Number Theoretical Remarks on the Paper "Some Structual Properties of Product Automata" by Kanaoka and Tomita

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Abstract

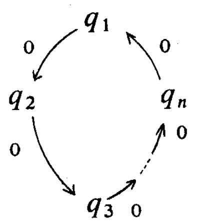
Number theoretical properties of automata are discussed. In particular, the following are discussed:
(1) the characterization of the existence of the cycle,
(2) the relation between the cycle of the product of relatively prime numbers and the cycles of these numbers,
(3) the relation between the cycle of the power of a prime number and the cycle of the prime number.
Lastly, the table of the cycles of prime numbers is given.

§0. Intorduction

In the recent paper [2] one of the authors studied the special finite automata which are said to be cyclic deterministic autonomous outputless automata (CDAOA).

This automaton is as follows:

(1) the input alphabet is consists of one element 0,
(2) the number of states is n,
(3) the move function is represented by the following graph



If A is a matrix represented by the above graph, that is,

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

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then $A^n=1$ and $A^k \neq 1$ for $1 \leq k < n$.

This matrix A is called a matrix representing the move function of CDAOA. In the paper [2] we introduced the self-product automata of CDAOA and studied the cycle of CDAOA.

Let A be a matrix representing the move function of CDAOA. If there is a natural number m such that $A^{2^m}=A$, it is said that A has a cycle. And the least natural number m such that $A^{2^m}=A$ is called the cycle of A. We studied the properties of the cycle in the paper [2], for example Proposition 4.1, Theorem 4.2 and Theorem 4.3.

The purpose of this paper is to give the detailed properties of the cycle by number theoretical method. First, we can easily prove the following theorems.

Let A be a square matrix of order n representing the move function of CDAOA, that is, $A^n=1$ and $A^k \neq 1$ for $1 \leq k < n$.

Theorem 1. A has a cycle if and only if n is an odd integer.

Theorem 2. $A^{2^k}=1$ for some k if and only if n is a power of 2.

Theorem 3. If $\{A^{2^k} \mid k=0, 1, 2,\dots\}=\{A, A^2,\dots, A^{n-1}\}$, then n is an odd prime number.

In §4 we shall give the more detailed properties of the cycle. Lastly we shall give the table of the cycle.

§1. $Z/(n)$

We denote by $\boldsymbol{Z}$ the set of all integers. $\boldsymbol{Z}$ is an infinite cyclic group with respect to the addition. For an integer n the set of all multiples of n

$$(n)=\{0, \pm n, \pm 2n,\dots\}$$

is an additive subgroup; the factor group $\boldsymbol{Z}/(n)$ consists of n elements;

$$\boldsymbol{Z}/(n)=\{\bar{0}, \bar{1},\dots, \overline{n-1}\},$$

where $\bar{k}$ is the coset containing k in $\boldsymbol{Z}$. The addition of $\boldsymbol{Z}/(n)$ is defined by

$$\bar{k}+\bar{\chi}=\overline{k+\chi}$$

for all k, χ in $\boldsymbol{Z}$.

$\boldsymbol{Z}/(n)$ has the multiplication defined by

$$\bar{k}\bar{\chi}=\overline{k\chi}.$$

A nonzero element $\bar{k}$ in $\boldsymbol{Z}/(n)$ is said to be invertible if there exists an element $\bar{\chi}$ in $\boldsymbol{Z}/(n)$ such that $\bar{k}\bar{\chi}=\bar{1}$. It is easy to see that the set of all invertible elements in $\boldsymbol{Z}/(n)$ forms a finite (multiplicative) group.

Lemma 1.1. Let $\bar{k}$ be an element in $\boldsymbol{Z}/(n)$. Then $\bar{k}$ is invertible if and only if k and n are relatively prime.

The proof of Lemma 1.1 follows from the fact that integers k and n are relatively prime if and only if there exist integers a and b such that $ak+bn=1$.

Corollary 1.2. The set of all nonzero elements of $\boldsymbol{Z}/(n)$ forms a group with respect to the multiplication if and only if n is a prime number.

In fact, n is a prime integer if and only if it is relatively prime with all positive integers less than n.

§2. Reduction to number theoretic argument

Let A be a matrix given in §0. Then the set

$$G=\{I, A, A^2,\dots, A^{n-1}\}$$

is a cyclic group of order n. The mapping $A^k \mapsto \bar{k}$ is a group isomorphism of G onto $\boldsymbol{Z}/(n)$. Under this mapping, in particular, A^{2^k} corresponds to $\bar{2}^k$. Therefore we have the following:

Proposition 2.1. (1) A has a cycle if and only if $\bar{2}^k=1$ for some k.

(2) $A^{2^k}=I$ for some k if and only if $\bar{2}^k=\bar{0}$ for some k.

(3) $\{A^{2^k} \mid k=0, 1,\dots\}=\{A, A^2,\dots, A^{n-1}\}$ if and only if $\{\bar{2}^k \mid k=0, 1,\dots\}=\{\bar{1}, \bar{2},\dots, \overline{n-1}\}$.

By this proposition the theorems in §0 can be restated as follows:

Theorem 1′. $\bar{2}^k=\bar{1}$ for some k if and only if n is an odd integer.

Theorem 2′. $\bar{2}^k=\bar{0}$ for some k if and only if n is a power of 2.

Theorem 3′. If $\{\bar{2}^k \mid k=0, 1,\dots\}=\{\bar{1}, \bar{2},\dots, \overline{n-1}\}$, then n is an odd prime integer.

§3. Proofs of Theorems

Proof of Theorem 1′. ($\Rightarrow$) The equation $\bar{2}^k=\bar{1}$ implies that 2^k-1 is a multiple of n. Since the integer 2^k-1 is odd, its divisors are all odd, in particular, n is odd.

($\Leftarrow$) Assume that n is an odd integer. Then 2 and n are relatively prime. Therefore Lemma 1.1 implies that the coset $\bar{2}$ is invertible. Thus $\bar{2}$ has a finite order, that is, $\bar{2}^k=\bar{1}$ for some k.

Proof of Theorem 2′. ($\Rightarrow$) $\bar{2}^k=\bar{0}$ implies that 2^k is a multiple of n. In particular, a divisor n of 2^k is also a power of 2.

($\Leftarrow$) Clear.

Proof of Theorem 3′. By the assumption we have $\bar{2}^k=\bar{1}$ for some k. It follows from Theorem 1′ that n is an odd integer. Since $\{\bar{2}^k \mid k=0, 1,\dots\}$ is a cyclic group, $\{\bar{1}, \bar{2},\dots, \overline{n-1}\}$ is a group. By Corollary 1.2, n is a prime number. This completes the proof.

§ 4. Cycles

By Theorem 1, the matrix given in §0 has a cycle if n is an odd integer. In this section we investigate the property of the cycle. The cycle is determined by only n, so we denote it by $s(n)$:

$$s(n)=\min\{k\in\boldsymbol{Z}\mid k>0,\ 2^k\equiv 1\ (\mathrm{mod}\ n)\}.$$

For integers a and b we use the notation $a\mid b$ if a can divide b.

The following lemma can be easily shown, so we omit the proof.

Lemma 4.1. $2^k\equiv 1\ (\mathrm{mod}\ n)$ if and only if $s(n)\mid k$.

Theorem 4.2. If m and n are relatively prime odd integers, then $s(mn)$ is equal to the least common multiple of $s(m)$ and $s(n)$.

Proof. Let s be the least common multiple of $s(m)$ and $s(n)$. Then we have $2^s\equiv 1\ (\mathrm{mod}\ m)$ and $2^s\equiv 1\ (\mathrm{mod}\ n)$ by Lemma 4.1. Since m and n are relatively prime it follows that $2^s\equiv 1\ (\mathrm{mod}\ mn)$ which implies, by Lemma 4.1, that $s(mn)\mid s$.

On the other hand, by definition, we have $2^{s(mn)}\equiv 1\ (\mathrm{mod}\ mn)$. In particular we have $2^{s(mn)}\equiv 1\ (\mathrm{mod}\ m)$ and $2^{s(mn)}\equiv 1\ (\mathrm{mod}\ n)$. By Lemma 4.1 we have $s(m)\mid s(mn)$ and $s(n)\mid s(mn)$. Therefore $s\mid s(mn)$, which implies that $s=s(mn)$. This completes the proof.

Corollary 4.3. Let $n=p_1^{e_1}\dots p_k^{e_k}$ be the decomposition into prime factors. Then the cycle of n is the least common multiple of $s(p_1^{e_1}),\dots,$ and $s(p_k^{e_k})$.

In the sequel, we consider the cycle of p^e for a fixed odd prime integer p and a natural number e.

For a natural number f (resp. e), we denote by $\tilde{k}$ (resp. $\bar{k}$) the coset in $\boldsymbol{Z}/(p^f)$ (resp. $\boldsymbol{Z}/(p^e)$) containing k in $\boldsymbol{Z}$. Let $e\leq f$. Then we have the natural mapping

$$\begin{array}{cccc} \rho\colon & \boldsymbol{Z}/(p^f) & \longrightarrow & \boldsymbol{Z}/(p^e) \\ & \Cup & & \Cup \\ & \tilde{k} & \longmapsto & \bar{k}. \end{array}$$

It is easily seen that ρ preserves addition and multiplication:

(i) $\rho(\tilde{k}+\tilde{\chi})=\bar{k}+\bar{\chi}$

(ii) $\rho(\tilde{k}\tilde{\chi})=\bar{k}\bar{\chi}$, $\rho(\tilde{1})=\bar{1}$.

We denote by U_f (resp. U_e) the multiplicative group of all invertible elements in $\boldsymbol{Z}/(p^f)$ (resp. $\boldsymbol{Z}/(p^e)$) and by T_f (resp. T_e) the cyclic subgroup generated by $\tilde{2}$ (resp. $\bar{2}$). Let ρ° (resp. ρ^*) be the restriction of ρ to U_f (resp. T_f). Then it can be shown that $\rho^\circ(U_f)=U_e$ and $\rho^*(T_f)=T_e$. Thus when we regard ρ° (resp. ρ^*) as a mapping of U_f (resp. T_f) onto U_e (resp. T_e), it is a group homomorphism by (ii);

$$\begin{array}{ccc} \boldsymbol{Z}/(p^f) & \xrightarrow{\ \rho\ } & \boldsymbol{Z}/(p^e) \\ \cup & & \cup \end{array}$$

$$\begin{array}{ccc} U_f & \xrightarrow{\rho^\circ} & U_e \\ \cup & & \cup \\ T_f & \xrightarrow{\rho^*} & T_e . \end{array}$$

It follows that

Lemma 4.4. $|U_f|=|U_e|\,|\mathrm{Ker}\,\rho^\circ|$ and $|T_f|=|T_e|\,|\mathrm{Ker}\,\rho^*|$.

Note that by Lemma 1.1 we can calculate the orders of U_f and U_e. In fact

$$|U_f|=p^{f-1}(p-1),\ |U_e|=p^{e-1}(p-1).$$

Thus we have from Lemma 4.4

Lemma 4.5 $|\mathrm{Ker}\,\rho^\circ|=p^{f-e}$.

By definition of cycle we have $|T_f|=s(p^f)$ and $|T_e|=s(p^e)$. Since ρ^* is a restriction of ρ° to T_f, $\mathrm{Ker}\,\rho^*$ is a subgroup of $\mathrm{Ker}\,\rho^\circ$. But when $f=e+1$ $\mathrm{Ker}\,\rho^\circ$ has no non-trivial subgroups since $|\mathrm{Ker}\,\rho^\circ|=p$ is a prime number. Thus

Lemma 4.6. If $f=e+1$, then $\mathrm{Ker}\,\rho^*=\{\tilde{1}\}$ or $\mathrm{Ker}\,\rho^*=\mathrm{Ker}\,\rho^\circ$.

Note that $\mathrm{Ker}\,\rho^*=\{\tilde{1}\}$ (resp. $\mathrm{Ker}\,\rho^*=\mathrm{Ker}\,\rho^\circ$) implies that $s(p^{e+1})=s(p^e)$ (resp. $s(p^{e+1})=ps(p^e)$).

Now consider the case where $\mathrm{Ker}\,\rho^*=\mathrm{Ker}\,\rho^\circ$. In this case we have

$$\tilde{1}+\widetilde{p^e}\in \mathrm{Ker}\,\rho^\circ=\mathrm{Ker}\,\rho^*\subset T_{e+1}.$$

This means the following congruence:

$$1+p^e\equiv 2^k \pmod{p^{e+1}}$$

for some k, that is,

$$1+p^e=2^k+\chi p^{e+1}$$

for some k, χ in $\boldsymbol{Z}$. Taking p-th power of both sides, we have a congruence

(iii) $1+p^{e+1}\equiv 2^{pk} \pmod{p^{e+2}}$.

Consider next the following diagram:

$$\begin{array}{ccccc} \boldsymbol{Z}/(p^{e+2}) & \xrightarrow{\rho_{e+1}} & \boldsymbol{Z}/(p^{e+1}) & \xrightarrow{\rho_e} & \boldsymbol{Z}/(p^e) \\ \cup & & \cup & & \cup \\ U_{e+2} & \xrightarrow{\rho^0_{e+1}} & U_{e+1} & \xrightarrow{\rho^0_e} & U_e \\ \cup & & \cup & & \cup \\ T_{e+2} & \xrightarrow{\rho^*_{e+1}} & T_{e+1} & \xrightarrow{\rho^*_e} & T_e \end{array}$$

Then (iii) means that the coset in $\boldsymbol{Z}/(p^{e+2})$ containing $1+p^{e+1}$ belongs to $\mathrm{Ker}\,\rho^*_{e+1}$. It follows that ρ^*_{e+1} is not injective. Thus we have

Lemma 4.7. If ρ^*_e is not injective, then ρ^*_{e+1} is not injective. That is to say, if $s(p^{e+1})=ps(p^e)$ then $s(p^{e+2})=ps(p^{e+1})$.

As a corollary we have

Corollary 4.8. There exists a natural number or infinity e_0 such that

$$s(p^e)=\begin{cases} s(p) & \text{for} \quad 1\le e\le e_0 \\ p^{e-e_0}s(p) & \text{for} \quad e>e_0 . \end{cases}$$

We can easily show as follows that $e_0=\infty$ can not happen. In fact, assume that $e_0=\infty$, that is, $s(p^e)=s(p)$ for all e. By definition, this implies that

$$2^{s(p)}\equiv 1 \pmod{p^e}$$

for all e. Let e be so large that

$$1<2^{s(p)}<p^e.$$

Clearly, for this e, we have

$$2^{s(p)}/\not\equiv 1 \pmod{p^e}$$

which contradicts the assumption. Therefore we can conclude:

Theorem 4.9. Let p be an odd prime integer. Then there exists a natural number e_0 such that

$$s(p^e)=\begin{cases} s(p) & \text{for} \quad 1\le e\le e_0 \\ p^{e-e_0}s(p) & \text{for} \quad e>e_0 . \end{cases}$$

The number e_0 is obtained by

$$e_0=\max\,\{e\,|\,2^{s(p)}\equiv 1 \pmod{p^e}\} .$$

Corollary 4.10. The following conditions are equivalent:

a) $e_0=1$,

b) $2^{s(p)}\not\equiv 1 \pmod{p^2}$,

c) $2^{p-1}\not\equiv 1 \pmod{p^2}$.

The equivalence of b) and c) follows from the fact that

$$s(p)\,|\,p-1 \quad \text{and} \quad s(p)\le p-1\le ps(p) .$$

Remark. $p=1093$ and 3511 are the only two prime integers such that $e_0>1$ and $p<3\times 10^9$ ([3], p. 115) In fact, $e_0=2$ for both prime numbers. In appendix we list the table of $s(p)$ for all odd prime numbers $p<10000$.

References

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2) T. Kanaoka and S. Tomita: "Some Structual Properties of Product Automata", Technology Reports of the Yamaguchi University, **2**, 5, 553–565 (1981).

3) H. Wada: "Introduction to Number Theory", Iwanami (1981) (in Japanese).

Table of the Cycles of Odd Prime Numbers

P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)
3	2	251	50	577	144	929	464	1291	1290	1667	1666	2083	2082	2473	618
5	4	257	16	587	586	937	117	1297	648	1669	1668	2087	1043	2477	2476
7	3	263	131	593	148	941	940	1301	1300	1693	1692	2089	29	2503	1251
11	10	269	268	599	299	947	946	1303	651	1697	848	2099	2098	2521	1260
13	12	271	135	601	25	953	68	1307	1306	1699	566	2111	1055	2531	2530
17	8	277	92	607	303	967	483	1319	659	1709	244	2113	44	2539	2538
19	18	281	70	613	612	971	194	1321	60	1721	215	2129	532	2543	1271
23	11	283	94	617	154	977	488	1327	221	1723	574	2131	2130	3549	2548
29	28	293	292	619	618	983	491	1361	680	1733	1732	2137	1068	2551	1275
31	5	307	102	631	45	991	495	1367	683	1741	1740	2141	2140	2557	2556
37	36	311	155	641	64	997	332	1373	1372	1747	1746	2143	51	2579	2578
41	20	313	156	643	214	1009	504	1381	1380	1753	146	2153	1076	2591	1295
43	14	317	316	647	323	1013	92	1399	233	1759	879	2161	1080	2593	81
47	23	331	30	653	652	1019	1018	1409	704	1777	74	2179	726	2609	1304
53	52	337	21	659	658	1021	340	1423	237	1783	891	2203	734	2617	1308
59	58	347	346	661	660	1031	515	1427	1426	1787	1786	2207	1103	2621	2620
61	60	349	348	673	48	1033	258	1429	84	1789	596	2213	2212	2633	1316
67	66	353	88	677	676	1039	519	1433	179	1801	25	2221	2220	2647	1323
71	35	359	179	683	22	1049	262	1439	719	1811	362	2237	2236	2657	166
73	9	367	183	691	230	1051	350	1447	723	1823	911	2239	1119	2659	2658
79	39	373	372	701	700	1061	1060	1451	1450	1831	305	2243	2242	2663	1331
83	82	379	378	709	708	1063	531	1453	1452	1847	923	2251	750	2671	445
89	11	383	191	719	359	1069	356	1459	486	1861	1860	2267	2266	2677	2676
97	48	389	388	727	121	1087	543	1471	245	1867	1866	2269	2268	2683	2682
101	100	397	44	733	244	1091	1090	1481	370	1871	935	2273	568	2687	79
103	51	401	200	739	246	1093	364	1483	1482	1873	936	2281	190	2689	224
107	106	409	204	743	371	1097	274	1487	743	1877	1876	2287	381	2693	2692
109	36	419	418	751	375	1103	29	1489	744	1879	939	2293	2292	2699	2698
113	28	421	420	757	756	1109	1108	1493	1492	1889	472	2297	1148	2707	2706
127	7	431	43	761	380	1117	1116	1499	1498	1901	1900	2309	2308	2711	1355
131	130	433	72	769	384	1123	1122	1511	755	1907	1906	2311	1155	2713	1356
137	68	439	73	773	772	1129	564	1523	1522	1913	239	2333	2332	2719	1359
139	138	443	442	787	786	1151	575	1531	1530	1931	1930	2339	2338	2729	1364
149	148	449	224	797	796	1153	288	1543	771	1933	644	2341	780	2731	26
151	15	457	76	809	404	1163	166	1549	1548	1949	1948	2347	782	2741	2740
157	52	461	460	811	270	1171	1170	1553	194	1951	975	2351	47	2749	916
163	162	463	231	821	820	1181	236	1559	779	1973	1972	2357	2356	2753	1376
167	83	467	466	823	411	1187	1186	1567	783	1979	1978	2371	2370	2767	461
173	172	479	239	827	826	1193	298	1571	1570	1987	1986	2377	1188	2777	1388
179	178	487	243	829	828	1201	300	1579	526	1993	996	2381	476	2789	2788
181	180	491	490	839	419	1213	1212	1583	791	1997	1996	2383	397	2791	465
191	95	499	166	853	852	1217	152	1597	532	1999	333	2389	2388	2797	2796
193	96	503	251	857	428	1223	611	1601	400	2003	286	2393	598	2801	1400
197	196	509	508	859	858	1229	1228	1607	803	2011	402	2399	1199	2803	2802
199	99	521	260	863	431	1231	615	1609	201	2017	336	2411	482	2819	2818
211	210	523	522	877	876	1237	1236	1613	52	2027	2026	2417	1208	2833	118
223	37	541	540	881	55	1249	156	1619	1618	2029	2028	2423	1211	2837	2836
227	226	547	546	883	882	1259	1258	1621	1620	2039	1019	2437	2436	2843	2842
229	76	557	556	887	443	1277	1276	1627	542	2053	2052	2441	305	2851	2850
233	29	563	562	907	906	1279	639	1637	1636	2063	1031	2447	1223	2857	102
239	119	569	284	911	91	1283	1282	1657	92	2069	2068	2459	2458	2861	2860
241	24	571	114	919	153	1289	161	1663	831	2081	1040	2467	2466	2879	1439

P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)
2887	1443	3331	222	3739	534	4177	87	4639	2319	5077	5076	5521	2760	5981	460
2897	1448	3343	557	3761	188	4201	525	4643	422	5081	635	5527	2763	5987	5986
2903	1451	3347	3346	3767	1883	4211	842	4649	2324	5087	2543	5531	5530	6007	1001
2909	2908	3359	1679	3769	1884	4217	1054	4651	1550	5099	5098	5557	5556	6011	6010
2917	972	3361	168	3779	3778	4219	4218	4657	388	5101	1700	5563	5562	6029	6028
2927	1463	3371	3370	3793	1896	4229	4228	4663	777	5107	5106	5569	464	6037	2012
2939	2938	3373	1124	3797	3796	4231	2115	4673	2336	5113	426	5573	5572	6043	318
2953	492	3389	484	3803	3802	4241	2120	4679	2339	5119	2559	5581	124	6047	3023
2957	2956	3391	113	3821	764	4243	4242	4691	4690	5147	5146	5591	2795	6053	6052
2963	2962	3407	1703	3823	637	4253	4252	4703	2351	5153	112	5623	2811	6067	6066
2969	371	3413	3412	3833	958	4259	4258	4721	295	5167	861	5639	2819	6073	3036
2971	110	3433	1716	3847	1923	4261	4260	4723	4722	5171	5170	5641	564	6079	1013
2999	1499	3449	431	3851	3850	4271	305	4729	788	5179	5178	5647	2823	6089	761
3001	1500	3457	576	3853	3852	4273	534	4733	364	5189	5188	5651	5650	6091	2030
3011	3010	3461	3460	3863	1931	4283	4282	4751	475	5197	1732	5653	1884	6101	6100
3019	3018	3463	577	3877	3876	4289	1072	4759	793	5209	217	5657	2828	6113	3056
3023	1511	3467	3466	3881	388	4297	537	4783	2391	5227	5226	5659	5658	6121	1530
3037	3036	3469	3468	3889	648	4327	2163	4787	4786	5231	2615	5669	436	6131	6130
3041	1520	3491	3490	3907	3906	4337	2168	4789	4788	5233	1308	5683	5682	6133	2044
3049	762	3499	3498	3911	1955	4339	1446	4793	2396	5237	748	5689	711	6143	3071
3061	204	3511	1755	3917	3916	4349	4348	4799	2399	5261	5260	5693	5692	6151	1025
3067	3066	3517	3516	3919	1959	4357	4356	4801	1200	5273	2636	5701	5700	6163	2054
3079	1539	3527	1763	3923	3922	4363	4362	4813	4812	5279	377	5711	571	6173	6172
3083	3082	3529	882	3929	1964	4373	4372	4817	1204	5281	2640	5717	5716	6197	6196
3089	772	3533	3532	3931	3930	4391	2195	4831	2415	5297	662	5737	239	6199	3099
3109	444	3539	3538	3943	219	4397	4396	4861	972	5303	2651	5741	5740	6203	6202
3119	1559	3541	236	3947	3946	4409	551	4871	487	5309	5308	5743	2871	6211	6210
3121	156	3547	3546	3967	1983	4421	340	4877	4876	5323	1774	5749	5748	6217	1036
3137	784	3557	3556	3989	3988	4423	737	4889	2444	5333	5332	5779	5778	6221	1244
3163	1054	3559	1779	4001	1000	4441	2220	4903	2451	5347	198	5783	2891	6229	6228
3167	1583	3571	3570	4003	4002	4447	2223	4909	1636	5351	2675	5791	2895	6247	3123
3169	1584	3581	3580	4007	2003	4451	4450	4919	2459	5381	1076	5801	2900	6257	3128
3181	1060	3583	1791	4013	4012	4457	1114	4931	986	5387	5386	5807	2903	6263	3131
3187	3186	3593	1796	4019	4018	4463	2231	4933	4932	5393	1348	5813	5812	6269	6268
3191	55	3607	601	4021	4020	4481	560	4937	1234	5399	2699	5821	388	6271	1045
3203	3202	3613	3612	4027	1342	4483	4482	4943	2471	5407	2703	5827	5826	6277	6276
3209	1604	3617	1808	4049	506	4493	4492	4951	2475	5413	1804	5839	2919	6287	3143
3217	804	3623	1811	4051	50	4507	4506	4957	4956	5417	2708	5843	5842	6299	6298
3221	644	3631	605	4057	169	4513	47	4967	2483	5419	42	5849	2924	6301	2100
3229	1076	3637	3636	4073	2036	4517	4516	4969	2484	5431	2715	5851	5850	6311	3155
3251	650	3643	3642	4079	2039	4519	753	4973	4972	5437	1812	5857	2928	6317	6316
3253	3252	3659	3658	4091	4090	4523	266	4987	4986	5441	544	5861	1172	6323	6322
3257	407	3671	1835	4093	4092	4547	4546	4993	624	5443	5442	5867	838	6329	3164
3259	1086	3673	918	4099	4098	4549	1516	4999	357	5449	908	5869	5868	6337	288
3271	545	3677	3676	4111	2055	4561	2280	5003	5002	5471	547	5879	2939	6343	3171
3299	3298	3691	3690	4127	2063	4567	761	5009	2504	5477	5476	5881	1470	6353	397
3301	660	3697	1848	4129	688	4583	2291	5011	5010	5479	2739	5897	2948	6359	3179
3307	3306	3701	3700	4133	4132	4591	2295	5021	1004	5483	5482	5903	2951	6361	53
3313	828	3709	3708	4139	4138	4597	1532	5023	2511	5501	5500	5923	5922	6367	3183
3319	1659	3719	1859	4153	346	4603	4602	5039	2519	5503	917	5927	2963	6373	6372
3323	3322	3727	1863	4157	4156	4621	4620	5051	5050	5507	5506	5939	5938	6379	6378
3329	1664	3733	3732	4159	2079	4637	4636	5059	5058	5519	2759	5953	992	6389	6388

P	S(P)	P	S (P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)	P	S(P)
6397	6396	6883	6882	7369	1228	7829	7828	8297	4148	8779	2926	9241	2310	9719	43
6421	2140	6899	6898	7393	264	7841	1960	8311	4155	8783	4391	9257	4628	9721	810
6427	2142	6907	6906	7411	7410	7853	7852	8317	308	8803	8802	9277	3092	9733	9732
6449	806	6911	3455	7417	3708	7867	874	8329	4164	8807	4403	9281	1160	9739	3246
6451	2150	6917	6916	7433	3716	7873	1312	8353	464	8819	8818	9283	9282	9743	4871
6469	6468	6947	6946	7451	7450	7877	7876	8363	8362	8821	8820	9293	9292	9749	9748
6473	3236	6949	6948	7457	3728	7879	3939	8369	2092	8831	883	9311	4655	9767	4883
6481	810	6959	497	7459	7458	7883	7882	8377	1396	8837	8836	9319	4659	9769	1221
6491	6490	6961	1160	7477	7476	7901	7900	8387	8386	8839	4419	9323	9322	9781	652
6521	1630	6967	1161	7481	1870	7907	7906	8389	2796	8849	4424	9337	2334	9787	3262
6529	102	6971	6970	7487	197	7919	3959	8419	2806	8861	8860	9341	9340	9791	4895
6547	6546	6977	3488	7489	468	7927	1321	8423	4211	8863	1477	9343	4671	9803	9802
6551	3275	6983	3491	7499	7498	7933	7932	8429	8428	8867	8866	9349	9348	9811	3270
6553	117	6991	3495	7507	7506	7937	3968	8431	4215	8887	4443	9371	9370	9817	1636
6563	386	6997	2332	7517	7516	7949	7948	8443	8442	8893	2964	9377	2344	9829	1404
6569	1642	7001	500	7523	7522	7951	3975	8447	4223	8923	8922	9391	4695	9833	4916
6571	1314	7013	7012	7529	941	7963	2654	8461	1692	8929	496	9397	9396	9839	4919
6577	3288	7019	7018	7537	1256	7993	999	8467	8466	8933	8932	9403	3134	9851	9850
6581	1316	7027	7026	7541	7540	8009	4004	8501	1700	8941	2980	9413	724	9857	4928
6599	3299	7039	3519	7547	7546	8011	2670	8513	4256	8951	4475	9419	9418	9859	9858
6607	3303	7043	7042	7549	7548	8017	4008	8521	4260	8963	8962	9421	9420	9871	4935
6619	6618	7057	392	7559	3779	8039	4019	8527	4263	8969	1121	9431	943	9883	9882
6637	6636	7069	7068	7561	3780	8053	8052	8537	2134	8971	8970	9433	4716	9887	4943
6653	6652	7079	3539	7573	7572	8059	2686	8539	8538	8999	4499	9437	9436	9901	9900
6659	6658	7103	3551	7577	1894	8069	8068	8543	4271	9001	2250	9439	1573	9907	9906
6661	2220	7109	7108	7583	3791	8081	1010	8563	8562	9007	4503	9461	1892	9923	9922
6673	1112	7121	890	7589	7588	8087	4043	8573	8572	9011	9010	9463	1577	9929	292
6679	159	7127	3563	7591	3795	8089	4044	8581	660	9013	3004	9467	9466	9931	3310
6689	836	7129	1782	7603	7602	8093	8092	8597	8596	9029	9028	9473	2368	9941	9940
6691	6690	7151	325	7607	3803	8101	100	8599	4299	9041	904	9479	4739	9949	9948
6701	6700	7159	3579	7621	7620	8111	4055	8609	1076	9043	3014	9491	9490	9967	4983
6703	3351	7177	3588	7639	1273	8117	8116	8623	4311	9049	4524	9497	4748	9973	3324
6709	6708	7187	7186	7643	7642	8123	8122	8627	8626	9059	9058	9511	317		
6719	3359	7193	3596	7649	3824	8147	8146	8629	2876	9067	3022	9521	476		
6733	6732	7207	3603	7669	7668	8161	408	8641	4320	9091	1818	9533	9532		
6737	3368	7211	7210	7673	3836	8167	1361	8647	4323	9103	4551	9539	9538		
6761	1690	7213	2404	7681	3840	8171	8170	8663	4331	9109	3036	9547	4546		
6763	6762	7219	7218	7687	3843	8179	8178	8669	8668	9127	4563	9551	4775		
6779	6778	7229	7228	7691	7690	8191	13	8677	8676	9133	3044	9587	9586		
6781	6780	7237	7236	7699	2566	8209	2052	8681	124	9137	1142	9601	2400		
6791	679	7243	7242	7703	3851	8219	8218	8689	4344	9151	4575	9613	9612		
6793	1698	7247	3623	7717	7716	8221	8220	8693	8692	9157	1308	9619	9618		
6803	6802	7253	7252	7723	2574	8231	4115	8699	8698	9161	4580	9623	283		
6823	3411	7283	7282	7727	3863	8223	2058	8707	2902	9173	9172	9629	9628		
6827	6826	7297	3648	7741	2580	8237	8236	8713	363	9181	9180	9631	1605		
6829	6828	7307	7306	7753	323	8243	8242	8719	1453	9187	3062	9643	9462		
6833	3416	7309	1044	7757	7756	8263	4131	8731	8730	9199	4599	9649	402		
6841	1710	7321	1220	7759	3879	8269	8268	8737	4368	9203	9202	9661	9660		
6857	857	7331	7330	7789	7788	8273	2068	8741	8740	9209	2302	9677	9676		
6863	3431	7333	2444	7793	1948	8287	1381	8747	8746	9221	9220	9679	4839		
6869	6868	7349	7348	7817	1954	8291	8290	8753	4376	9227	9226	9689	4844		
6871	687	7351	525	7823	3911	8293	8292	8761	365	9239	4619	9697	2424		