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Three-Dimensional Multicounter Automata

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Abstract

This paper introduces a three-dimensional multicounter automaton whose input tapes are restricted to cubic ones, and investigates some of its properties. We first show the difference between the accepting powers of five-way and six-way three-dimensional multicounter automata, and between the accepting powers of deterministic and nondeterministic five-way three-dimensional multicounter automata. We then show that hierarchies can be obtained by varying the number of counters or the amount of space allowed, for classes of sets accepted by five-way three-dimensional multicounter automata.

1 INTRODUCTION

Inoue and Takanami [5] introduced a three-way two-dimensional multicounter automaton and investigated its basic properties. Szepietowski also investigated some of its properties [9]. A four-way two-dimensional k-counter automaton (2-kCA) M is a two-dimensional finite automaton [1] that has k counters. The action of M is similar to that of the one-dimensional offline k-counter machine [3], except that the input head of M can move up, down, right, or left on a two-dimensional input tape. A three-way two-dimensional k-counter automaton is a 2-kCA whose input head can move right, left, or down, but not up.

By the way, during the past thirty years, several automata on a two-dimensional tape have been proposed and many properties of them have been obtained [6]. On the other hand, few properties of automata on a three-dimensional tape have been obtained [7,8,10-12].

In this paper, we introduce six-way and five-way three-dimensional multicounter automata. A six-way three-dimensional k-counter automaton (3-kCA), which can be considered as a natural extension of the 2-kCA to three dimensions, consists of a finite control, k counters, a read-only three-dimensional input tape, k counter heads, and an input tape head which can move north, east, south, west, up, or down. A five-way three-dimensional k-counter automaton ($FV3$-kCA) is a 3-kCA whose input tape head can move north, east, south, west, or down, but not up. It has often been noticed that we

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can easily get several properties of three-dimensional automata by directly applying the results of one or two dimensional case, if the three-dimensional input tapes are not restricted to cubic ones. So we let the three-dimensional input tapes, throughout this paper, be restricted to cubic ones in order to increase the theoretical interest.

This paper has three sections in addition to this Introduction. Section 2 contains some definitions and notation. Section 3 investigates the difference between the accepting powers of (counter-bounded) five-way and six-way three-dimensional multicounter automata. Section 4 investigates the difference between the accepting powers of (counter-bounded) deterministic and nondeterministic five-way three -dimensional multicounter automata. Section 5 shows that hierarchies can be obtained by varying the number of counters or the amount of space allowed, for classes of sets accepted by five- way three-dimensional multicounter automata.

2 PRELIMINARIES

Definition 2.1. Let Σ be a finite set of symbols. A *three-dimensional tape* over Σ is a three-dimensional rectangular array of elements of Σ. The set of all three-dimensional tapes over Σ is denoted by $\Sigma^{(3)}$.

Given a tape $x \in \Sigma^{(3)}$, for each j($1\leq j \leq 3$), we let $l_j(x)$ be the length of x along the j-th axis. The set of all $x \in \Sigma^{(3)}$ with $l_1(x)=m_1$, $l_2(x)=m_2$, and $l_3(x)=m_3$ is denoted by $\Sigma^{(m1,m2,m3)}$. When $1\leq i_j\leq l_j(x)$ for each j($1\leq j \leq 3$), let $x(i_1, i_2, i_3)$ denote the symbol in x with coordinates (i_1, i_2, i_3). Furthermore, we define

$$x[(i_1, i_2, i_3), (i'_1, i'_2, i'_3)],$$

when $1\leq i_j\leq i'_j\leq l_j(x)$ for each integer $j(1\leq j \leq 3)$, as the three-dimensional tape y satisfying the following (i) and (ii) :

(i) for each $j(1\leq j \leq 3)$, $l_j(y) = i'_j - i_j + 1$;

(ii) for each r_1, r_2, r_3 $(1\leqq r_1\leqq l_1(y), 1\leqq r_2\leqq l_2(y), 1\leqq r_3\leqq l_3(y))$, $y(r_1, r_2, r_3) = x(r_1 + i_1 - 1, r_2 + i_2 - 1, r_3 + i_3 - 1)$.

(We call $x[(i_1, i_2, i_3), (i'_1, i'_2, i'_3)]$ the $[(i_1, i_2, i_3), (i'_1, i'_2, i'_3)]$-*segment* of x.) For each $x \in \Sigma^{(m1,m2,m3)}$ and for each $1\leq i_1\leq m_1$, $1\leq i_2 \leq m_2$, $1 \leq i_3 \leq m_3$, $x[(i_1, 1, 1), (i_1, m_2, m_3)]$, $x[(1, i_2, 1), (m_1, i_2, m_3)]$, $x[(1, 1, i_3), (m_1, m_2, i_3)]$, $x[(i_1, 1, i_3), (i_1, m_2, i_3)]$, and $x[(1, i_2, i_3), (m_1, i_2, i_3)]$ are called the i_1-*th* (2-3) *plane of* x, the i_2-*th* (1-3) *plane of* x, the i_3-*th* (1-2) *plane of* x, the i_1-*th row on the* i_3-*th* (1-2) *plane*, and the i_2-*th column on the* i_3-*th* (1-2) *plane*, and are denoted by $x(2\text{-}3)_{i_1}$, $x(1\text{-}3)_{i_2}$, $x(1\text{-}2)_{i_3}$, $x[i_1, *, i_3]$, and $x[*, i_2, i_3]$, respectively.

We now introduce a five-way three-dimensional multicounter automaton. A *three -dimensional k-counter automaton* (3-*kCA*) M, $k \geq 1$, has a read-only three -dimensional input tape with *boundary symbols # and k counters*. (Of course, M has a *finite control, an input head, and k counter heads.*) The action of M is similar to that of the two-dimensional multicounter automaton [5], except that the input head of M can move east, west, south, north, up or down. That is, when an input tape $x \in \Sigma^{(3)}$ (where Σ is the set of input symbols of M and the boundary symbol # is not in Σ) is presented to M, M determines the next state of the finite control, the move direction (east, west, south, north, up, down, or no move) of the input head, and the move direction (right, left, or no move) of each counter head, depending on the present state of the finite control, the symbol read by the input head, and whether or not the content of each counter is zero (i.e., whether or not each counter head is on the bottom symbol Z_0 of the counter). If the input head falls off the tape x with boundary symbols, M can make no further move. M starts in its initial state, with the input head on position (1, 1,1) of the tape x, and with the contents of each counter zero (i.e., with each counter on the bottom symbol Z_0 of the counter). We say that M *accepts* the tape x if M eventually halts in a specified state (accepting state) on the bottom boundary symbol # of the input. We denote by $T(M)$ the set of all three-dimensional tapes accepted by M. A *five-way three-dimensional k-counter automaton* ($FV3$-kCA) is a 3-kCA whose input head can move east, west, south, north, or down, but not up (see Fig. 1).

Let $L(m) : \mathbf{N} \to \mathbf{R}$ (where $\mathbf{N}$ is the set of all positive integers and $\mathbf{R}$ is the set of all

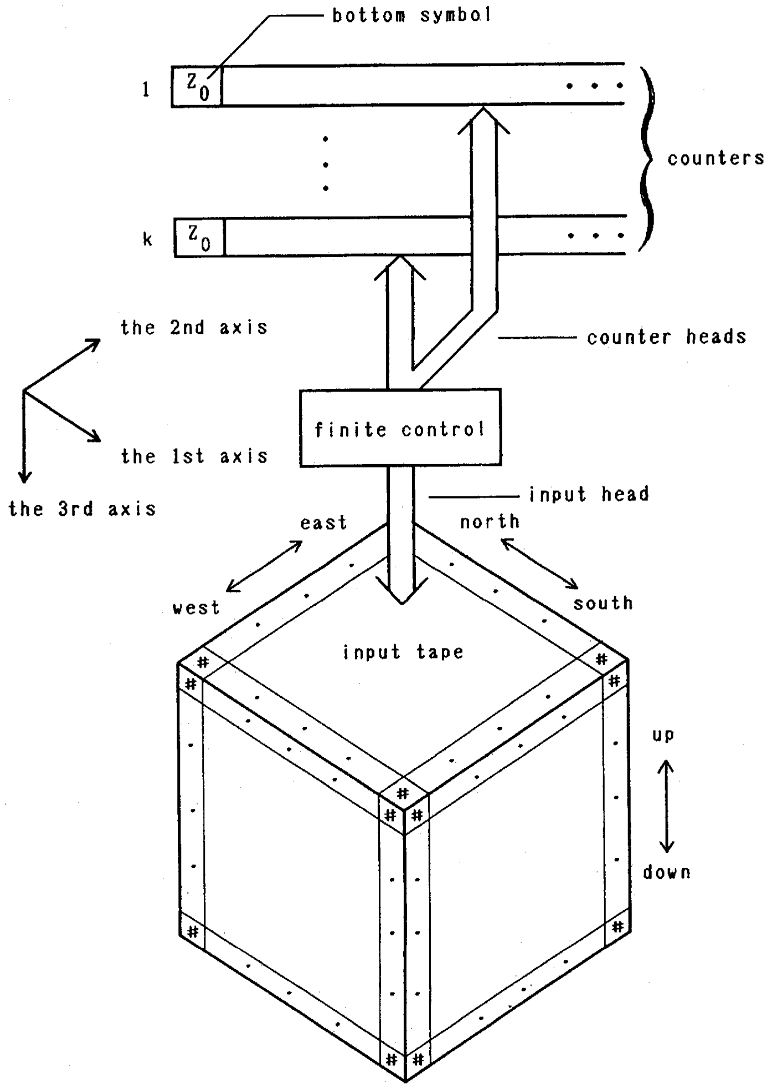

Fig. 1. Three-dimensional k-counter automaton.

nonnegative real numbers) be a function with one variable m. A 3-kCA ($FV3$-kCA) M whose input tapes are restricted to cubic ones is said to be $L(m)$ *counter-bounded* if for each $m \geq 1$ and each input tape x (accepted by M) with $l_1(x) = l_2(x) = l_3(x) = m$, each counter of M requires space not exceeding $L(m)$[1]. As usual, we define nondeterministic and deterministic 3-kCA*'s* ($FV3$-kCA*'s*). By $N3$-$kCA^c(L(m))$ (respectively, $D3$-$kCA^c(L(m))$, $NFV3$-$kCA^c(L(m))$, and $DFV3$-$kCA^c(L(m))$), we denote a nondeterministic 3-kCA (respectively, deterministic 3-kCA, nondeterministic $FV3$-kCA, and deterministic $FV3$-kCA) whose input tapes are restricted to cubic ones and which is $L(m)$ counter-bounded. Let $\pounds[N3\text{-}kCA^c(L(m))] = \{T \mid T = T(M)$ *for some* $N3\text{-}kCA^c(L(m))\ M\}$. $\pounds[D3\text{-}kCA^c(L(m))]$, $\pounds[NFV3\text{-}kCA^c(L(m))]$, and $\pounds[DFV3\text{-}kCA^c(L(m))]$ have similar meanings.

We briefly recall *five-way three-dimensional Turing machines* [10]. A five-way three-dimensional Turing machine M has a read-only three-dimensional input tape with boundary symbols # and one semiinfinite *storage tape*. (Of course, M has a *finite control*, an *input head*, and a *storage-tape head*.) The action of M is similar to that of the two-dimensional Turing machine [6] which has a read-only input tape with boundary symbols # and one semiinfinite storage tape, except that the input head of M can move east, west, south, north, or down, but not up. M starts in its initial state, with the input head on position (1,1,1) of an input tape x, and with all cells of the storage tape blank. We say that M accepts the tape x if M eventually halts in an accepting state. Let $L(m): \mathbf{N} \rightarrow \mathbf{R}$ be a function with one variable m. By $NFV3\text{-}TM^c(L(m))$ ($DFV3\text{-}TM^c(L(m))$) we denote a nondeterministic (deterministic) five-way three-dimensional Turing machine whose input tapes are restricted to cubic ones and which does not scan more than $L(m)$ cells on the storage tape for any input tape x (accepted by M) with $l_1(x) = l_2(x) = l_3(x) = m$. Let $\pounds[NFV3\text{-}TM^c(L(m))]$ ($\pounds[DFV3\text{-}TM^c(L(m))]$) denote the class of sets accepted by $NFV3\text{-}TM^c(L(m))$*'s* ($DFV3\text{-}TM^c(L(m))$*'s*).

We denote a nondeterministic (deterministic) three-dimensional finite automaton by $N3$-FA ($D3$-FA). A five-way $N3$-FA (five-way $D3$-FA) is an $N3$-FA ($D3$-FA) whose input tape head can move east, west, south, north, or down, but not up. By $N3\text{-}FA^c$ ($D3\text{-}FA^c$, $NFV3\text{-}FA^c$, $DFV3\text{-}FA^c$) we denote an $N3$-FA ($D3$-FA, five-way $N3$-FA, five-way $D3$-FA) whose input tapes are restricted to cubic ones [10]. For example, let $\pounds[D3\text{-}FA^c]$ denote the class of sets accepted by $D3\text{-}FA^c$'s. As is easily seen, it follows that for any constant k, $\pounds[D3\text{-}FA^c] = \pounds[D3\text{-}1CA^c(k)]$, $\pounds[DFV3\text{-}FA^c] = \pounds[DFV3\text{-}1CA^c(k)]$, and so on.

We conclude this section by giving a relationship between five-way three-dimensional multicounter automata and five-way three-dimensional Turing machines, which will be used in the latter sections.

Theorem 2.1.

(1) $\bigcup_{1 \leq k < \infty} \pounds[XFV3\text{-}kCA^c(L(m))] \subseteq \pounds[XFV3\text{-}TM^c(\log L(m))]$ for any $L(m): \mathbf{N} \rightarrow \mathbf{R}$ and any $X \in \{D, N\}$,

(2) $\bigcup_{1 \leq k < \infty} \pounds[XFV3\text{-}kCA^c(L(m))] = \pounds[XFV3\text{-}TM^c(\log L(m)]$ for any $X \in \{D,$

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[1]Rigorously, "exceeding $L(m)$" should be replaced with "exceeding $\lceil L(m) \rceil$", where $\lceil r \rceil$ means the smallest integer greater than or equal to r. Below we omit $\lceil\ \rceil$, if no confusion occurs.

$N\}$.

Proof. (1) : Let M be an $XFV3\text{-}kCA^c(L(m))$. The set $T(M)$ is also accepted by the $XFV3\text{-}TM^c(\log L(m))$ which divides the storage tape into k tracks and makes each track play a role of the corresponding counter of M.

(2) : From (1), $\cup_{1\leq k<\infty}$ £$[XFV3\text{-}kCA^c(m)]\subseteq$ £$[XFV3\text{-}TM^c(\log\ m)]$. It is well known that any log m tape-bounded one-dimensional off-line Turing machine can be simulated by a one-dimensional two-way multihead finite automaton [4]. By using the same argument as in the proof of this fact, we can easily show that any $XFV3\text{-}TM^c$ (log m) can be simulated by an $XFV3\text{-}kCA^c(m)$ for some $k \geq 1$. Thus £$[XFV3\text{-}TM^c(\log\ m)]\subseteq\cup_{1\leq k<\infty}$£$[XFV3\text{-}kCA^c(m)]$. ∎

3 FIVE-WAY VERSUS SIX-WAY

In this section, we investigate the difference between the accepting powers of counter-bounded six-way and five-way three-dimensional multicounter automata whose input tapes are restricted to cubic ones.

We need the following two lemmas.

Lemma 3.1. Let $T_1=\{x\in\{0,1\}^{(3)} \mid \exists\ m\geq 2[l_1(x)=l_2(x)=l_3(x)=m\ \ \&\ \ x(1\text{-}2)_1=x(1\text{-}2)_2]\}$, and let $L_1(m)$: **N** → **R** be a function such that $\lim_{m\to\infty}[(\log L_1(m))/m^2]=0$. Then,

(1) $T_1\in$ £$[3\text{-}DA^c]=$ £$[D3\text{-}1CA^c(0)]$, and

(2) $T_1\notin\cup_{1\leq k<\infty}$£$[NFV3\text{-}kCA^c(L_1(m))]$.

Proof. The proof of (1) is omitted here, since it is obvious. As shown in Lemma 3.6 (2) in [10], T_1 is not in £$[NFV3\text{-}TM^c(L'_1(m))]$, where $L'_1(m)$: **N** → **R** is a function such that $\lim_{m\to\infty}[L'_1(m)/m^2] = 0$. From this fact and from the condition that $\lim_{m\to\infty}[(\log L_1(m))/m^2] = 0$, it follows that T_1 is not in £$[NFV3\text{-}TM^c(\log L_1(m))]$. Part (2) of the lemma follows from this fact and Theorem 2.1(1). ∎

Lemma 3.2. Let $T_2 = \{x\in\{0,1\}^{(3)} \mid \exists\ m\geq 1[l_1(x) = l_2(x) = l_3(x) = 2m\&x[(1,1,1),(2m,m,2m)] =x[(1,m+1,1),(2m,2m,2m)]]\}$, and let $L_2(m)$: **N** → **R** be a function such that $\lim_{m\to\infty}[(\log L_2(m))/m^3] = 0$. Then,

(1) $T_2\in$ £$[D3\text{-}1CA^c(m)]$, and

(2) $T_2\notin\cup_{1\leq k<\infty}$ £$[NFV3\text{-}kCA^c(L_2(m))]$.

Proof. The proof of (1) is omitted here, since it is obvious. By using the same ideas as in the proof of Lemma 4.2 (2) in [10], we can easily show that T_2 is not in £$[NFV3\text{-}TM^c(L'_2(m))]$, where $L'_2(m)$: **N** → **R** is a function such that $\lim_{m\to\infty}[L'_2(m)/m^3] = 0$. From this fact and from the condition that $\lim_{m\to\infty}[(\log L_2(m))/m^3] = 0$, it follows that T_2 is not in $L[NFV3\text{-}TM^c(\log L_2(m))]$. Part (2) of the lemma follows from this fact and Theorem 2.1 (1). ∎

From Lemmas 3.1 and 3.2, we can get the following theorem.

Theorem 3.1. (1) Let $L(m)$: **N** → **R** be a function such that $\lim_{m\to\infty}[(\log L(m))/m^2] = 0$. Then, £$[D3\text{-}FA^c]-\cup_{1\leq k<\infty}$£$[NFV3\text{-}kCA^c(L(m))]\neq\phi$. (2) *Let* $L'(m)$: **N** → **R** be a function such that $\lim_{m\to\infty}[(\log L'(m))/m^3] = 0$. Then, £$[D3\text{-}1CA^c(m)]-\cup_{1\leq k<\infty}$£$[NFV3\text{-}kCA^c(L'(m))]\neq\phi$.

4 NONDETERMINISM VERSUS DETERMINISM

In this section, we investigate the difference between the accepting powers of counter-bounded deterministic and nondeterministic five-way three-dimensional multicounter automata whose input tapes are restricted to cubic ones.

We need the following two lemmas.

Lemma 4.1. Let $T_3 = \{x \in \{0,1\}^{(3)} \mid \exists m \geq 2[l_1(x) = l_2(x) = l_3(x) = m \& x(1\text{-}2)_1 \neq x(1\text{-}2)_2]$, and $L_1(m) : \mathbf{N} \to \mathbf{R}$ be a function such that $\lim_{m\to\infty}[(\log L_1(m))/m^2] = 0$. Then,

(1) $T_3 \in \pounds[NFV3\text{-}FA^c] = \pounds[NFV3\text{-}1CA^c(0)]$, and

(2) $T_3 \notin \cup_{1\leq k<\infty}\pounds[DFV3\text{-}kCA^c(L_1(m))]$.

Proof. The proof of (1) is omitted here, since it is obvious. We prove (2). Suppose that there is a $DFV3\text{-}kCA^c(L_1(m))$ M, $k \geq 1$, accepting T_3, and that s is the number of states of the finite control of M. For each $m \geq 2$, let

$V(m) = \{x \in \{0,1\}^{(3)} \mid l_1(x) = l_2(x) = l_3(x) = m \& x[(1,1,2),(m,m,m)] \in \{0\}^{(3)}\}$.

For each x in $V(m)$, let $conf(x)$ be the configuration[2] of M just after the input head left the first plane $x(1\text{-}2)_1$ of x. Then the following must hold.

Proposition 4.1. For any two different tapes x, y in $V(m)$,

$$conf(x) \neq conf(y).$$

[For suppose that $conf(x) = conf(y)$. Consider two tapes z, z' with side-length m which satisfy the following :

(1) $z(1\text{-}2)_1 = x(1\text{-}2)_1$ and $z'(1\text{-}2)_1 = y(1\text{-}2)_1$;

(2) $z(1\text{-}2)_2 = z'(1\text{-}2)_2 = y(1\text{-}2)_1$;

(3) $z[(1,1,3),(m,m,m)] = z'[(1,1,3),(m,m,m)]$.

Clearly, z is in T_4, and so z is accepted by M. Since $conf(x) = conf(y)$, it follows that z' is also accepted by M. This contradicts the fact that z' is not in T_3.]

Clearly, $|V(m)| = 2^{m^2}$. On the other hand, let $c(m)$ be the number of possible configurations of M just after the input head left the top planes of tapes in $V(m)$. Then $c(m) \leq s(m+2)^2(L_1(m))^k$. Since $\lim_{m\to\infty}[(\log L_1(m))/m^2] = 0$, $|V(m)| > c(m)$ for large m. Therefore, it follows that for large m there must be two different tapes x, y *in* $V(m)$ such that $conf(x) = conf(y)$. This contradicts Proposition 4.1. This completes the proof of (2).∎

Lemma 4.2. Let $T_4 = \{x \in \{0,1\}^{(3)} \mid \exists m \geq 2[l_1(x) = l_2(x) = l_3(x) = 2m \& x[(1,1,1),(2m,2m,m)] \neq x[(1,1,m+1),(2m,2m,2m)]]\}$, and let $L_2(m) : \mathbf{N} \to \mathbf{R}$ be a function such that $\lim_{m\to\infty}[(\log L_2(m))/m^3] = 0$. Then,

(1) $T_4 \in \pounds[NFV3\text{-}1CA^c(m)]$, and

(2) $T_4 \notin \cup_{1\leq k<\infty}\pounds[DFV3\text{-}kCA^c(L_2(m))]$.

Proof. (1) : We consider the $NFV3\text{-}1CA^c(m)$ M which acts as follows. Suppose that

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[2]For any (five-way) three-dimensional multicounter automaton M, we define the *configuration* of M to be a combination of the (1) state of the finite control, (2) position of the input head within the input tape, (3) contents of each counter.

an input tape x with $l_1(x) = l_2(x) = l_3(x) = 2m$ is presented to M. First of all, M starts on position (1,1,1) of x, and adds the number one by one in the counter for every two east moves of the input head along the 2nd axis. When the input head reaches the eastmost cell, M stores the number m in the counter. M then chooses some r_1, r_2 ($1 \leq r_1, r_2 \leq 2m$) nondeterministically, and moves the input head downwards along the 3 rd axis, starting from the position $(r_1,r_2,1)$. During this action, M chooses some r_3 ($1 \leq r_3 \leq m$) nondeterministically, picks up $x(r_1,r_2,r_3)$, and stores it in the finite control. Then, by using the number m stored in the counter, M picks up $x(r_1,r_2,r_3+m)$, compares it with $x(r_1,r_2,r_3)$ stored in the finite control, and accepts x if and only if $x(r_1,r_2,r_3) \neq x(r_1,r_2,r_3+m)$. [After M has picked up $x(r_1,r_2,r_3)$, M subtracts one from the counter for every down move of the input head. $x(r_1,r_2,r_3+m)$ is the symbol under the input head when the contents of the counter is zero. If the input head arrives at the bottom boundary symbol # before the contents of the counter is zero, then M fails in the choice of r_3 and enters the rejecting state.] It will be obvious that $T(M)=T_4$.

(2) : By using the same ideas as in the proof of part (2) of Lemma 4.1, we can easily show that T_5 is not in $\cup_{1\leq k<\infty}\pounds[DFV3\text{-}kCA^c(L_2(m))]$. The proof is left to the reader. ∎

From Lemmas 4.1 and 4.2, we can get the following theorem.

Theorem 4.1. (1) Let $L(m)$: $\mathbf{N} \rightarrow \mathbf{R}$ be a function such that $\lim_{m\to\infty}[(\log L(m))/m^2] = 0$. Then, $\pounds[NFV3\text{-}FA^c] - \cup_{1\leq k<\infty}\pounds[DFV3\text{-}kCA^c(L(m))] \neq \phi$. (2) Let $L'(m)$: $\mathbf{N} \rightarrow \mathbf{R}$ be a function such that $\lim_{m\to\infty}[(\log L'(m))/m^3] = 0$. Then, $\pounds[NFV3\text{-}1CA^c(m)] - \cup_{1\leq k<\infty}\pounds[DFV3\text{-}kCA^c(L'(m))] \neq \phi$.

5 HIERARCHIES BASED ON THE NUMBER OF COUNTERS OR THE SPACE ALLOWED

This section investigates how the number of counters or the space allowed (of five-way three-dimensional multicounter automata whose input tapes are restricted to cubic ones) affects the accepting power.

To do this we need to consider the following sets. For each $j \leq 1$, let $A(j)$ be the set of all cubic tapes $x \in \{0,1\}^{(3)}$ such that :

(a) $l_1(x)=l_2(x)=l_3(x) \geq j$.

(b) There are exactly j 1's in the first row of the first (1-2) plane.

(c) All the rows from the second to the last in the first (1-2) plane contain only 0.

(d) All the (1-2) planes from the second to the last but one contain only 0.

(e) The last (1-2) plane is equal to the first.

The following three lemmas show that the set $A(j)$ can be accepted by an m^r counter-bounded deterministic (or nondeterministic) k-counter automaton if $j \leq (k-1)r+1$, $k \geq 2$, and $r \geq 1$, but not by any m^r counter-bounded nondeterministic k-counter automaton if $j>kr$ or by any deterministic one if $j>(k-1)r+2$.

Lemma 5.1. For each $k \geq 2$ and $r \geq 1$, $A(j)$ can be accepted by a $DFV3\text{-}kCA^c(m^r)$ if $j \leq (k-1)r+1$.

Proof. We show how $A(j)$ can be accepted by a $DFV3\text{-}kCA^c(m^r)$ M if $j=(k-1)r+1$. The case when $j<(k-1)r+1$ can be proved similarly. Suppose that an input tape x

with $l_1(x)=l_2(x)=l_3(x)=m$ is presented to M. First M checks if x satisfies conditions (a), (b), and (c) above (in the definition of $A(j)$). Let $d(i)$ denote the position of the i-th 1 in the first row of the first (1-2) plane. All $d(i)$, for $1 \leq i \leq j-1 = (k-1)r$, are stored in k-1 counters in groups of r in each counter. The first r numbers from $d(1)$ to $d(r)$ are stored as

$$\Sigma_{i=1}^{r} d(i)(m+1)^{r-i}$$

in the first counter. First M stores d(1) on the first counter, and then, using the k-th counter and going from one end of the first row to another; multiplies the first counter by $(m+1)$, then adds $d(2)$, multiplies again, and so on. Similarly the rest of $d(i)$, for $r+1 \leq i \leq (k-1)r$ are stored in the counters from the second to the $(k-1)$-th. Then M stores on the k-th counter the position of the j-th 1 and checks if all (1-2) planes from the second to the last but one contain only 0.

M assumes that the (1-2) plane that contains the first 1 below the first (1-2) plane is the last (1-2) plane (M will reject the input if it finds another (1-2) plane below). Next, M checks if there are exactly j 1's in the first row of the last (1-2) plane, and all the rows from the second to the last in the last (1-2) plane contain only 0. After that, unloading the k-th counter M checks if the last 1 in the first row of the last (1-2) plane stands on the $d(j)$ position and then, using the empty k-th counter, unloads one by one the numbers $d(i)$ and checks if there is 1 in the $d(i)$ position of the first row. Unloading is done in the following way : If a number $s(m+1)+d(i)$, with $d(i) \leq m$ is stored on a counter (say the g-th), then M goes from the first cell of the first row to the # symbol standing on the other end, decreasing the g-th counter by 1 after each step, and after reaching # it adds 1 to the k-th counter, comes back to the beginning of the row, and repeats the process until the g-th counter is empty. At this moment M stands on the $d(i)$ position of the first row and keeps s on the k-th counter. It is obvious that in this way all numbers $d(i)$ can be unloaded (in reverse order to when they were loaded) and the positions of all 1's in the first row checked. ∎

Lemma 5.2. $A(j)$ cannot be accepted by any $NFV3\text{-}kCA^c(m^r)$ *if* $j>kr$.

Proof. Suppose that there is an $NFV3\text{-}kCA^c(m^r)$ M accepting $A(j)$ *and* $j>kr$. For each $m \geq j$, let

$$A(m,j)=\{x \in A(j) \mid l_1(x)=l_2(x)=l_3(x)=m\}.$$

Any accepting computation of M reading any $x \in A(m,j)$ has to visit $x(1,1,2)$, the northmost and westmost cell in the second (1-2) plane. Otherwise, if there is an $x \in A(m,j)$ accepted without visiting $x(1,1,2)$, then, putting $x(1,1,2) = 1$, we obtain the tape that is not in $A(j)$ but is accepted by M. Let $conf(x)$ be the set of configurations of M while visiting $x(1,1,2)$ in the accepting computations on x. For any two different $x,y \in A(m,j)$, $conf(x) \cap conf(y) = \phi$. Otherwise, replacing the last (1-2) plane in x by the last (1-2) plane of y, we obtain the tape that is not in $A(j)$ but is accepted by M.

Clearly

$$| A(m,j) | = \binom{m}{j}.$$

Let $c(m)$ be the number of possible configurations of M while visiting $x(1,1,2)$. Then $c(m) \leq sm^{kr}$, where s is the number of states of the finite control. Since $j>kr$, there exists m such that $| A(m,j) | > c(m)$, and there must be two different $x,y \in A(m,j)$ such that $conf(x) \cap conf(y) \neq \phi$. This contradicts the above. ∎

Lemma 5.3. $A(j)$ cannot be accepted by any $DFV3\text{-}kCA^c(m^r)$ if $j>(k-1)r+2$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.2, but it has to be observed that if a deterministic automaton M visits the cell $x(1,1,2)$ of an input x then there is a moment when it visits $x(1,1,2)$ and at least one of its counters does not exceed cm^2 for some constant c. Suppose that at a moment t, M stands on $x(1,1,2)$ and each of its counter exceeds $2s(m+2)^2+1$, where s is the number of states of the finite control. This means that the last moment when at least one of the counters is empty (say t_0) is before $t-2s(m+2)^2-1$ and there are two moments t_1 and t_2, $t_0<t_1<t_2 \leq t_0+2s(m+2)^2+1 \leq t$, such that M stands at t_1 and t_2 with the same state on the same cell of the first or second (1-2) plane (since M is five-way it cannot visit the third (1-2) plane before t). If each counter of M is nonempty, then the next move depends only on the state of the finite control and the symbol scanned by the input head, and since M is deterministic, the moves from t_1 to t_2 are repeated in a loop until one of the counters becomes empty (if each counter never becomes empty after t_0, then M never stops). So there is a moment t_3, $t_1 \leq t_3 \leq t_2$, when M visits $x(1,1,2)$ [otherwise $x(1,1,2)$ cannot be visited at t], and since $t_3-t_0 \leq 2s(m+2)^2$ the counter that is empty at t_0 cannot contain more than $2s(m+2)^2$ at t_3. So there is the moment t_3 when M stands in $x(1,1,2)$ with one counter not exceeding $2s(m+2)^2 \leq cm^2$, for some constant c. ∎

We are now ready to prove the following theorems. First we show that for every $r \geq 1$ there exists an infinite hierarchy, with respect to the number of counters, of languages accepted by m^r counter-bounded (deterministic or nondeterministic) five-way three-dimensional k-counter automata.

Theorem 5.1. For each $r \geq 1$, $k \geq 1$, and $X \in \{D,N\}$, $\pounds[XFV3\text{-}kCA^c(m^r)] \subsetneq \pounds[XFV3\text{-}(k+1)CA^c(m^r)]$.

Proof. From Lemmas 5.1 (1) and (2) it follows that for any $r \geq 1$, $k \geq 1$, *and* $X \in \{D,N\}$,

$$A(kr+1) \in \pounds[XFV3\text{-}(k+1)CA^c(m^r)]$$

and

$$A(kr+1) \notin \pounds[XFV3\text{-}kCA^c(m^r)]. \blacksquare$$

Next we show that for any $k \geq 3$ there is an infinite hierarchy, with respect to the amount of space allowed, of the powers of deterministic five-way three-dimensional k-counter automata.

Theorem 5.2. For each $k \geq 3$ and $r \geq 1$, $\pounds[DFV3\text{-}kCA^c(m^r)] \subsetneq \pounds[DFV3\text{-}kCA^c(m^{r+1})]$.

Proof. From Lemmas 5.1 (1) and (3) it follows that for each $r \geq 1$ *and* $k \geq 3$,

$$A((r+1)(k-1)+1) \in \pounds[DFV3\text{-}kCA^c(m^{r+1})]$$

and

$$A((r+1)(k-1)+1) \notin \pounds[DFV3\text{-}kCA^c(m^r)]. \blacksquare$$

Finally, we show that for every $k \geq 2$ and $r < k$ there is an infinite hierarchy, with respect to the amount of space allowed, of the powers of nondeterministic five-way three-dimensional k-counter automata.

Theorem 5.3. For each $k \geq 2$ and $r < k$, $\pounds[NFV3\text{-}kCA^c(m^r)] \subsetneq \pounds[NFV3\text{-}kCA^c(m^{r+1})]$.

Proof. From Lemma 5.1 (1) we have

$$A((r+1)(k-1)+1) \notin \pounds[NFV3\text{-}kCA^c(m^{r+1})],$$

and if $r < k$, then $(r+1)(k-1)+1 > rk$, and by Lemma 5.1(2),

$$A((r+1)(k-1)+1) \notin \pounds[NFV3\text{-}kCA^c(m^r)]. \blacksquare$$

6 CONCLUSION

In this paper, we have showed the differences between the accepting powers of six-way and five-way three-dimensional multicounter automata, and between the accepting powers of nondeterministic and deterministic three-dimensional multicounter automata. Furthermore, we showed that hierarchies can be obtained by varying the number of counters or the amount of space allowed, for classes of sets accepted by five-way three-dimensional multicounter automata.

It will be also interesting to investigate the accepting powers of "alternating" three-dimensional multicounter automata (see [2] for the concept of "alternation").

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