The Decomposition of Stochastic Systems

By Taiho KANAOKA and Shingo TOMITA

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Abstract

This paper proposes a method to decompose any n-state stochastic system into m r-state component stochastic systems.

The basic tool in our method is a partition with substitution property which exists on the set of states of the interconnected stochastic system. Using our method it is mainly shown that there exists a relationship among n, m and r, and for one outer input, some specific number of transition matrices are assigned to each component.

1. Introduction

On the decomposition theory of stochastic systems (or automata), since Bacon (1) first applied the concept of substitution property (SP) which was introduced by Hartmanis and Stearns (2, 3) for deterministic case, several papers are appeared (4–8).

For automorphism groups of stochastic systems, an iterative decomposition of Giorgadze and Safiulina (7) is avairable, however, when we investigate the decomposition for any stochastic system the concept of SP plays very important role.

A partition with SP on the set of states dose not always exist for any transition matrix. To improve such weak point Fujimoto and Fukao (4) suggested a idea of state splitting. After that, based on the concept of state splitting Paz(5) gave a result that it is possible to decompose any n-state stochastic system into interconnected n–1 two-state stochastic systems. Moreover, extending the idea of Paz, Kikuchi and Fujino (8) suggested that any stochastic system is decomposable into some interconnected q-state r-neighbor component stochastic systems, in which some of transition matrices may be pseud stochastic matrices.

Furthermore, based on the SP, authors (10) gave a new method to decompose any n-state stochastic system into m 2-state component stochastic systems, and using the method some interesting results were shown.

In this paper, we propose a method that any n-state stochastic system can be decompose into m r-state component stochastic systems. Basic idea of our method is a partition with SP which exists on the set of states of the interconnected stochastic system. Using our method it is mainly shown that there exists a relationship among n, m, and r, and for one outer input, some specific number of transition matrices are assigned to each component.

Department of Electronics Engineering

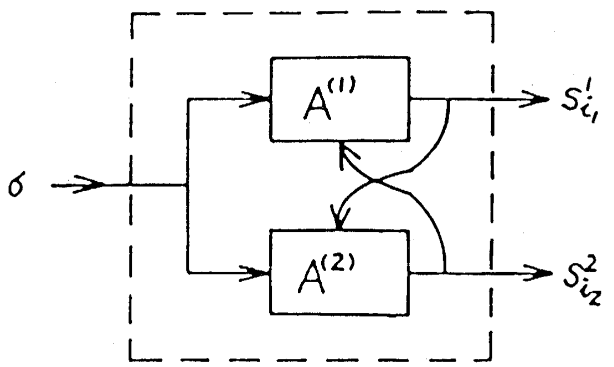

Fig. 1 Interconnection of stochastic systems.

2. Preliminary

The stochastic system is defined as follows.

(Definition 1) A stochastic system is a three-tuple $A=[S, \Sigma, \{A(\sigma)\}]$, where S is a finite set of states, Σ is a finite set of inputs and $\{A(\sigma)\}$ $(\sigma \in \Sigma)$ is a finite set of stochastic matrices.

If, at time t, the system is in a state $s_i \in S$ and receives an input $\sigma \in \Sigma$, then it moves to a state $s_j \in S$ with probability $a_{i,j}(\sigma)$, where $A(\sigma)=[a_{i,j}(\sigma)]$. Thus $A(\sigma)$ is an $|S|$-dimensional stochastic matrix, where $|U|$ denotes the number of element in a set U.

(Definition 2) A component stochastic system is defined as follows.

$$A^{(i)}=[S^i, \Sigma\times S^1\times S^2\times\cdots\times S^{i-1}\times S^{i+1}\times\cdots\times S^m, \{A^{(i)}(\sigma, s^1_{i_1}, s^2_{i_2},\ldots, s^{i-1}_{i_{i-1}}, s^{i+1}_{i_i}, \ldots, s^m_{i_{m-1}})\}] \quad (1\leqq i\leqq m,\ m\geqq 2)$$

where S^i is a finite set of states, $\Sigma\times S^1\times S^2\times\cdots\times S^{i-1}\times S^{i+1}\times\cdots\times S^m=S^{\triangle}$ is a finite set of inputs and $\{A^{(i)}(\sigma, s^1_i, s^2_{i_2},\ldots, s^{i-1}_{i_{i-1}}, s^{i+1}_{i_i},\ldots, s^m_{i_{m-1}})\}$ $((\sigma, s^1_{i_1}, s^2_{i_2},\ldots, s^{i-1}_{i_{i-1}}, s^{i+1}_{i_i},\ldots, s^m_{i_{m-1}})\in S^{\triangle})$ is a finite set of stochastic matrices which designates the transition structure of $A^{(i)}$.

The component stochastic system $A^{(i)}$ is interconnected with all other component stochastic systems, so the transition matrix of $A^{(i)}$ is determined by a input σ and all of the neighboring states. From now on, to distinguish from the inputs as the neighboring states, σ is called an outer input. And for simplicity we often say component instead of component stochastic system.

(Definition 3) A stochastic system which is interconnected with m r-state components $A^{(1)}, A^{(2)},\ldots$ and $A^{(m)}$ is defined as $B=[Z, \Sigma, \{B(\sigma)\}]$, where $Z=S^1\times S^2\times\cdots\times S^m$, $|S^i|=r$ $(1\leqq i\leqq m)$ and

$$B(\sigma)=[b_{u,v}(\sigma)] \qquad (0\leqq u,\ v\leqq r^m-1) \tag{1}$$

$$b_{u,v}(\sigma)=\prod_{k=1}^{m} a^{(k)}_{i_k,j_k}(\sigma, s^1_i, s^2_{i_2},\ldots, s^{k-1}_{i_{k-1}}, s^{k+1}_{i_{k+1}},\ldots, s^m_{i_m}). \tag{2}$$

Here, $b_{u,v}(\sigma)$ designates the transition probability such that if, at time t, the system B is in a state $z_u\in Z$ and receives an outer input σ then it moves to a state $z_v\in Z$. $z_u=$

$(s_i^1, s_{i_2}^2, \ldots, s_{i_m}^m)$ and $z_v=(s_j^1, s_{j_2}^2, \ldots, s_{j_m}^m)$ are m dimenional vectors, where $(u)_{10}=(i_1 i_2 \cdots i_m)_r$, $(v)_{10}=(j_1 j_2 \cdots j_m)_r$ and $(\ )_q$ denotes q-adic number.

(Definition 4) When a state of stochastic system B is $z_i((i)_{10}=(i_1 i_2 \cdots i_m)_r)$, for each k $(1 \leqq k \leqq m)$, $i_k \in \{0, 1, \ldots, r-1\})$, a probability from a state $s_{i_v}^v$ to $s_{j_v}^v$ in the component $A^{(v)}$, for a outer input σ, is denoted by $h_i^{(v)}(i_v \sim j_v)(\sigma)$.

(Definition 5) A partition Π on a set K is defined as follows.

$$\Pi=\{\Pi_i \mid \Pi_i \subset K,\ r \neq q \longrightarrow \Pi_r \cap \Pi_q=\phi,\ \cup \Pi_i=K\}$$

Now, we give the most important definition for the decomposition of stochastic system in this paper.

(Definition 6) For a stochastic system $A=[S, \Sigma, \{A(\sigma)\}]$, a partition $\Pi=\{\Pi_i \mid 1 \leqq i \leqq t\}$ on the state set S is said to have the substitution property (SP) if and only if it satisfies the following condition;

for each Π_k, Π_l and each outer input $\sigma \in \Sigma$, if $s_i, s_j \in \Pi_k$, then

$$\sum_{s_f \in \Pi_l} a_{i,f}(\sigma)=\sum_{s_f \in \Pi_l} a_{j,f}(\sigma). \tag{3}$$

(Definition 7) Let $A=[S, \Sigma, \{A(\sigma)\}]$ be a stochastic system and $\Pi=\{\Pi_i \mid 1 \leqq i \leqq t\}$ be a partition with SP.

(7.1) $A^*=[S^*, \Sigma, \{A^*(\sigma)\}]$ is called a system merged with the partition Π, where $S^*=\{\Pi_1, \Pi_2, \ldots, \Pi_t\}$ and for each i, j $(1 \leqq i, j \leqq t)$

$$a_{i,j}^*(\sigma)=\sum_{s_f \in \Pi_j} a_{l,f}(\sigma) \qquad (S_l \in \Pi_i) \tag{4}$$

It must be noted that, for each μ, $v(s_\mu, s_v \in \Pi_i)$, since Π is satisfied with SP,

$$\sum_{s_f \in \Pi_j} a_{\mu,f}(\sigma)=\sum_{s_f \in \Pi_j} a_{v,f}(\sigma).$$

(7.2) $\bar{A}(\sigma)=[\bar{a}_{i,j}(\sigma)]$ $(1 \leqq i \leqq |S|,\ 1 \leqq j \leqq t)$ is a matrix merged with partition Π, where $\bar{a}_{i,j}(\sigma)=\sum_{s_f \in \Pi_j} a_{i,f}(\sigma)$.

(Definition 8) Stochastic systems $A_1=[S_1, \Sigma, \{A_1(\sigma)\}]$ and $A_2=[S_2, \Sigma, \{A_2(\sigma)\}]$ are isomorphic if there exists a one to one mapping f between S_1 and S_2, and for any $s_i, s_j \in S_1$, $\sigma \in \Sigma$

$$a_{1_{i,j}}(\sigma)=a_{2_{f(i),f(j)}}(\sigma). \tag{5}$$

In the following definition, we give a decomposability of a stochastic system.

(Definition 9) Let $A=[S, \Sigma, \{A(\sigma)\}]$ be a stochastic system and $B=[Z, \Sigma, \{B(\sigma)\}]$ be a stochastic system interconnected by components $A^{(1)}, A^{(2)}, \ldots$ and $A^{(m)}$, and on the set $Z=S^1 \times S^2 \times \cdots \times S^m$, let there be a partition Π with SP. And let B^* be a stochastic system merged with Π. Then, A is said to be decomposable into interconnected component stochastic systems $A^{(1)}, A^{(2)}, \ldots$ and $A^{(m)}$, if A if isomorphic to B^* and $|S^i|<|S|$ for each i $(1 \leqq i \leqq m)$.

In the decomposition of stochastic systems, if the theory of decomposition is established for only one outer input, then for other inputs it is entirely the same as that one. So, in the subsequence section, we discuss for only one outer input σ and for simplicity, we often use the notations $a_{i,j}$, $b_{i,j}$, $\bar{b}_{i,j}$, etc instead of $a_{i,j}(\sigma)$, $b_{i,j}(\sigma)$, $\bar{b}_{i,j}(\sigma)$, etc.

3. Partition Π^{τ} and Decomposition of Stochastic Systems

For each $u(0 \leqq u \leqq r^{m-1}-1)$, we define the transition matrix of a component $A^{(v)}$ $(1 \leqq v \leqq m)$ as follows.

$$A^{(v)}(\sigma, \rho_u) = [a_{i,j}^{v,u}] \qquad (0 \leqq i, j \leqq r-1) \tag{6}$$

where $\rho_u = (s_{i_1}^1, s_{i_2}^2, \ldots, s_{i_{v-1}}^{v-1}, s_{i_{v+1}}^{v+1}, \ldots, s_{i_m}^m)$ and $(u)_{10} = (i_1 i_2 \ldots i_{v-1} i_{v+1} \ldots i_m)_r$.

We now give a following lemma concerning with a partition on Z.

(Lemma 1) Let $\Pi = \{\Pi_i | 1 \leqq i \leqq t\} (t > r)$ be a partition on $Z$and $\eta_i = (\bar{b}_{i,1}, \bar{b}_{i,2}, \ldots, \bar{b}_{i,t})$ be an i-th row of the matrix $\bar{B}(\sigma) = [\bar{b}_{i,j}(\sigma)]$ $(0 \leqq i \leqq r^m - 1,\ 1 \leqq j \leqq t)$. Then, for any stochastic vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_t)$, there exist solutions $a_{i_v,j}^{v,u_v}$ $(1 \leqq v \leqq m, 0 \leqq j \leqq r-1)$ of a equation $\eta_i = \boldsymbol{c}$ only if $t \leqq m(r-1)+1$. Where $(i)_{10} = (i_1 i_2 \cdots i_m)_r$.

Proof. For each $v(1 \leqq v \leqq m)$

$$\sum_{j=0}^{r-1} a_{i_v,j}^{v,u_v} = 1, \quad \text{where } (u_v)_{10} = (i_1 i_2 \cdots i_{v-1} i_{v+1} \cdots i_m)_r$$

$$\text{and} \quad \sum_{j=1}^{t} \bar{b}_{i,j} = \sum_{j=1}^{t} c_j = 1.$$

From this, finding at most $m(r-1)$ unknown quantities $a_{i_v,j}^{v,u_v}$ $(0 \leqq v \leqq m,\ 0 \leqq j \leqq r-2)$ reduce to solve the equations as follows.

$$\begin{cases} \bar{b}_{i,1} = c_1 \\ \bar{b}_{i,2} = c_2 \\ \quad \vdots \\ \bar{b}_{i,t-1} = c_{t-1} \end{cases} \tag{7}$$

Thus, obviously (7) has solutions for any $\boldsymbol{c}$ only if $t \leqq m(r-1)+1$. Q.E.D.

From Lemma 1 following theorem is easily derived. The proof is ommited here.

(Theorem 1) For each $k(k > m(r-1)+1)$, there exists some k-state stochastic system which is impossible to realize by interconnected stochastic system with m r-state components.

In the subsequent discussion, for each $k(r < k \leqq m(r-1)+1)$ any k-state stochasitc system is decomposable into m r-state component stochastic systems.

Firstly we introduce a partition Π^{τ} on the set of states $Z = \{z_0, z_1, \ldots, z_{r^m-1}\}$ as follows.

$$\Pi^{\tau} = \{\Pi_i^{\tau} | 1 \leqq i \leqq r+\tau\} \quad (1 \leqq \tau \leqq (m-1)(r-1)) \tag{8}$$

where for any $z_u, z_v (z_u \in \Pi_\rho^\tau, z_v \in \Pi_\gamma^\tau)$ $\rho < \gamma$ if and only if $u < v$. And for each $k(1 \leqq k \leqq m-1)$, when $(k-1)(r-1)+1 \leqq \tau \leqq k(r-1)$

$$|\Pi_i^\tau| = \begin{cases} r^{m-t}, & \text{if } (t-1)(r-1)+1 \leqq i \leqq t(r-1), \quad (1 \leqq t \leqq k) \\ [r-\{\tau-(k-1)(r-1)\}]r^{m-k-1}, & \text{if } i=k(r-1)+1 \\ r^{m-k-1}, & \text{if } k(r-1)+2 \leqq i \leqq r+\tau \end{cases} \tag{9}$$

For example, in the case of $r=3$ and $m=3$,

$$\Pi^1 = \{\{z_0, z_1, \ldots, z_8\}, \{z_9, z_{10}, \ldots, z_{17}\}, \{z_{18}, z_{19}, \ldots, z_{23}\}, \{z_{24}, z_{25}, z_{26}\}\},$$

$$\Pi^2 = \{\{z_0, z_1, \ldots, z_8\}, \{z_9, z_{10}, \ldots, z_{17}\}, \{z_{18}, z_{19}, z_{20}\}, \{z_{21}, z_{22}, z_{23}\}, \{z_{24}, z_{25}, z_{26}\}\},$$

$$\Pi^3 = \{\{z_0, z_1, \ldots, z_8\}, \{z_9, z_{10}, \ldots, z_{17}\}, \{z_{18}, z_{19}, z_{20}\}, \{z_{21}, z_{22}, z_{23}\}, \{z_{24}, z_{25}\}, \{z_{26}\}\},$$

$$\Pi^4 = \{\{z_0, z_1, \ldots, z_8\}, \{z_9, z_{10}, \ldots, z_{17}\}, \{z_{18}, z_{19}, z_{20}\}, \{z_{21}, z_{22}, z_{23}\}, \{z_{24}\}, \{z_{25}\}, \{z_{26}\}\},$$

And let $P = m - \log_m |\Pi_{r+\tau}^\tau|$.

Then concerning the partition Π^τ, we obtain the result as follows

(Theorem 2) Let $B=[Z, \Sigma, \{B(\sigma)\}]$ be a interconnected stochastic system which is obtained from m r-state component $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$. Then, $\bar{B}(\sigma)=[\bar{b}_{i,j}(\sigma)]$ $(0 \leqq i \leqq r^m-1, 1 \leqq j \leqq |\Pi^\tau|)$ which is the matrix merged with the partition Π^τ is dependent on P components and independent on $m-P$ components.

Proof. A transition probability such that the interconnected stochastic system enters a state z_j from $z_i((i)_{10}=(i_1 i_2 \cdots i_m)_r, (j)_{10}=(j_1 j_2 \cdots j_m)_r$ and for each $k(1 \leqq k \leqq m)$ $i_k, j_k \in \{0, 1, \ldots, r-1\})$

$$b_{i,j} = \prod_{\nu=1}^{m} h_i^{(\nu)}(i_\nu \sim j_\nu) \tag{10}$$

Let $\eta_i = (\bar{b}_{i,1}, \bar{b}_{i,2}, \ldots, \bar{b}_{i,r+\tau})$ be an i-th row of the matrix $\bar{B}(\sigma)$. Then for $\Pi_l^\tau \in \Pi^\tau$ such that $|\Pi_l^\tau| = tr^{m-k}(1 \leqq t \leqq r-1, 1 \leqq k \leqq m)$, we can put

(1) in the case of $t=1$

$$\Pi_l^\tau = \{z_{(\overbrace{r-1,\ldots,r-1}^{k-1},\alpha,\overbrace{0,\ldots,0}^{m-k})_r}, \ldots, z_{(r-1,\ldots,r-1,\alpha,r-1,\ldots,r-1)_r}\} \qquad (1 \leqq \alpha \leqq r-1)$$

(2) in the case of $t \neq 1$

$$\Pi_l^\tau = \{z_{(\overbrace{r-1,\ldots,r-1}^{k-1},0,\overbrace{0,\ldots,0}^{m-k})_r}, \ldots, z_{(r-1,\ldots,r-1,0,r-1,\ldots,r-1)_r}, z_{(r-1,\ldots,r-1,1,0,\ldots,0)_r}, \ldots,$$

$$z_{(r-1,\ldots,r-1,1,r-1,\ldots,r-1)r},$$
$$\vdots$$
$$z_{(r-1,\ldots,r-1,t-1,0,\ldots,0)r},\ldots,$$
$$z_{(r-1,\ldots,r-1,t-1,r-1,\ldots,r-1)r}\}$$

So the transition probability from a state $z_i \in Z$ to all states in Π_l^τ is

$$\bar{b}_{i,l}=\begin{cases}\displaystyle\sum_{j_{k+1},j_{k+2},\ldots,j_m\in M} h_i^{(1)}(i_1\sim r-1)h_i^{(2)}(i_2\sim r-1)\cdots \\ \qquad h_i^{(k-1)}(i_{k-1}\sim r-1)h_i^{(k)}(i_k\sim\alpha)h_i^{(k+1)}(i_{k+1}\sim i_{k+1})\cdots \\ \qquad h_i^{(m)}(i_m\sim j_m), \qquad if \quad t=1 \\ \displaystyle\sum_{j_{k+1},j_{k+2},\ldots,j_m\in M} h_i^{(1)}(i_1\sim r-1)h_i^{(2)}(i_2\sim r-1)\cdots \\ \qquad h_i^{(k-1)}(i_{k-1}\sim r-1)\{h_i^{(k)}(i_k\sim 0)+h_i^{(k)}(i_k\sim 1)+\cdots+ \\ \qquad h_i^{(k)}(i_k\sim t-1)\}h_i^{(k+1)}(i_{k+1}\sim j_{k+1})\cdots h_i^{(m)}(i_m\sim j_m), \qquad if \quad t\neq 1\end{cases} \tag{11}$$

, where $M=\{0, 1,\ldots, r-1\}$
It follows from for each $v(k+1\leqq v\leqq m)$

$$\sum_{j_v\in M} h_i^{(v)}(i_v\sim j_v)=1 \tag{12}$$

that

$$\sum_{j_{k+1},j_{k+2},\ldots,j_m\in M} h_i^{(k+1)}(i_{k+1}\sim j_{k+1})\cdots h_i^{(m)}(i_m\sim j_m)=1 \tag{13}$$

Thus, it is easily seen from (11) and (12) that $\bar{b}_{i,j}$ is dependent on the components $A^{(1)}, A^{(2)},\ldots, A^{(k)}$ and independent on $A^{(k+1)}\ldots, A^{(m)}$. In $|\Pi_i^\tau|=tr^{m-k}$, k is the largest when $i=r+\tau$, namely $|\Pi_{r+\tau}^\tau|=r^{m-P}$. From this, $\bar{B}(\sigma)$ is dependent on P components $A^{(1)}, A^{(2)},\ldots, A^{(P)}$ and independent on $m-P$ components. Q. E. D.

Furthermore, concerning each row of matrix $\bar{B}(\sigma)$ we can show the following theorem.
(Theorem 3) Let $\eta_i=(\bar{b}_{i,1}, \bar{b}_{i,2},\ldots, \bar{b}_{i,r+\tau})$ be an i-th row of the matrix $\bar{B}(\sigma)=[\bar{b}_{i,j}(\sigma)]$ $(0\leqq i\leqq r^m-1, 1\leqq j\leqq r+\tau)$. Then, there exist the components $A^{(1)}, A^{(2)},\ldots, A^{(m)}$ such that for any stochastic vector $\mathbf{c}=(c_1, c_2,\ldots, c_{r+\tau})$

$$\boldsymbol{\eta}_i=\boldsymbol{c} \tag{14}$$

Proof. Let

$$(|\Pi_1^\tau|, |\Pi_2^\tau|,\ldots, |\Pi_{r+\tau}^\tau|)=(\overbrace{r^{m-1},\ldots, r^{m-1}}^{r-1}, \overbrace{r^{m-2},\ldots, r^{m-2}}^{r-1},\ldots, \underbrace{r^{m-k},\ldots, r^{m-k}}_{r-1}, \alpha r^{m-(k+1)}, \underbrace{r^{m-(k+1)},\ldots, r^{m-(k+1)}}_{r-\alpha}) \tag{15}$$

, where $1\leqq\alpha\leqq r-1$, $1\leqq k\leqq m-1$, $k(r-1)+r-\alpha+1=|\Pi^\tau|$

Then, from (12) we get the following equations

$$\bar{b}_{i,1}=x_0^1,\ \bar{b}_{i,2}=x_1^1,\ldots,\ \bar{b}_{i,r-1}=x_{r-2}^1$$
$$\bar{b}_{i,r}=x_{r-1}^1x_0^2,\ \bar{b}_{i,r+1}=x_{r-1}^1x_1^2,\ldots,\ \bar{b}_{i,2(r-1)}=x_{r-1}^1x_{r-2}^2$$
$$\bar{b}_{i,(k-1)(r-1)+1}=x_{r-1}^1\cdots x_{r-1}^{k-1}x_0^k,\ldots,\ \bar{b}_{i,k(r-1)}=x_{r-1}^1\cdots x_{r-1}^{k-1}x_{r-2}^k$$
$$\bar{b}_{i,k(r-1)+1}=x_{r-1}^1\cdots x_{r-1}^k(x_0^{k+1}+x_1^{k+1}+\cdots+x_{\alpha-1}^{k+1})$$
$$\bar{b}_{i,k(r-1)+2}=x_{r-1}^1\cdots x_{r-1}^k x_\alpha^{k+1},\ldots,\ \bar{b}_{i,k(r-1)+r-\alpha+1}=x_{r-1}^1\cdots x_{r-1}^k x_{r-1}^{k+1} \qquad (16)$$

where $h_i^{(\nu)}(i_\nu\sim i_\nu)=x_{j_\nu}^\nu \quad (1\leqq\nu\leqq m,\ 0\leqq i_\nu,\ j_\nu\leqq r-1)$.

And from

$$\sum_{i=0}^{r-1}x_i^\nu=1 \quad (1\leqq\nu\leqq k+1) \qquad (17)$$
$$\sum_{i=1}^{k(r-1)+r-d+1}c_i=1,\quad \text{where}\quad c_j=\bar{b}_{i,j},$$

we obtain the solutions of (16) as follows.

$$\begin{cases} x_j^\mu=\begin{cases} c_{j+1}, & \text{if}\quad \mu=1,\quad 0\leqq j\leqq r-2 \\ \dfrac{c_{(\mu-1)(r-1)+j+1}}{1-\sum_{i=1}^{(\mu-1)(r-1)}c_i}, & \text{if}\quad 2\leqq\mu\leqq k,\quad 0\leqq j\leqq r-2 \end{cases} \\ X=\dfrac{c_{k(r-1)+1}}{1-\sum_{i=1}^{k(r-1)}c_i} \\ x_{\alpha+j}^{k+1}=\dfrac{c_{k(r-1)+j+2}}{1-\sum_{i=1}^{k(r-1)}c_i}, \quad \text{if}\quad 0\leqq j\leqq r-2-\alpha, \end{cases} \qquad (18)$$

, where $X=x_0^{k+1}+x_1^{k+1}+\cdots+x_{\alpha-1}^{k+1}$.

In (18), when a denominator is zero the solution is arbitrary, and $x_{r-1}^\mu(1\leqq\mu\leqq k+1)$ are easily derived from (17) and (18). Also, x_0^{k+1}, $x_1^{k+1},\ldots,x_{\alpha-1}^{k+1}$ are arbitrary real number (non zero less than 1) such that $\sum_{i=0}^{\alpha-1}x_i^{k+1}=X$. Thus, for any stochastic vector $\boldsymbol{c}$, the solutions of $\eta_i=\boldsymbol{c}$, namely, the components $A^{(1)}$, $A^{(2)},\ldots,A^{(m)}(A^{(\nu)}(1\leqq\nu\leqq k)$ is unique) always exist. Q.E.D.

Furthermore, we get the following theorem.

(Theorem 4) In the matrix $B(\sigma)=[b_{i,j}]$ $(0\leqq i,\ j\leqq r^m-1)$, if $i\neq i'$ then for any j, $j'(0\leqq j,$ $j'\leqq r^m-1)$ $b_{i,j}$ and $b_{i',j'}$ are independent.

Proof, $b_{i,j}$, $b_{i',j'}$ can be denote as follows.

$$b_{i,j}=a_{\mu_1,\nu_1}^{1,u_1}a_{\mu_2,\nu_2}^{2,u_2}\cdots a_{\mu_m,\nu_m}^{m,u_m}$$

$$b_{i',j'}=a^{1,u_1'}_{\mu_1',v_1'}a^{2,u_2'}_{\mu_2',v_2'}\cdots a^{m,u_m'}_{\mu_m',v_m'}$$

Now let $(i)_{10}=(i_1i_2\cdots i_m)_r$, $(i')_{10}=(i_1'i_2'\cdots i_m')_r$. If $i\neq i'$, for at least one $k(1\leqq k\leqq m)$ $i_k\neq i_k'$. It follows from this and (6) that for each $k(1\leqq k\leqq m)$ $u_k\neq u_k'$ or $\mu_k=\mu_k'$. Thus, it is easily seen that if $i\neq i'$, $b_{i,j}$ and $b_{i',j'}$ are independent. Q.E.D.

From Theorem 4, following theorem is easily derived. So the proof is omitted.
(Theorem 5) For each i, $i'(0\leqq i, i'\leqq r-1)$, if $i\neq i'$, η_i and η_i' which are the i-th and i'-th rows of $B(\sigma)$ repectively are independent.
(Theorem 6) Let $B=[Z, \Sigma, \{B(\sigma)\}]$ be a interconnected stochastic system obtained from m r-state components $A^{(1)}, A^{(2)},\ldots, A^{(m)}$. Then, we can make a partition Π^τ on Z to be satisfied SP. And let $B^*=[Z, \Sigma, \{B^*(\sigma)\}]$ be a stochastic system merged with the partition Π^τ which is satisfied SP. Then, for any stochastic matrix $N=[n_{i,j}](1\leqq i.\ j\leqq r+\tau)$, there exist m r-state components $A^{(1)}, A^{(2)},\ldots, A^{(m)}$ such that

$$B^*(\sigma)=N \tag{19}$$

Proof. From Theorem 3 and 5, the proof is straightforward, and so is omitted here. Q.E.D.

From the above discussion, we can easily show a decomposition theory as follows. The proof is omitted here.
(Theorem 7) Any n-state stochastic system is decomposable into interconnected $m(m\geqq 2)$ r-state $(2\leqq r\leqq n-1)$ component stochastic systems. Where among n, r and m the following condition is satsisfied.

$$r<n\leqq m(r-1)+1 \tag{20}$$

Eventually, when we decompose a n-state stochastic system $A=[S, \Sigma, \{A(\sigma)\}]$ into m r-state components, each component is determined by solving the equation

$$B^*(\sigma)=A(\sigma) \tag{21}$$

In that time we adopt a partition Π^τ such that $|\Pi^\tau|=n$. And from Theorem 2 if $n=|\Pi^{(m-1)(r-1)}|=m(r-1)+1$ we must determine all m components, however, in general $\log_r |\Pi^\tau_{r+\tau}|$ components are arbitrary.

Furthermore, concerning a number of requied transition martices of each component for one outer input we can show the following theorem.
(Theorem 8) Let $\Pi^\circ=\{\Pi^0_i \mid i\leqq 1\leqq r\}$ (for each $i(1\leqq i\leqq r)$, $|\Pi^0_i|=r^{m-1}$), and $\tilde{\Pi}^\tau=\{\Pi^\tau_i \mid r\leqq i\leqq r+\tau\}$ in $\Pi^\tau=\{\Pi^\tau_i \mid 1\leqq i\leqq r+\tau\}$ be a partition on Π^0_r. Where, similar to Π^τ, Π° is a partition on Z and for any $z_u\in\Pi^0_\rho$, $z_v\in\Pi^0_\gamma$, $\rho<\gamma$ if and only if $u<v$. Then, a number of required transition matrices of each component $A^{(\nu)}$ $(1\leqq\nu\leqq P)$ for one outer input is $|\tilde{\Pi}^\tau|(=\tau+1)$.

Proof. We pay attention to a component $A^{(1)}$. If the interconnected stochastic system B is in a state $z_i((i)_{10}=(i_1i_2\cdots i_m)_r)$, input of $A^{(1)}$ as the neighboring states is

$$\rho_u\colon (u)_{10}=(i_2i_3\cdots i_m)_r.$$

And, from (12) and the fact that $A^{(1)}$ has all inputs $\rho_u(0\leqq u\leqq r^{m-1}-1)$ for each Π_t^0 $(1\leqq t\leqq r)$, for each $t(1\leqq t\leqq r)$

$$\bar{b}_{i,t}=\begin{cases} a_{1,t}^{1,i}, & \text{if} \quad z_i\in\Pi_1^0 \\ a_{2,t}^{1,i-r^{m-1}}, & \text{if} \quad z_i\in\Pi_2^0 \\ \vdots & \quad\quad \vdots \\ a_{r,t}^{1,i-(r-1)r^{m-1}}, & \text{if} \quad z_i\in\Pi_r^0 \end{cases} \tag{23}$$

, where $\bar{B}(\sigma)=[\bar{b}_{i,j}]$ $(0\leqq i\leqq r^m-1,\ 1\leqq j\leqq r)$ is a matrix merged with the partition Π^0. Now, let B^* be a stochastic system merged with Π^0. Since Π^0 has to be satisfied SP, for each t, $j(1\leqq t, j\leqq r)$ if z_u, $z_v\in\Pi_t^0$, $\bar{b}_{\mu,j}=\bar{b}_{v,j}$. Thus, in (23) considering

$$\{i\mid z_i\in\Pi_1^0\}=\{i-r^{m-1}\mid z_i\in\Pi_2^0\}=\cdots$$
$$=\{i-(r-1)r^{m-1}\mid z_i\in\Pi_m^0\}=\{i\mid 0\leqq i\leqq r^{m-1}-1\} \tag{24}$$

for each ρ_u, ρ_v $(0\leqq u<v\leqq r^{m-1}-1)$

$$A^{(1)}(\sigma,\ \rho_u)=A^{(1)}(\sigma,\ \rho_v) \tag{25}$$

Namely, a number of required transition matrices of the component $A^{(1)}$ is only one.

From above discussion and for each $v(1\leqq v\leqq r-1)$ $\Pi_v^0=\Pi_v^\tau$, it is easily seen that under the partition Π^τ, $A^{(1)}$ has $|\tilde{\Pi}^\tau|$ transition matrices for one outer input.

Each component $A^{(v)}(1\leqq v\leqq m)$ is interconnected with all other components, that is, each component is connected symmetrical. So, for the components $A^{(2)}$, $A^{(3)},\ldots,A^{(p)}$ the proof is analogous to the case of $A^{(1)}$. Q. E. D.

Intuitively, for each component, we must determine r^{m-1} transition matrices for one outer input. Theorem 8 implies that $U=\{\rho_u\mid 0\leqq u\leqq r^{m-1}-1\}$ is classified by the equivalence relation $\sim$ as follows.

$$\rho_u\sim\rho_v\longleftrightarrow A^{(v)}(\sigma,\ \rho_u)=A^{(v)}(\sigma,\ \rho_v) \tag{26}$$

4. Example

In this section, using the decomposition theory in the previous section, we try to decompose a 7-state stochastic system $A=[S,\ \Sigma,\ \{A(\sigma)\}]$ $(\Sigma=\{\sigma\})$ into three 3-state component stochastic systems. Where

$$A(\sigma)=\begin{pmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.3 & 0 \\ 0.4 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.3 & 0 & 0 & 0.2 & 0 & 0.4 & 0.1 \\ 0 & 0.2 & 0 & 0.3 & 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.4 \\ 0.2 & 0.2 & 0 & 0.2 & 0.1 & 0.2 & 0.1 \\ 0.6 & 0 & 0 & 0 & 0 & 0.4 & 0 \end{pmatrix}. \tag{27}$$

Let $B=[Z, \Sigma, \{B(\sigma)\}]$ $(Z=\{z_0, z_1,\ldots, z_{26}\})$ be a stochastic system which is interconnected with three 3-state components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ whose transition matrices are given in (6). And let Π^4 be a partition on Z as follows.

$$\Pi^4=\{\Pi_1^4, \Pi_2^4,\ldots, \Pi_7^4\}$$

$$, \text{ where } \Pi_1^4=\{z_0, z_1,\ldots, z_8\},\ \Pi_2^4=\{z_9, z_{10},\ldots, z_{17}\},$$

$$\Pi_3^4=\{z_{18}, z_{19}, z_{20}\},\ \Pi_4^4=\{z_{21}, z_{22}, z_{23}\},$$

$$\Pi_5^4=\{z_{24}\},\ \Pi_6^4=\{z_{25}\},\ \Pi_7^4=\{z_{26}\}.$$

Then, since Π^4 has the SP the following equations are derived.

$$\begin{cases}
A^{(1)}(\sigma, \rho_0)=A^{(1)}(\sigma, \rho_1)=A^{(1)}(\sigma, \rho_2)=\begin{pmatrix} a_{00}^{10} & a_{01}^{10} & a_{02}^{10} \\ a_{10}^{10} & a_{11}^{10} & a_{12}^{10} \\ a_{20}^{10} & a_{21}^{10} & a_{22}^{10} \end{pmatrix} \\
A^{(1)}(\sigma, \rho_3)=A^{(1)}(\sigma, \rho_4)=A^{(1)}(\sigma, \rho_5)=\begin{pmatrix} a_{00}^{10} & a_{01}^{10} & a_{02}^{10} \\ a_{10}^{10} & a_{11}^{10} & a_{12}^{10} \\ a_{20}^{13} & a_{21}^{13} & a_{22}^{13} \end{pmatrix} \\
A^{(1)}(\sigma, \rho_6)=\begin{pmatrix} a_{00}^{10} & a_{01}^{10} & a_{02}^{10} \\ a_{10}^{10} & a_{11}^{10} & a_{12}^{10} \\ a_{20}^{16} & a_{21}^{16} & a_{22}^{16} \end{pmatrix}, \quad A^{(1)}(\sigma, \rho_7)=\begin{pmatrix} a_{00}^{10} & a_{01}^{10} & a_{02}^{10} \\ a_{10}^{10} & a_{11}^{10} & a_{12}^{10} \\ a_{20}^{17} & a_{21}^{17} & a_{22}^{17} \end{pmatrix} \\
A^{(1)}(\sigma, \rho_8)=\begin{pmatrix} a_{00}^{10} & a_{01}^{10} & a_{02}^{10} \\ a_{10}^{10} & a_{11}^{10} & a_{12}^{10} \\ a_{20}^{18} & a_{21}^{18} & a_{22}^{18} \end{pmatrix}
\end{cases}$$

$$\begin{cases}
A^{(2)}(\sigma, \rho_0)=A^{(2)}(\sigma, \rho_1)=A^{(2)}(\sigma, \rho_2)=\begin{pmatrix} a_{00}^{20} & a_{01}^{20} & a_{02}^{20} \\ a_{00}^{20} & a_{01}^{20} & a_{02}^{20} \\ a_{00}^{20} & a_{01}^{20} & a_{02}^{20} \end{pmatrix} \\
A^{(2)}(\sigma, \rho_3)=A^{(2)}(\sigma, \rho_4)=A^{(2)}(\sigma, \rho_5)=\begin{pmatrix} a_{00}^{23} & a_{01}^{23} & a_{02}^{23} \\ a_{00}^{23} & a_{01}^{23} & a_{02}^{23} \\ a_{00}^{23} & a_{01}^{23} & a_{02}^{23} \end{pmatrix} \\
A^{(2)}(\sigma, \rho_6)=\begin{pmatrix} a_{00}^{26} & a_{01}^{26} & a_{02}^{26} \\ a_{10}^{26} & a_{11}^{26} & a_{12}^{26} \\ a_{20}^{26} & a_{21}^{26} & a_{22}^{26} \end{pmatrix}, \quad A^{(2)}(\sigma, \rho_7)=\begin{pmatrix} a_{00}^{26} & a_{01}^{26} & a_{02}^{26} \\ a_{10}^{26} & a_{11}^{26} & a_{12}^{26} \\ a_{20}^{27} & a_{21}^{27} & a_{22}^{27} \end{pmatrix}
\end{cases}$$

$$
\left.\begin{array}{l}
A^{(2)}(\sigma, \rho_8)=\begin{pmatrix} a^{26}_{00} & a^{26}_{01} & a^{26}_{02} \\ a^{26}_{10} & a^{26}_{11} & a^{26}_{12} \\ a^{28}_{20} & a^{28}_{21} & a^{28}_{22} \end{pmatrix}
\end{array}\right.
$$

$$
\left\{\begin{array}{l}
A^{(3)}(\sigma, \rho_0)=A^{(3)}(\sigma, \rho_1)=A^{(3)}(\sigma, \rho_2)=\begin{pmatrix} a^{30}_{00} & a^{30}_{01} & a^{30}_{02} \\ a^{30}_{00} & a^{30}_{01} & a^{30}_{02} \\ a^{30}_{00} & a^{30}_{01} & a^{30}_{02} \end{pmatrix} \\
A^{(3)}(\sigma, \rho_3)=A^{(3)}(\sigma, \rho_4)=A^{(3)}(\sigma, \rho_5)=\begin{pmatrix} a^{33}_{00} & a^{33}_{01} & a^{33}_{02} \\ a^{33}_{00} & a^{33}_{01} & a^{33}_{02} \\ a^{33}_{00} & a^{33}_{01} & a^{33}_{02} \end{pmatrix} \\
A^{(3)}(\sigma, \rho_6)=\begin{pmatrix} a^{36}_{00} & a^{36}_{01} & a^{36}_{02} \\ a^{36}_{00} & a^{36}_{01} & a^{36}_{02} \\ a^{36}_{00} & a^{36}_{01} & a^{36}_{02} \end{pmatrix}, \quad A^{(3)}(\sigma, \rho_7)=\begin{pmatrix} a^{37}_{00} & a^{37}_{01} & a^{37}_{02} \\ a^{37}_{00} & a^{37}_{01} & a^{37}_{02} \\ a^{37}_{00} & a^{37}_{01} & a^{37}_{02} \end{pmatrix} \\
A^{(3)}(\sigma, \rho_8)=\begin{pmatrix} a^{38}_{00} & a^{38}_{01} & a^{38}_{02} \\ a^{38}_{10} & a^{38}_{11} & a^{38}_{12} \\ a^{38}_{20} & a^{38}_{21} & a^{38}_{22} \end{pmatrix}
\end{array}\right. \tag{28}
$$

, where we use the notation a^{ij}_{kl} instead of $a^{i,j}_{k,l}$.

From (28) $B(\sigma)$ can be constructed, and the following matrix $B^*(\sigma)$ which is merged with Π^4 is obtained

$$
B^*(\sigma)=\begin{pmatrix}
a^{10}_{00} & a^{10}_{01} & a^{10}_{02}a^{20}_{00} & a^{10}_{02}a^{20}_{01} & a^{10}_{02}a^{20}_{02}a^{30}_{00} & a^{10}_{02}a^{20}_{02}a^{30}_{01} & a^{10}_{02}a^{20}_{02}a^{30}_{02} \\
a^{10}_{10} & a^{10}_{11} & a^{10}_{12}a^{23}_{00} & a^{10}_{12}a^{23}_{01} & a^{10}_{12}a^{23}_{02}a^{33}_{00} & a^{10}_{12}a^{23}_{02}a^{33}_{01} & a^{10}_{12}a^{23}_{02}a^{33}_{02} \\
a^{10}_{20} & a^{10}_{21} & a^{10}_{22}a^{16}_{00} & a^{10}_{22}a^{16}_{01} & a^{10}_{22}a^{26}_{02}a^{36}_{00} & a^{10}_{22}a^{26}_{02}a^{36}_{01} & a^{10}_{22}a^{26}_{02}a^{36}_{02} \\
a^{13}_{20} & a^{13}_{21} & a^{13}_{22}a^{26}_{01} & a^{13}_{22}a^{26}_{11} & a^{13}_{22}a^{26}_{12}a^{37}_{00} & a^{13}_{22}a^{26}_{12}a^{37}_{01} & a^{13}_{22}a^{26}_{12}a^{37}_{02} \\
a^{16}_{20} & a^{16}_{21} & a^{16}_{22}a^{26}_{20} & a^{16}_{22}a^{26}_{21} & a^{16}_{22}a^{26}_{22}a^{38}_{00} & a^{16}_{22}a^{26}_{22}a^{38}_{01} & a^{16}_{22}a^{26}_{22}a^{38}_{02} \\
a^{17}_{20} & a^{17}_{21} & a^{17}_{22}a^{27}_{20} & a^{17}_{22}a^{27}_{21} & a^{17}_{22}a^{27}_{22}a^{38}_{10} & a^{17}_{22}a^{27}_{22}a^{38}_{11} & a^{17}_{22}a^{27}_{22}a^{38}_{12} \\
a^{18}_{20} & a^{18}_{21} & a^{18}_{22}a^{28}_{20} & a^{18}_{22}a^{28}_{21} & a^{18}_{22}a^{28}_{22}a^{38}_{20} & a^{18}_{22}a^{28}_{22}a^{38}_{21} & a^{18}_{22}a^{28}_{22}a^{38}_{22}
\end{pmatrix} \tag{29}
$$

From (27) and (29), solving a equation $B^*(\sigma)=A(\sigma)$ we get three components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ such as (28).

5. Conclusions

We have shown a method to decompose any n-state stochastic system into interconnected m r-state component stochastic systems. Using our method it is mainly derived that there exists a relationship among n, m and r, and for one outer input, some specific number of transition matrices are assigned to each component.

Problems to be considered in the future are to decompose a given stochastic system into components whose transition matrices are identical and to develop a decomposition method when a number of the connected neighboring components are restricted.

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