Mutual Inductance between the Semicircular Bus and the Straight Bus (Ⅲ)

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Abstract

In the case that two conductors are disconnected, it is difficult to calculate mutual inductance by means of elementary functions. Therefore the authors use Bessel function. They show mutual inductance expression of infinite power series.

1. Introduction

In this report, as well as first report,[1] two conductors lie in the plane. But they are disconnected. Diameter of the semicircular bus is perpendicular to the straight bus. One end of the straight bus lies in the extension line of diameter.

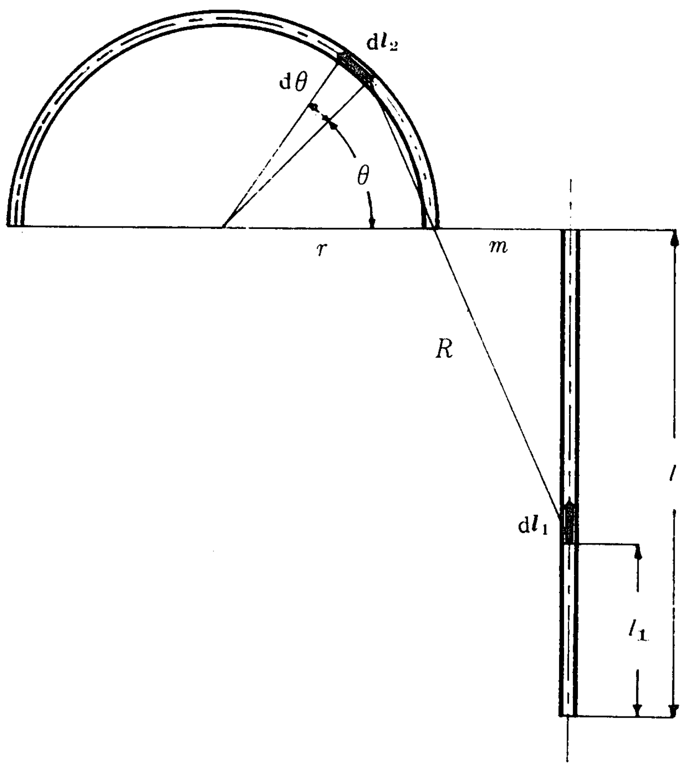

Fig. 1 Arrangement of the conductors

2. Calculation of mutual inductance

In Fig. (1)

$$R^2=(l-l_1+r\sin\theta)^2+(r+m-r\cos\theta)^2$$

Therefore

$$R=\sqrt{(l-l_1+r\sin\theta)^2+(a-r\cos\theta)^2} \qquad (1)$$

where $a=r+m$

On the other hand, scalar product $dl_1 \cdot dl_2$ is

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$$dl_1 \cdot dl_2 = dl_1\ dl_2 \cos\theta = r\cos\theta d\theta dl_1 \tag{2}$$

By Neumann's formula, mutual inductance M between the semicircular bus and the straight bus in Fig.(1) is

$$M=\frac{\mu_0}{4\pi}\int\int\frac{dl_1\cdot dl_2}{R}$$
$$=\frac{\mu_0}{4\pi}\int_{\theta=0}^{\theta=\pi}\int_{l_1=0}^{l_1=l}\frac{r\cos\theta\, d\theta\, dl_1}{\sqrt{(l-l_1+r\sin\theta)^2+(a-r\cos\theta)^2}} \tag{3}$$

Integrating with l_1

$$\int_0^l\frac{dl_1}{\sqrt{(l-l_1+r\sin\theta)^2+(a-r\cos\theta)^2}}$$
$$=-\left[\log_\varepsilon\{l-l_1+r\sin\theta+\sqrt{(l-l_1+r\sin\theta)^2+(a-r\ \mathrm{co}\ \theta)^2}\}\right]_0^l$$
$$=\log_\varepsilon\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}$$
$$-\log_\varepsilon\{r\sin\theta+\sqrt{(r\sin\theta)^2+(a-r\cos\theta)^2} \tag{4}$$

Using Eq.(4), Eq.(3) becomes

$$M=\frac{\mu_0 r}{4\pi}\int_0^\pi\cos\theta\log_\varepsilon\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}\,d\theta$$
$$-\frac{\mu_0 r}{4\pi}\int^\pi\cos\theta\log_\varepsilon\{r\sin\theta+\sqrt{(r\sin\theta)^2+(a-r\cos\theta)^2}\}\,d\theta$$
$$=\frac{\mu_0 r}{4\pi}(M_1-M_2) \tag{5}$$

where

$$M_1=\int_0^\pi\cos\theta\log_\varepsilon\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}\,d\theta$$
$$M_2=\int_0^\pi\cos\theta\log_\varepsilon\{r\sin\theta+\sqrt{(r\sin\theta)^2+(a-r\cos\theta)^2}\}\,d\theta$$

2.1. Calculation of M_1

Integrating by parts, M_1 is followed

$$M_1=\int_0^\pi\cos\theta\log_\varepsilon\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}\,d\theta$$
$$=\left[\sin\theta\log_\varepsilon\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\right]_0^\pi$$
$$-r\int_0^\pi\sin\theta\frac{\cos\theta\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}+(a-r\cos\theta)\sin\theta}{\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\{l+r\sin\theta+\sqrt{(l+r\sin\theta)^2+(a-r\cos\theta)^2}\}}\,d\theta$$
$$=-\int_0^\pi\frac{r\sin\theta\cos\theta\,d\theta}{\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}}$$
$$-\int_0^\pi\frac{r\sin^2\theta d\theta}{a-r\cos\theta}$$
$$+\int_0^\pi\frac{r\sin^2\theta(l+r\sin\theta)d\theta}{(a-r\cos\theta)\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}}$$
$$=M_{1A}+M_{1B}+M_{1C} \tag{6}$$

where $\rho=\sqrt{a^2+l^2}$, $\phi=\arctan\dfrac{l}{a}$ (7)

$$M_{1A}=-r\int_0^\pi\frac{\cos\theta\sin\theta\,d\theta}{\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}} \tag{8}$$
$$M_{1B}=-r\int\frac{\sin^2\theta\,d\theta}{a-r\cos\theta} \tag{9}$$
$$M_{1C}=r\int_0^\pi\frac{\sin^2\theta(l+r\sin\theta)d\theta}{(a-r\cos\theta)\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}} \tag{10}$$

It is difficult to obtain M_{1A}, M_{1C} by means of elementary functions. So we use Bessel function. By Lipschitz[2)]

$$\frac{1}{\mathfrak{R}}=\int_0^\infty J_0(\mathfrak{R}t)\mathrm{d}t \tag{11}$$

where $\mathfrak{R}^2=\rho^2+r^2-2\rho r\cos(\theta+\phi)$ (12)

By Neumann's addition theorem[3)]

$$J_0(\mathfrak{R}t)=J_0(\rho t)\,J_0(rt)+2\sum_{n=1}^{\infty}J_n(\rho t)J_n(rt)\cos n(\theta+\phi) \tag{13}$$

Substituting $J_0(\mathfrak{R}t)$ into Eq.(11)

$$\frac{1}{\mathfrak{R}}=\int_0^\infty J_0(\rho t)J_0(rt)\mathrm{d}t$$

$$+2\sum_{n=1}^{\infty}\int_0^\infty J_n(\rho t)J_n(rt)\cos n(\theta+\phi)\mathrm{d}t \tag{14}$$

By a complete elliptic integral of the first kind, the first term of Eq.(14) becomes.

$$\int_0^\infty J_0(\rho t)\,J_0(rt)\mathrm{d}t=\frac{2}{\rho\pi}K\left(\frac{r}{\rho}\right) \tag{15}$$

On the other hand, by hypergeometric function, the second term of Eq.(14) is followed[4)]

$$2\sum_{n=1}^{\infty}\int_0^\infty J_n(\rho t)J_n(rt)\cos n(\theta+\phi)\mathrm{d}t$$

$$=2\sum_{n=1}^{\infty}\cos n(\theta+\phi)\frac{r_n\,\Gamma\left(n+\frac{1}{2}\right)}{\rho^{n+1}\,\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}\,{}_2F_1\left(n+\frac{1}{2},\ \frac{1}{2}:n+1;\ \frac{r^2}{\rho^2}\right) \tag{16}$$

Hypergeometric function ${}_2F_1$ is expanded into hypergeometric series[5)].

$${}_2F_1\left(n+\frac{1}{2},\ \frac{1}{2};\ n+1;\frac{r^2}{\rho^2}\right)$$

$$=\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\eta=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}+\eta\right)\Gamma\left(\frac{1}{2}+\eta\right)}{\Gamma(n+1+\eta)\quad \eta!}\left(\frac{r^2}{\rho^2}\right)^\eta$$

$$=\frac{\pi\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\eta=0}^{\infty}\frac{(2n+2\eta)!(2\eta)!}{\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^\eta \tag{17}$$

Using Eq.(15), (16) and(17), Eq.(14) is given by

$$\frac{1}{\mathfrak{R}}=\frac{2}{\rho\pi}K\left(\frac{r}{\rho}\right)+2\sum_{n=1}^{\infty}\sum_{\eta=0}^{\infty}\frac{r^n}{\rho^{n+1}}\cos n(\theta+\phi)\frac{(2n+2\eta)!\,(2\eta)!}{\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^\eta \tag{18}$$

Eq.(18) substituted in Eq.(8) gives

$$M_{1A}=-r\int_0^\pi\frac{\sin\theta\cos\theta\,\mathrm{d}\theta}{\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}}$$

$$=-\frac{2r}{\rho\pi}K\left(\frac{r}{\rho}\right)\int_0^\pi\cos\theta\sin\theta\mathrm{d}\theta$$

$$-2\sum_{n=1}^{\infty}\sum_{\eta=0}^{\infty}\frac{r^n}{\rho^{n+1}}\frac{(2\eta+2n)!(2\eta)!}{\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^\eta\int_0^\pi r\cos\theta\sin\theta\cos n(\theta+\phi)\mathrm{d}\theta$$

$$=-2\sum_{n=1}^{\infty}\sum_{\eta=0}^{\infty}\frac{r^n}{\rho^{n+1}}\frac{(2\eta+2n)!(2\eta)!}{\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^\eta I_{An} \tag{19}$$

where $I_{An}=\int_0^\pi r\cos\theta\sin\theta\cos n(\theta+\phi)\mathrm{d}\theta$ (20)

Similarly Eq.(18) substituted in Eq.(10) gives

$$M_{1C}=r\int_0^\pi\frac{\sin^2\theta(l+r\sin\theta)\mathrm{d}\theta}{(a-r\cos\theta)\sqrt{\rho^2+r^2-2\rho r\cos(\theta+\phi)}}$$

$$=\frac{2}{\pi\rho}K\left(\frac{r}{\rho}\right)\int_0^\pi\frac{r\sin^2\theta(l+r\sin\theta)}{a-r\cos\theta}\mathrm{d}\theta$$

$$+2\sum_{n=1}^{\infty}\sum_{\eta=0}^{\infty}\frac{r^n(2n+2\eta)!\,(2\eta)!}{\rho^{n+1}\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^{\eta}\int_0^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta$$

$$=\frac{2r}{\pi\rho}K\left(\frac{r}{\rho}\right)I_{CO}$$

$$+2\sum_{n=1}^{\infty}\sum_{\eta=0}^{\infty}\frac{r^n\,(2n+2\eta)\,(2\eta)!}{\rho^{n+1}\{(n+\eta)!\eta!\}^2\,4^{2\eta+n}}\left(\frac{r^2}{\rho^2}\right)^{\eta}I_{cn} \tag{21}$$

where $$I_{co}=\int_0^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)}{a-r\cos\theta}d\theta \tag{22}$$

$$I_{cn}=\int_0^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta \tag{23}$$

2.1.1 Calculation of I_{An}

$$I_{An}=\int_0^{\pi}r\cos\theta\sin\theta\cos n(\theta+\phi)\,d\theta$$

$$=\frac{r}{4}\int_0^{\pi}\left[\sin\{(n+2)\theta+n\phi\}-\sin\{(n-2)\theta+n\phi\}\right]d\theta$$

If $n\neq 2$, this becomes

$$I_{An}=\frac{r}{4}\left[-\frac{1}{n+2}\cos\{(n+2)\theta+n\phi\}+\frac{1}{n-2}\cos\{(n-2)\theta+n\phi\}\right]_0^{\pi}$$

$$=\begin{cases}-\dfrac{2r}{n^2-4}\cos n\phi & (n:\text{ odd integer})\\[2mm] 0 & (n:\text{ even integer}(\neq 2))\end{cases} \tag{24}$$

And if $n=2$, I_{A2} is followed

$$I_{A2}=-\frac{\pi r}{4}\sin 2\phi \tag{25}$$

2. 1. 2. Calculation of I_{C0}

$$I_{C0}=\int_0^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)}{a-r\cos\theta}d\theta$$

Introducing the new variable

$$\left.\right|_0^{\pi}\ a-r\cos\theta=U\ \left|\begin{matrix}a+r\\a-r\end{matrix}\right.$$

$$r\sin\theta d\theta=dU$$

$$\sin\theta=\frac{\sqrt{r^2-a^2+2aU-U^2}}{r}$$

Then

$$I_{C0}=\frac{1}{r}\int_{a-r}^{a+r}\frac{l\sqrt{r^2-a^2+2aU-U^2}+r^2-a^2+2aU-U^2}{U}dU$$

$$=\frac{l}{r}\pi(a-\sqrt{a^2-r^2})-\frac{1}{r}(a^2-r^2)\log_e\frac{a+r}{a-r}+2a \tag{26}$$

2. 1. 3. Calculation of I_{Cn}

$$I_{Cn}=\int_0^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta$$

$$+\int_{\frac{\pi}{2}}^{\pi}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta \tag{27}$$

Let θ substitute for $\pi-\theta$ in the second term of Eq.(27)

$$I_{Cn}=\int_0^{\frac{\pi}{2}}\frac{r\sin^2\theta(l+r\sin\theta)\cos n(\theta+\phi)}{a-r\cos\theta}d\theta$$

$$+\int_0^{\frac{\pi}{2}}\frac{r\sin^2\theta(l+r\sin\theta)\cos n\pi\cos n(\theta-\phi)}{a+r\cos\theta}d\theta$$

$$=\int_0^{\frac{\pi}{2}} r\sin^2\theta(l+r\sin\theta)\left\{\frac{\cos n(\theta+\phi)}{a-r\cos\theta}+\frac{\cos n\pi\cos n(\theta-\phi)}{a+r\cos\theta}\right\}d\theta \tag{28}$$

If n is odd, Eq.(28) becomes

$$I_{cn(odd)}=\int_0^{\frac{\pi}{2}} r\sin^2\theta(l+r\sin\theta)\frac{2r\cos\theta\cos n\phi\cos n\theta-2a\sin n\phi\sin n\theta}{a^2-r^2\cos^2\theta}d\theta \tag{29}$$

If n is even, Eq.(28) becomes

$$I_{cn(even)}=\int_0^{\frac{\pi}{2}} r\sin^2\theta(l+r\sin\theta)\frac{2a\cos n\phi\cos n\theta-2r\cos\theta\sin n\phi\sin n\theta}{a^2-r^2\cos^2\theta}d\theta \tag{30}$$

First consider Eq.(29). We make next substitution.

$$n=2k-1\quad(k=1,2,\cdots\cdots) \tag{31}$$

So using the formulas of trigonometrical function, $\cos n\theta$, $\sin n\theta$ are respectively[6]

$$\cos n\theta=\cos(2k-1)\theta=\sum_{h=0}^{k-1}(-1)^h\frac{(2k-1)}{2(2k-h-1)}\binom{2k-h-1}{h}(2\cos\theta)^{2k-2h-1} \tag{32}$$

$$\sin n\theta=\sin(2k-1)\theta=\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-2}{h}\sin\theta\,(2\cos\theta)^{2k-2h-2} \tag{33}$$

Inserting Eq.(32) and Eq.(33) into Eq.(29)

$$\begin{aligned}I_{cn(odd)}&=\frac{2r^2}{a^2}\cos(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\frac{(2k-1)}{2(2k-h-1)}\binom{2k-h-1}{h}2^{2k-2h-1}\\&\qquad\times\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta(l+r\sin\theta)\cos^{2k-2h}\theta}{1-z\cos^2\theta}d\theta\\&\quad-\frac{2r}{a}\sin(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-2}{h}2^{2k-2h-2}\int_0^{\frac{\pi}{2}}\frac{\sin^3\theta(l+r\sin\theta)\cos^{2k-2h-2}\theta}{1-z\cos^2\theta}d\theta\\&=\frac{2r^2}{a^2}\cos(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\frac{(2k-1)}{2(2k-h-1)}\binom{2k-h-1}{h}\times2^{2k-2h-1}\\&\quad\times\left\{\frac{l}{2}\int_0^1\frac{T^{k-h-\frac{1}{2}}(1-T)^{\frac{1}{2}}dT}{1-z}+\frac{r}{2}\int_0^1\frac{T^{k-h-\frac{1}{2}}(1-T)dT}{1-zT}\right\}\\&\quad-\frac{2r}{a}\sin(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-2}{n}2^{2k-2h-2}\\&\quad\times\left\{\frac{l}{2}\int_0^1\frac{T^{k-h-\frac{3}{2}}(1-T)}{1-zT}dT+\frac{r}{2}\int_0^1\frac{T^{k-h-\frac{3}{2}}(1-T)^{\frac{3}{2}}}{1-zT}dT\right\}\end{aligned} \tag{34}$$

where $z=\frac{r^2}{a^2}=\left(\frac{r}{r+m}\right)^2<1$

In Eq.(34) we use next substitution

$$\left.\begin{matrix}\frac{\pi}{2}\\0\end{matrix}\right|\cos^2\theta=T\left|\begin{matrix}0\\1\end{matrix}\right.$$

$$d\theta=-\frac{1}{2}T^{-\frac{1}{2}}(1-T)^{-\frac{1}{2}}dT$$

The integral expression of Gaussian hypergeometric function is given by[7]

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1T^{\alpha-1}(1-T)^{\gamma-\alpha-1}(1-zT)^{-\beta}dT \tag{35}$$

Using hypergeometric series, Eq.(35) is rewritten

$$\begin{aligned}&\int_0^1T^{\alpha-1}(1-T)^{\gamma-\alpha-1}(1-zT)^{-\beta}dT\\&=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}\sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha+\lambda)\,\Gamma(\beta+\lambda)}{\Gamma(\lambda+\gamma)\,\lambda!}z^\lambda\end{aligned} \tag{36}$$

Comparing Eq.(36) with the integral term of Eq.(34), $I_{cn(odd)}$ is given by

$$\begin{aligned}I_{cn(odd)}&=\frac{2r^2}{a^2}\cos(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\frac{(2k-1)}{2(2k-h-1)}\binom{2k-h-1}{h}2^{2k-2h-1}\\&\quad\times\left\{\frac{l}{2}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(k-h+\lambda+\frac{1}{2}\right)\Gamma(\lambda+1)}{\Gamma(k-h+\lambda+2)}z^\lambda\right.\end{aligned}$$

$$+\frac{r}{2}\frac{\Gamma(2)}{\Gamma(1)}\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(k-h+\lambda+\frac{1}{2}\right)\Gamma(\lambda+1)}{\Gamma\left(k-h+\lambda+\frac{5}{2}\right)\lambda!}z^{\lambda}\Big\}$$

$$-\frac{2r}{a}\sin(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-2}{h}2^{2k-2h-2}$$

$$\times\Big\{\frac{l}{2}\frac{\Gamma(2)}{\Gamma(1)}\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(k-h+\lambda-\frac{1}{2}\right)\Gamma(\lambda+1)}{\Gamma\left(k-h+\lambda+\frac{3}{2}\right)\lambda!}z^{\lambda}$$

$$+\frac{r}{2}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(1)}\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(k-h+\lambda-\frac{1}{2}\right)\Gamma(\lambda+1)}{\Gamma(k-h+\lambda+2)\lambda!}z^{\lambda}\Big\}$$

$$=\frac{2r^2}{a^2}\cos(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\frac{(2k-1)}{2(2k-h-1)}\binom{2k-h-1}{h}2^{2k-2h-1}$$

$$\times\Big\{\frac{\pi l}{4}\sum_{\lambda=0}^{\infty}\frac{(2k-2h+2\lambda)!\,z^{\lambda}}{4^{k-h+\lambda}(k-h+\lambda)!(k-h+\lambda+1)!}$$

$$+8l\sum_{\lambda=0}^{\infty}\frac{(k-h+\lambda+2)!(2k-2h+2\lambda)!}{(k-h+\lambda)!(2k-2h+2\lambda+4)!}z^{\lambda}\Big\}$$

$$-\frac{2r}{a}\sin(2k-1)\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-2}{h}2^{2k-2h-2}$$

$$\times\Big\{8l\sum_{\lambda=0}^{\infty}\frac{(k-h+\lambda+1)!(2k-2h+2\lambda-2)!}{(k-h+\lambda-1)!(2k-2h+2\lambda+2)!}z^{\lambda}$$

$$+\frac{3\pi r}{8}\sum_{\lambda=0}^{\infty}\frac{(2k-2h+2\lambda-2)!\,z^{\lambda}}{4^{k-h+\lambda-1}(k-h+\lambda-1)!(k-h+\lambda+1)}\Big\} \quad (37)$$

Next consider Eq.(30) In this case $n=2k$ ($k=1, 2, \cdots\cdots$)

As well as $I_{cn(odd)}$, we obtain

$$I_{cn(even)}=\frac{2r}{a}\cos 2k\phi\sum_{h=0}^{k}\frac{(-1)^h 2k}{2(2k-h)}\binom{2k-h}{h}2^{2k-2h}$$

$$\times\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta(l+r\sin\theta)\cos^{2k-2h}\theta\,d\theta}{1-z\cos^2\theta}$$

$$-\frac{2r^2}{a^2}\sin 2k\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-1}{h}2^{2k-2h-1}$$

$$\times\int_0^{\frac{\pi}{2}}\frac{\sin^3\theta(l+r\sin\theta)\cos^{2k-2h-1}\theta}{1-z\cos^2\theta}d\theta$$

$$=\frac{r}{a}\cos 2k\phi\sum_{h=0}^{k}\frac{(-1)^h k}{(2k-h)}\binom{2k-h}{h}2^{2k-2h}$$

$$\times\int_0^1\frac{lT^{k-h-\frac{1}{2}}(1-T)^{\frac{1}{2}}+rT^{k-h-\frac{1}{2}}(1-T)}{1-zT}dT$$

$$-\frac{r^2}{a^2}\sin 2k\phi\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-1}{h}2^{2k-2h-1}$$

$$\times\int_0^1\frac{lT^{k-h-1}(1-T)+rT^{k-h-1}(1-T)^{\frac{3}{2}}}{1-zT}dT$$

$$=\frac{r}{a}\cos 2k\phi\sum_{\lambda=0}^{\infty}\sum_{h=0}^{k}(-1)^h\frac{k}{2k-h}\binom{2k-h}{h}2^{2k-2h}$$

$$\times\Big\{\frac{l\pi}{2}\frac{(2k-2h+2\lambda)!\,z^{\lambda}}{4^{k-h+\lambda}(k-h+\lambda)!(k-h+\lambda+1)!}$$

$$+16r\frac{(2k-2h+2\lambda)!(k-h+\lambda+2)!}{(2k-2h+2\lambda+4)!(k-h+\lambda)!}z^{\lambda}\Big\}$$

$$-\frac{r^2}{a^2}\sin 2k\phi\sum_{\lambda=0}^{\infty}\sum_{h=0}^{k-1}(-1)^h\binom{2k-h-1}{h}2^{2k-2h-1}$$

$$\times\left\{\frac{l(k-h+\lambda-1)!z^\lambda}{(k-h+\lambda+1)!}+3r\frac{4^{k-h+\lambda+1}(k-h+\lambda-1)!\,(k-h+\lambda+2)!z^\lambda}{(2k-2h+2\lambda+4)!}\right\} \tag{38}$$

M_{1A} is obtained inserting Eq.(24), Eq.(25) into Eq.(19) and M_{1C} is obtained inserting Eq.(26), (37), (38) into Eq.(21).

We do not calculate M_{1B} because M_{1B} cancels M_{2B} of M_2.

These M_{1A}, M_{1B}, and M_{1C} are shown in **2.3.**

2.2. Calculation of M_2

$$M_2=\int_0^\pi \cos\theta \log_e(r\sin\theta+\sqrt{a^2+r^2-2ar\cos\theta})\,d\theta$$

$$=\Big[\sin\theta\log_e(r\sin\theta+\sqrt{a^2+r^2-2ar\cos\theta})\Big]_0^\pi$$

$$-r\int_0^\pi \sin\theta\frac{\cos\theta(r\sin\theta+\sqrt{a^2+r^2-2ar\cos\theta})+(a-r\cos\theta)\sin\theta}{\sqrt{a^2+r^2-2ar\cos\theta}(r\sin\theta+\sqrt{a^2+r^2-2ar\cos\theta})}d\theta$$

$$=-\int_0^\pi\frac{r\sin\theta\cos\theta\,d\theta}{\sqrt{a^2+r^2-2ar\cos\theta}}$$

$$-\int_0^\pi\frac{r\sin^2\theta\,d\theta}{a-r\cos\theta}$$

$$+\int_0^\pi\frac{r^2\sin^3\theta\,d\theta}{(a-r\cos\theta)\sqrt{a^2+r^2-2ar\cos\theta}}$$

$$=M_{2A}+M_{2B}+M_{2C} \tag{39}$$

where M_{2A}, M_{2B}, M_{2C} are the first, second and third term of Eq.(39)

2.2.1. Calculation of M_{2A}

We make next substitution

$$\left.{}_0^\pi\right|\; a^2+r^2-2ar\cos\theta=X^2\;\left|\begin{matrix}(a+r)^2\\(a-r)^2\end{matrix}\right.$$

$$ar\sin\theta\,d\theta=XdX$$

So that M_{2A} is given by

$$M_{2A}=-\frac{1}{2a^2r}\int_{a-r}^{a+r}(a^2+r^2-X)dX$$

$$=-\frac{2r^2}{3a^2} \tag{40}$$

2.2.2. Calculation of M_{2C}

We replace $a-r\cos\theta$ with U. We have

$$M_{2C}=\frac{1}{r}\int_{a-r}^{a+r}\frac{2aU-a^2+r^2-U^2}{U\sqrt{2aU-a^2+r^2}}dU$$

$$=\frac{1}{r}\left[2\sqrt{2aU-a^2+r^2}-2\sqrt{a^2-r^2}\arctan\sqrt{\frac{2aU-a^2+r^2}{a^2-r^2}}\right]_{a-r}^{a+r}$$

$$-\frac{1}{r}\left[\frac{1}{3a^2}(aU+a^2-r^2)\sqrt{2aU-a^2+r^2}\right]_{a-r}^{a+r}$$

$$=2+\frac{2r^2}{3a^2}-\frac{2}{r}\sqrt{a^2-r^2}\left(\arctan\sqrt{\frac{a+r}{a-r}}-\arctan\sqrt{\frac{a-r}{a+r}}\right) \tag{41}$$

2.3. M

In 2.1.3. we have referred to the method of calculation of M_1. M_2 can be obtained as the sum of Eq.(40) and Eq.(41).

Let M show the expression of infinite power series.

$$M=\frac{\mu_0 r}{4\pi}(M_1-M_2)=\frac{\mu_0 r}{4\pi}(M_{1A}+M_{1B}+M_{1C}-M_{2A}-M_{2B}-M_{2C})$$

$$=\frac{\mu_0 r}{4\pi}(M_{1A}+M_{1C}-M_{2A}-M_{2C})$$

$$
\begin{aligned}
&= \frac{\mu_0 r}{4\pi}\Bigg[\frac{\pi r^3}{2\rho^3}\sin 2\phi \sum_{\eta=0}^{\infty}\frac{(2\eta+4)!(2\eta)!}{\{(\eta+2)!\eta!\}^2 4^{2\eta}}\left(\frac{r^2}{\rho^2}\right)^{\eta} \\
&\quad + 4\sum_{\eta=0}^{\infty}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\phi}{4k^2-4k-3}\,\frac{(2\eta+4k-2)!(2\eta)!}{\{(\eta+2k-1)!\eta!\}^2 4^{2\eta+2k-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k} \\
&+\frac{\pi l r}{a^2}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\cos(2k-1)\phi}{4^{2\eta+2k+\lambda}}\left(\frac{r}{\rho}\right)^{2\eta+2k} z^{\lambda}\,\frac{2k-1}{2k-h-1} \\
&\qquad\times\frac{(2\eta+4k-2)!(2\eta)!(2k-h-1)!(2k-2h+2\lambda)!}{\{(\eta+2k-1)!\eta!\}^2 h!(2k-2h-1)!(k-h+\lambda)!(k-h+\lambda+1)!} \\
&+\frac{8r^2}{a^2}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\cos(2k-1)\phi}{4^{2\eta+k+h-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k} z^{\lambda}\,\frac{2k-1}{2k-h-1} \\
&\qquad\times\frac{(2\eta+4k-2)!(2\eta)!(2k-h-1)!(k-h+\lambda+2)!(2k-2h+2\lambda)!}{\{(\eta+2k-1)!\eta!\}^2 h!(2k-2h-1)!(k-h+\lambda)!(2k-2h+2\lambda+4)!} \\
&-\frac{8l}{a}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\sin(2k-1)\phi}{4^{2\eta+k+h-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k} z^{\lambda} \\
&\qquad\times\frac{(2\eta+4k-2)!(2\eta)!(2k-h-2)!(k-h+\lambda+1)!(2k-2h+2\lambda-2)!}{\{(\eta+2k-1)!\eta!\}^2(2k-2h-2)!h!(k-h+\lambda-1)!(2k-2h+2\lambda+2)!} \\
&-\frac{3\pi r}{2a}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\sin(2k-1)\phi}{4^{2\eta+2k+\lambda-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k} z^{\lambda} \\
&\qquad\times\frac{(2\eta+4k-2)!(2\eta)!(2k-h-2)!(2k-2h+2\lambda-2)!}{\{(\eta+2k-1)!\eta!\}^2(2k-2h-2)!h!(k-h+\lambda+1)!(k-h+\lambda-1)!} \\
&+\frac{\pi l}{a}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k}(-1)^h\frac{\cos 2k\phi}{4^{2\eta+2k+\lambda}}\left(\frac{r}{\rho}\right)^{2\eta+2k+1} z^{\lambda}\,\frac{k}{2k-h} \\
&\qquad\times\frac{(2\eta+4k)!(2\eta)!(2k-h)!(2k-2h+2\lambda)!}{\{(\eta+2k)!\eta!\}^2 h!(2k-2h)!(k-h+\lambda+1)!(k-h+\lambda)!} \\
&+\frac{8r}{a}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k}(-1)^h\frac{\cos 2k\phi}{4^{2\eta+k+h-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k+1} z^{\lambda}\,\frac{k}{2k-h} \\
&\qquad\times\frac{(2\eta+4k)!(2\eta)!(2k-h)!(2k-2h+2\lambda)!(k-h+\lambda+2)!}{\{(\eta+2k)!\eta!\}^2 h!(2k-2h)!(2k-2h+2\lambda+4)!(k-h+\lambda)!} \\
&-\frac{lr}{a^2}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\sin 2k\phi}{4^{2\eta+k+h}}\left(\frac{r}{\rho}\right)^{2\eta+2k+1} z^{\lambda}\,\frac{(2\eta+4k)!(2\eta)!(2k-h-1)!(k-h+\lambda-1)!}{\{(\eta+2k)!\eta!\}^2 h!(2k-2h-1)!(k-h+\lambda+1)!} \\
&-\frac{3r^2}{a^2}\sum_{\eta=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{k=1}^{\infty}\sum_{h=0}^{k-1}(-1)^h\frac{\sin 2k\phi}{4^{2\eta+2h-\lambda-1}}\left(\frac{r}{\rho}\right)^{2\eta+2k+1} z^{\lambda} \\
&\qquad\times\frac{(2\eta+4k)!(2\eta)!(2k-h-1)!(k-h+\lambda-1)!(k-h+\lambda+2)!}{\{(\eta+2k)!\eta!\}^2 h!(2k-2h-1)!(2k-2h+2\lambda+4)!}\Bigg] \\
&-\frac{\mu_0 r}{4\pi}\left\{2-\frac{2}{r}\sqrt{a^2-r^2}\left(\arctan\sqrt{\frac{a+r}{a-r}}-\arctan\sqrt{\frac{a-r}{a+r}}\right)\right\} \qquad (42)
\end{aligned}
$$

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(昭和44年8月14日受理)