y) \equiv \rho_i'(X, Y) \qquad (A, B) \in \mathfrak{M}. \quad (X, Y) \in \mathfrak{N}$$

give on $T(P)$ $(T(P/H), T(H))$ invariant distribution Γ, corresponding to the reductive decomposition of $T(G)/T(H)$.

Proof We can easily prove the relations

$$\tilde{\rho}_1\cdot \text{ad}\,(h, V)\,(A+X, B+Y) = \text{ad}\,(h)\,\tilde{\rho}_1\,(A+X, B+Y)$$
$$\tilde{\rho}_2\cdot \text{ad}\,(h, V)\,(A+X, B+Y) = \text{ad}(h)\,\tilde{\rho}_2\,(A+X, B+Y) - \text{ad}(h)[\tilde{\rho}_1(A+X, B+Y), V]$$
$$\text{for } (h, V) \in H\times\mathfrak{H},\ (A, B) \in \mathfrak{M},\ (X, Y) \in \mathfrak{N}$$

Proposition 5 The connection mapping which defines any other decomposition, if we fix (D_H), will be denoted with $\tilde{\rho}$, then $\tilde{\rho} = \bar{\rho} + \rho'$ where

$$\bar{\rho} : \mathfrak{M} \to \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)} \qquad \bar{\rho}\cdot \text{ad}\,(h, V) = \text{ad}\,(h, V)\cdot\bar{\rho}$$
$$\rho' : \mathfrak{N} \to \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)} \qquad \rho'\cdot \text{ad}\,(h, V) = \text{ad}\,(h, V)\cdot\rho'$$

Conversely any such $\tilde{\rho}$ gives decomposition of $T(P)/T(H)$.

Proof An element of $\widetilde{\mathfrak{M}}$ will be the form $(A+X, B+Y)$, $(A, B) \in \mathfrak{M}$, $(X, Y) \in \mathfrak{N}$ then R-linearlity of $\tilde{\rho}$ shows

$$\tilde{\rho}\cdot \text{ad}\,(h, V)\,(A, B) + \tilde{\rho}\cdot \text{ad}\,(h, V)\ X, Y) = \text{ad}\,(h, V)\cdot\tilde{\rho}(A, B) + \text{ad}\,(h, V)\cdot\tilde{\rho}(X, Y)$$
$$\text{for any } (A, B) \in \mathfrak{M},\ (X, Y) \in \mathfrak{N}$$

let $\tilde{\rho}\,|\,\mathfrak{M} \equiv \bar{\rho}$. $\tilde{\rho}\,|\,\mathfrak{N} \equiv \rho'$, then we have the result.

Thus we have

Theorem. Let $T(P)/T(G)$, $T(G)/T(H)$ be reductive, and fix decompositions

$$(D_G) \qquad \mathfrak{P}^{(1)} + \mathfrak{P}^{(2)} = \mathfrak{G}^{(1)} + \mathfrak{G}^{(2)} + \mathfrak{M} \quad \text{ad}\,(T(G)) \quad \mathfrak{M} = \mathfrak{M}$$
$$(D') \qquad \mathfrak{G}^{(1)} + \mathfrak{G}^{(2)} = \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)} + \mathfrak{N} \quad \text{ad}\,(T(H)) \quad \mathfrak{N} = \mathfrak{N},$$

then there corresponds a reductive decomposition of $T(P)/T(H)$,

$$(D_H) \qquad \mathfrak{P}^{(1)} + \mathfrak{P}^{(2)} = \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)} + \widetilde{\mathfrak{M}} \quad \text{ad}\,(T(H))\,\widetilde{\mathfrak{M}} = \widetilde{\mathfrak{M}},$$

Any other reductive decomposition can be obtained and defines left invariant distribution corresponding to the following pair of mappings

$$\rho_i' : \mathfrak{N} \to \mathfrak{H} \begin{cases} \rho_1'\cdot \text{ad}\,(h, V) = \text{ad}\,(h)\rho_1' \\ \rho_2'\cdot \text{ad}\,(h, V) = \text{ad}\,(h)\rho_2' - \text{ad}\,(h)[\rho_1', V] \end{cases}$$
$$\rho_i : \mathfrak{M} \to \mathfrak{G} \begin{cases} \rho_1\cdot \text{ad}\,(g, U) = \text{ad}\,(g)\rho_1 \\ \rho_2\cdot \text{ad}\,(g, U) = \text{ad}\,(g)\rho_2 - \text{ad}\,(g)[\rho_1, U] \end{cases}$$
$$\text{for } (h, V) \in H\times\mathfrak{H},\ (g, U) \in G\times\mathfrak{G}.$$

Proof $\bar{\rho}$ in Prop. 5, is obtained by putting $\bar{\rho}=(\rho_1, \rho_2)_{(1)+(2)}$, that is $\mathfrak{H}^{(1)}+\mathfrak{H}^{(2)}$ component of $\rho=(\rho_1, \rho_2)$ in the decomposition (D').

If C^∞-distribution $\tilde{\Lambda}$ of horizontal spaces of non-linear connection on $T(P/H)$ satisfies $\pi_M \tilde{\Lambda}=\Lambda$, then $\tilde{\Lambda}$ is called π_M-related with Λ.

In this case we have induced (Cl)-connection with $\tilde{\Lambda}$ and $\tilde{\Gamma}$ on $T(P)$ $(T(P/H), T(H))$.

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(Received 1 June 1965)