Observation of the State Vector of a Discrete-Time Linear System from the Actual Output Data

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Abstract

The observation of the state vector of a discrete-time linear deterministic system is considered under the assumption of complete observability.

The entire state vector is determined exactly from observations of the actual outputs in p_M measurement points where p_M is the observability index. The main result shows that the important concept of the observability index in the field of the modern control theory is easily obtained from the simple ordinary algebraic operation.

I. Introduction

The Luenberger observer in the case of deterministic measurements [1] and the Kalman filter in the case of noisy measurements [2] are well known methods for system state estimation. In the deterministic system, the Luenberger observer permits determination of unknown state variables via an $(n-m)$-dimensional observer which is called minimal observer, where n is the dimension of the state vector and m is the dimension of the measurement vector.

This paper considers about observing the state vector of a discrete-time deterministic linear time-invariant system under the mild assumption of well known complete observability. It is shown how the available system inputs and outputs may be used to determine the system state vector. The results show that the exact determination of the state vector is possible from the actual output vectors of the same number as the observability index, although this concept has been developed by the pole configurations of the observer.

The organization of this paper is as follows. The problem statement is given in Section II. In Section III, the canonical transformation is introduced, and in Section IV, the equation for the unknown state vector is developed. Section V is devoted to conclusions.

II. Problem Statement

Consider a discrete-time linear time-invariant system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \tag{1}$$

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where

$\mathbf{x}$ is an $n \times 1$ state vector
$\mathbf{u}$ is a $r \times 1$ input vector
A is an $n \times n$ system matrix
B is an $n \times r$ input matrix.

The observation of this system is governed by

$$\mathbf{y}_k = C\mathbf{x}_k \tag{2}$$

where

$\mathbf{y}$ is an $m \times 1$ observation vector
C is an $m \times n$ observation matrix.

It is assumed that the initial state $\mathbf{x}_0$ is unknown and the dimension of the observation m is less than or equal to n. The fundamental assumption imposed on this system is that of complete system observability, namely the $n \times m$ observability matrix,

$$[C^T, (CA)^T, \dots, (CA^{n-1})^T]$$

has rank n where T denotes the transpose.

$$\operatorname{rank}[C^T, (CA)^T, \dots, (CA^{n-1})^T] = n \tag{3}$$

We assume for simplicity in forming the canonical transformation described in Section III that the columns of C are linearly independent. The purpose of this paper is to determine the entire state vector in Eq. (1) from the output data in Eq. (2).

III. Canonical Transformation

Define the observation matrix is

$$C = \begin{pmatrix} C_1^T \\ C_2^T \\ \vdots \\ C_m^T \end{pmatrix} \tag{4}$$

where each C_i^T is an n-dimensional row vector.

The first step in the development of a canonical form is the selection of n linearly independent vector from the following vector sequences [3]:

$$\begin{array}{lll} C_1^T, & C_1^T A, & C_1^T A^2, \dots \\ C_2^T, & C_2^T A, & C_2^T A^2, \dots \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} \\ C_m^T, & C_m^T A, & C_m^T A^2, \dots \end{array} \tag{5}$$

If the new column vector is linearly independent of all the previously selected column vectors, retain it and otherwise omit it from selection. This selection procedure terminates when n linearly independent column vectors are found.

Arranging the n column vectors, the matrix P is defined as follows

$$\begin{aligned} P=[&C_1^T,\ \ C_1^T A, \ldots, C_1^T A^{p_1-1}, \\ &C_2^T,\ \ C_2^T A, \ldots, C_2^T A^{p_2-1}, \\ &\ldots\ldots\ldots\ldots, \\ &C_m^T\ \ C_m^T A, \ldots, C_m^T A^{p_m-1}]^T \end{aligned} \tag{6}$$

where p_i's, $i=1, 2, \ldots, m$ are referred to as observability subindices, which satisfy

$$\sum_{i=1}^{m} p_i = n \tag{7}$$

from the assumption of observability condition.

We consider the same canonical form as applied in the works of [4] and [5]. A change of coordinates from state $\mathbf{x}$ to $\mathbf{w}$ defined by

$$\mathbf{w} = P\mathbf{x} \tag{8}$$

transforms the system Eq. (1) and (2) to

$$\mathbf{w}_{k+1} = \Phi \mathbf{w}_k + G\mathbf{u}_k \tag{9}$$

$$\mathbf{y}_k = H\mathbf{w}_k \tag{10}$$

In Eq. (9) and (10), Φ, G and H are given as follows:

$$\Phi = PAP^{-1} = \begin{array}{cl} \left(\begin{array}{c:c:c:c} \begin{matrix} \bigcirc \quad I \\ \phi_{11}^T \end{matrix} & \begin{matrix} \bigcirc \\ \phi_{12}^T \end{matrix} & \cdots & \begin{matrix} \bigcirc \\ \phi_{1m}^T \end{matrix} \\ \hdashline \begin{matrix} \bigcirc \\ \phi_{21}^T \end{matrix} & \begin{matrix} \bigcirc \quad I \\ \phi_{22}^T \end{matrix} & \cdots & \begin{matrix} \bigcirc \\ \phi_{2m}^T \end{matrix} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \begin{matrix} \bigcirc \\ \phi_{m1}^T \end{matrix} & \begin{matrix} \bigcirc \\ \phi_{m2}^T \end{matrix} & \cdots & \begin{matrix} \bigcirc \quad I \\ \phi_{mm}^T \end{matrix} \end{array}\right) & \begin{matrix} \}\, p_1 \\ \\ \}\, p_2 \\ \\ \\ \}\, p_m \end{matrix} \\ \begin{matrix} \underbrace{\qquad}_{p_1} & \underbrace{\qquad}_{p_2} & \qquad & \underbrace{\qquad}_{p_m} \end{matrix} & \end{array} \tag{11}$$

where, ϕ_{ij}, i, $j=1, 2, \ldots, m$ is the $p_j \times 1$ matrix defined as

$$\phi_{ij}^T = [\phi_{ij}^1, \phi_{ij}^2, \ldots, \phi_{ij}^{p_j}] \tag{12}$$

$$G=PB=\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} \tag{13}$$

$$H=CP^{-1}=\left(\underbrace{\begin{matrix} 1 & \bigcirc \\ & \\ \bigcirc & \end{matrix}}_{p_1} \;\vdots\; \underbrace{\begin{matrix} \bigcirc & \\ 1 & \bigcirc \\ \bigcirc & \end{matrix}}_{p_2} \;\vdots\; \cdots \;\vdots\; \underbrace{\begin{matrix} \bigcirc & \\ & \\ 1 & \bigcirc \end{matrix}}_{p_m}\right) \tag{14}$$

IV. Determination of State Vector

Stacking up outputs of Eq. (10), the following equations are obtained.

$$\begin{aligned} \mathbf{y}_k &= H\mathbf{w}_k \\ \mathbf{y}_{k+1} &= H\Phi\mathbf{w}_k + HG\mathbf{u}_k \\ &\vdots \\ \mathbf{y}_{k+l} &= H\Phi^l\mathbf{w}_k + H\sum_{j=0}^{l-1}\Phi^j G\mathbf{u}_{k+l-1-j} \end{aligned} \tag{15}$$

According to Eq. (15), the following construction is obtained for representation of the ith element of $\mathbf{y}_k$ as $y_{i,k}$ and the ith row vector of H as h_i.

$$\begin{pmatrix} y_{1,k} \\ y_{1,k+1} \\ \vdots \\ y_{1,k+p_1-1} \\ \vdots \\ y_{m,k} \\ y_{m,k+1} \\ \vdots \\ y_{m,k+p_m-1} \end{pmatrix} = \begin{pmatrix} h_1^T \\ h_1^T\Phi \\ \vdots \\ h_1^T\Phi^{p_1-1} \\ \vdots \\ h_m^T \\ h_m^T\Phi \\ \vdots \\ h_m^T\Phi^{p_m-1} \end{pmatrix} \mathbf{w}_k + \begin{pmatrix} 0 \\ h_1^T G\mathbf{u}_k \\ \vdots \\ h_1^T\sum_{j=0}^{p_1-2}\Phi^j G\mathbf{u}_{k+p_1-2-j} \\ \vdots \\ 0 \\ h_m^T G\mathbf{u}_k \\ \vdots \\ h_m^T\sum_{j=0}^{p_m-2}\Phi^j G\mathbf{u}_{k+p_m-2-j} \end{pmatrix} \tag{16}$$

It is easily shown that the coefficient matrix of $\mathbf{w}_k$ in Eq. (16) is the $n\times n$ identity matrix according to the relation between Eq. (11) and Eq. (14). Then $\mathbf{w}_k$ is represented as

$$\mathbf{w}_k = \begin{pmatrix} y_{1,k} \\ y_{1,k+1} \\ \vdots \\ y_{1,k+p_1-1} \end{pmatrix} - \begin{pmatrix} 0 \\ h_1^T G\mathbf{u}_k \\ \vdots \\ h_1^T\sum_{j=0}^{p_1-2}\Phi^j G\mathbf{u}_{k+p_1-2-j} \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} \vdots \\ y_{m,k} \\ y_{m,k+1} \\ \vdots \\ y_{m,k+p_m-1} \end{pmatrix} \quad \begin{pmatrix} \vdots \\ 0 \\ h_m^T G\mathbf{u}_k \\ \vdots \\ h_m^T \sum_{j=0}^{p_m-2} \Phi^j G\mathbf{u}_{k+p_m-2-j} \end{pmatrix}$$

Eq. (10) indicates that the following m elements of $\mathbf{w}_k$ vector are obtained directly from the observation of $\mathbf{y}_k$.

$$\begin{pmatrix} w_{1,k} \\ w_{v_1,k} \\ \vdots \\ w_{v_{m-1}+1,k} \end{pmatrix} = \begin{pmatrix} y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{m,k} \end{pmatrix} \tag{18}$$

where $w_{i,k}$ is defined as the ith element of $\mathbf{w}_k$ and

$$v_i = \sum_{j=1}^{i} p_j, \qquad i=1, 2,\dots, m \tag{19}$$

where v_0 is equal to 0 and v_m means n from Eq. (7).

Eliminating the directly observed element of the state vector $\mathbf{w}_k$ of Eq. (18) from Eq. (17), the $(n-m)$ dimensional unknown state vector is given by

$$\begin{pmatrix} w_{2,k} \\ \vdots \\ w_{v_1,k} \\ w_{v_1+2,k} \\ \vdots \\ w_{v_{m-1}+2,k} \\ \vdots \\ w_{v_m,k} \end{pmatrix} = \begin{pmatrix} y_{1,k+1} \\ \vdots \\ y_{1,k+p_1-1} \\ y_{2,k+1} \\ \vdots \\ y_{m,k+1} \\ \vdots \\ y_{m,k+p_m-1} \end{pmatrix} = \begin{pmatrix} h_1^T G\mathbf{u}_k \\ \vdots \\ h_1^T \sum_{j=0}^{p_1-2} \Phi^j G\mathbf{u}_{k+p_1-2-j} \\ h_2^T G\mathbf{u}_k \\ \vdots \\ h_m^T G\mathbf{u}_k \\ \vdots \\ h_m^T \sum_{j=0}^{p_m-2} \Phi^j G\mathbf{u}_{k+p_m-2-j} \end{pmatrix} \tag{20}$$

Defining the observability index as p_M,

$$p_M = \max\{p_1, p_2,\dots, p_m\} \tag{21}$$

Eq. (2) is expressed as

$$\begin{pmatrix} w_{v_1,k} \\ w_{v_2,k} \\ \vdots \\ w_{v_m,k} \end{pmatrix} = \begin{pmatrix} Y_{1,k+p_1-1} \\ Y_{2,k+p_2-1} \\ \vdots \\ Y_{m,k+p_m-1} \end{pmatrix} - \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_m \end{pmatrix} \begin{pmatrix} \mathbf{u}_k \\ \mathbf{u}_{k+1} \\ \vdots \\ \mathbf{u}_{k+p_M-2} \end{pmatrix} \tag{22}$$

where

$$w_{\nu_i,k}=\begin{pmatrix} w_{\nu_{i-1}+2,k} \\ w_{\nu_{i-1}+3,k} \\ \vdots \\ w_{\nu_i,k} \end{pmatrix}, \qquad \text{for} \quad i=1, 2,\dots, m \tag{23}$$

$$Y_{i,k+p_i-1}=\begin{pmatrix} y_{i,k+1} \\ y_{i,k+2} \\ \vdots \\ y_{i,k+p_i-1} \end{pmatrix}, \qquad \text{for} \quad i=1, 2,\dots, m \tag{24}$$

$$\Delta p_i=\left.\underbrace{\begin{pmatrix} g_{\nu_{i-1}+1} & & 0 \\ g_{\nu_{i-1}+2} & \ddots & \\ \vdots & & \ddots \\ g_{\nu_{i-1}+p_i-1} & \cdots\cdots & g_{\nu_{i-1}+1} \end{pmatrix}}_{(p_M-1)r}\right\} p_i-1, \qquad \text{for} \quad i=1, 2,\dots, m \tag{25}$$

The available data for the determination of the unknown state sub-vector $w_{\nu_i,k}$ are the outputs from the kth time point to the $(k+p_i-1)$th time point for $i=1, 2,\dots, m$.

Therefore, p_M measurement time points are required to determine the value of the state vector and within this interval n elements of the observed output vector are available for the determination of the entire state vector. After observing outputs at p_M sequential time points, all we have to do in order to determine the unknown state vector is to calculate the $(n-m)$-dimensional vector in Eq. (22), which m elements of the state vector are directly obtained as the observed outputs shown in Eq. (18).

Finally the original state $\mathbf{x}_k$ is given by

$$\mathbf{x}_k=P^{-1}\mathbf{w}_k. \tag{26}$$

V. Conclusion

The determination of the state vector of a discrete-time linear system is revisited by the simple algebraic approach under the assumption of complete observability.

It has been shown that the entire state vector, including its initial, present and future values, is exactly obtained by the n elements of the observed output with p_M measurement time points using the canonical transformation. This concepts is demonstrated by selecting the special values of poles of observer, i.e. the pole configurations.

References

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