Eigen Value Problems of Operators on a Product Hilbert Bundle

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Abstract

Let S^1 be a one–dimensional torus and H be a Hilbert space. We discuss the properties of eigen values of operators associated with bundle homomorphisms on a product Hilbert bundle ($S^1 \times H, S^1, \pi$).

It is important to study operators and oparator algebras on Hilbert bundles for applications to relatively quantum mechanics.

The purpose of this paper is to show the properties of eigen values of special operators associated with bundle homomorphism of a product Hilbert bundle.

It remains to be solved to investigate the operators and operator algebras on general Hilbert bundles.

Let H be a Hilbert space and X be a topological space. The set of all bounded linear operators on H is denoted by B(H). A product Hilbert bundle is a triplet $\xi = (E, X, \pi)$ where E is the product space of X and H equipped with product topology and π is the projection of E to X.

We set $E_x = \pi^{-1}(x)$ and the Hilbert space $H = E_x$ is said to be the fibre at x.

A continuous map $\sigma : X \to E$ is called a continuous cross section on ξ if

$$\pi \sigma(x) = x \qquad \text{for all } x \in X.$$

We deonote the set of all continuous cross sections on ξ by $\Gamma(\xi)$. The space $\Gamma(\xi)$ is a vector space by pointwise scalar multiplication and pointwise addition, that is,

$$(\alpha \sigma)(x) = \alpha \sigma(x)$$

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x) \qquad \text{for } \sigma, \tau \in \Gamma(\xi),\ \alpha \in \mathbf{C},\ x \in X.$$

If X is a compact space, $\Gamma(\xi)$ is a Banach space by supremum norm.

A bundle homomorphism in $\xi = (E, X, \pi)$ is a pair (f, f_0) of two continuous maps such that

(i) the map f is a continuous map from E to E and the map f_0 is a continuous map from X to X

(ii) $\pi f = f_0 \pi$

(iii) the restriction f_x of f to E_x is a bounded linear operator from $E_x = H$ to $E_{f_0}(x) = H$

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(iv) the map $X \ni x \to f_x \in B(H)$ is continuous in a suitable operator topology.

Let (f, f_0) be a bundle homomorphism in ξ. If f_0 is a homeomorphism, we can define a linear operator T associeted with (f, f_0) on $\Gamma(\xi)$ as follows :

$$T\sigma = f\sigma f_0^{-1} \qquad \text{for all } \sigma \in \Gamma(\xi).$$

The eigen vector $\sigma \in \Gamma(\xi)$ of the operator T is specially called an eigen cross section of T. It is important to study eigen value problems of operators associated with bundle homomorphisms on Hilbert bundles.

This paper deals with a special case that a topological space X is a one-dimensional torus and f_x is an identity operator for each $x \in X$.

Let S^1 be a one-dimensional torus, that is,

$$S^1 = \{z \in C : |z| = 1\}$$

and ω be in $X = S^1$.

Let $\xi = (E, X, \pi)$ be a product Hilbert bundle where $X = S^1$ and $E = S^1 \times H$. We define a bundle homomorphism (f, f_0) on product Hilbert bundle ξ as follows :

$$f_0(x) = \omega x \qquad \text{for } x \in S^1$$
$$f(x, h) = (f_0(x), h) \qquad \text{for } (x, h) \in E$$

i. e., f_x is an identity operator.

Since $\Gamma(\xi)$ is a Banach space and the operator T associated with (f, f_0) is isometry, the absolute value of each eigen value of T is 1. Furthermore the following theorem holds.

Theorem Let a product Hilbert bundle $\xi = (E, X, \pi)$ and bundle homomorphisms (f, f_0) be as above and T be the operator associated with (f, f_0).

If ω is a primitive n-th root of 1, the set of all eigen values of T is $\{\omega^k : 0 \leqq k \leqq n-1\}$.

If ω is not a roof of 1, then we get the following :

(i) for each integer l, ω^l is an eigen value of T and the associated eigen space is $\{x^{-l}h : h \in H\}$

(ii) if $\lambda \neq 1$ is an eigen value of T, λ is not a root of 1.

Proof If σ is a cross section on ξ,

$$(T\sigma)(x) = f\sigma f_0^{-1}(x) = \sigma(\omega^{-1}x).$$

Hence λ is an eigen value and σ is an eigen cross section cssociated with λ if

$$\sigma(x) = \lambda\, \sigma(\omega x) \qquad \text{for } x \in X = S^1$$

If for a non-zero vector $h \in H$ we put

$$\sigma(x) = x^{-k}h \qquad \text{for } x \in X,$$
$$\omega^k \sigma(\omega x) = \sigma(x) \qquad \text{for } x \in X \text{ and integer } k.$$

Hence for each integer k, ω^k is an eigen value and $\sigma(x) = x^k h$ is an eigen cross section associated with ω^k.

First let ω be a primitive n–th root, λ be an eigen value and σ be an eigen croos section associated with λ. Since σ is an eigen croos section, there is a point $x_0 \in X$ such that $\sigma(x_0) \neq 0$. Since $\sigma(x_0) = \lambda^n \sigma(\omega^n x_0) = \lambda^n \sigma(x_0)$, so $\lambda^n = 1$ and therefore λ is in $\{\omega^k : 0 \leqq k \leqq n-1\}$.

Next let ω be not a root of 1. If σ is an eigen croos section associated with an eigen value ω^l, we get

$$\sigma(\omega^k) = \omega^{-kl} \sigma(1) \qquad \text{for all integer k.}$$

Since the set $\{\omega^k : k \in Z\}$ is dence in $X = S^1$, for each $x \in X$ there is a sequence $\{\omega^{m(k)}\}_k$ such that

$$x = \lim_{k \to \infty} \omega^{m(k)}.$$

Then we get

$$\sigma(x) = \lim_{k \to \infty} \sigma(\omega^{m(k)}) = (\lim_{k \to \infty} \omega^{-m(k)l}) \sigma(1)$$
$$= x^{-l} \sigma(1).$$

Hence $\sigma(x) = x^{-l} h$ if we set $h = \sigma(1) \in H$.

We have proved that the eigen space asociated with an eigen value w^l is the set $\{x^{-l} h : h \in H\}$ if ω is not a root of 1.

Let $\lambda \neq 1$ be an eigen value and σ be an associated eigen croos section.

If λ were root of 1, there is an integer k such that $\lambda^k = 1$. Then $\sigma(\omega^{lk}) = \lambda^{-lk} \sigma(1) = \sigma(1)$ for all integer l.

Since ω^k is not a root of 1, the set $\{\omega^{lk} : l \in Z\}$ is dence in $X = S^1$.

Hence we have

$$\sigma(x) = \sigma(1) \qquad \text{for all } x \in X.$$

Since $\sigma(1) = \sigma(x) = \lambda \sigma(\omega x) = \lambda \sigma(1)$, we get

$$\sigma(1) = \lambda \sigma(1) \quad \text{and so} \quad \sigma(1) = 0.$$

Thus $\sigma(x) = 0$ for all $x \in X$.

This constradicts the condition of σ being an eigen croos section. Therefore λ is not a root of 1. This completes the proof.

Remark 1

Even if H is a general locally convex Hausdorff space, the absolute value of each eigen value of T is 1 and Theorem holds.

Remark 2

Let H be of one–dimension. If ω is not a root of 1, the set of all eigen values of T is the set $\{\omega^n : n \in Z\}$.

References

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