Mutual Inductance between the Semicircular Bus and the Straight Bus (Ⅱ)

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Abstract

The authors calculate mutual inductance between the semicircular busses and the straight busses in more general cases.

1. Introduction

In this report, the straight bus and the semicircular bus lie in the two separate planes. Two planes are parallel each other. Straight bus and a diameter of semicircular bus lie in the same plane which is perpendicular to above planes.

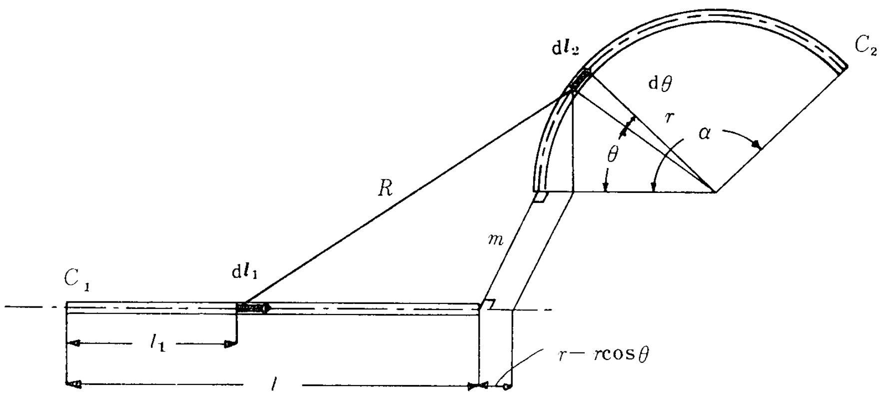

Fig. 1 Arrangement of the conductors

2. Calculation of mutual inductance

In Fig. 1

$$R^2=(l-l_1+r-r\cos\theta)^2+r^2\sin^2\theta+m^2$$

therefore

$$R=\sqrt{(l-l_1+r-r\cos\theta)^2+r^2\sin^2\theta+m^2} \tag{1}$$

and moreover scalar product $dl_1 \cdot dl_2$ is

$$dl_1\cdot dl_2=dl_1\,dl_2\cos\left(\frac{\pi}{2}-\theta\right)=r\sin\theta\,d\theta\,dl_1 \tag{2}$$

By Neumann's formula, mutual inductance M between the semicircular bus and the straight bus in Fig. 1 is

$$M=\frac{\mu_0}{4\pi}\int\int\frac{dl_1\cdot dl_2}{R}$$

$$=\frac{\mu_0}{4\pi}\int_{\theta=0}^{\theta=\alpha}\int_{l_1=0}^{l_1=l}\frac{r\sin\theta\,d\theta dl_1}{\sqrt{(l-l_1+r-r\cos\theta)^2+r^2\sin^2\theta+m^2}} \tag{3}$$

We substitute

$$\Big|_0^l\; l-l_1+r-r\cos\theta=x\Big|_{l+r-r\cos\theta}^{r-r\cos\theta},\quad -dl_1=dx \tag{4}$$

And first integrating with l_1

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$$\int_0^l \frac{dl_1}{\sqrt{(l-l_1+r-r\cos\theta)^2+r^2\sin^2\theta+m^2}}$$
$$=\int_{l+r-r\cos\theta}^{r-r\cos\theta} \frac{-dx}{\sqrt{x^2+r^2\sin^2\theta+m^2}}$$
$$=\log_\varepsilon\{l+r-r\cos\theta+\sqrt{(l+r-r\cos\theta)^2+r^2\sin^2\theta+m^2}\}$$
$$-\log_\varepsilon\{r-r\cos\theta+\sqrt{2r^2(1-\cos\theta)+m^2}\} \tag{5}$$

Using Eq.(5), Eq.(3) becomes

$$M=\frac{\mu_0 r}{4\pi}\int_0^\alpha \sin\theta\ \log_\varepsilon\{l+r-r\cos\theta+\sqrt{(l+r-r\cos\theta)^2+r^2\sin^2\theta+m^2}\}d\theta$$
$$-\frac{\mu_0 r}{4\pi}\int_0^\alpha \sin\theta \log_\varepsilon\{r-r\cos\theta+\sqrt{2r^2(1-\cos\theta)+m^2}\}d\theta$$
$$=M_1-M_2 \tag{6}$$

where we put the first and second term of Eq.(6) into M_1, M_2 respectively.

We calculate M_1.

Substituting $1-\cos\theta$ for t and integrating with t

$$M_1=\frac{\mu_0 r}{4\pi}\int_0^{1-\cos\alpha} \log_\varepsilon\{l+rt+\sqrt{2r(r+l)t+m^2+l^2}\}dt$$
$$=\frac{\mu_0 r}{4\pi}\Big[t\log_\varepsilon\{l+rt+\sqrt{2r(r+l)t+m^2+l^2}\}\Big]_0^{1-\cos\alpha}$$
$$-\frac{\mu_0 r}{4\pi}\int_0^{1-\cos\alpha}\frac{rt+\dfrac{r(r+l)t}{\sqrt{2r(r+l)t+m^2+l^2}}}{l+rt+\sqrt{2r(r+l)t+m^2+l^2}}dt$$
$$=I_a-I_b \tag{7}$$

where we put the first and second term of Eq.(7) into I_a, I_b respectively.

I_a is followed

$$I_a=\frac{\mu_0 r(1-\cos\alpha)}{4\pi}\log_\varepsilon\{l+r-r\cos\alpha+\sqrt{2r(r+l)(1-\cos\alpha)+m^2+l^2}\} \tag{8}$$

We calculate I_b by changing of variable

$$\left.\begin{aligned} &\Big|_0^{1-\cos\alpha}\ \sqrt{2r(r+l)t+m^2+l^2}=T\ \Big|_{\sqrt{m^2+l^2}}^{\sqrt{2r(r+l)(1-\cos\alpha)+m^2+l^2}} \\ &r(r+l)dt=TdT \end{aligned}\right\} \tag{9}$$

$$I_b=\frac{\mu_0}{4\pi(r+l)}\int_{\eta_2}^{\eta_1}\{T-(r+l)\}dT$$
$$+\frac{\mu_0 r}{2\pi}\int_{\eta_2}^{\eta_1}\frac{dT}{T^2+2(r+l)T+2rl+l^2-m^2}$$
$$+\frac{\mu_0(rl-m^2)}{2\pi}\int_{\eta_2}^{\eta_1}\frac{dT}{T^2+2(r+l)T+2rl+l^2-m^2}$$
$$=\frac{\mu_0}{4\pi(r+l)}\Big[\frac{T^2}{2}-(r+l)T+\frac{1}{2}(l^2+2rl+2r^2+m^2)$$
$$\times\ \log_\varepsilon\{T^2+2(r+l)T+2rl+l^2-m^2\}-\frac{(r+l)\sqrt{r^2+m^2}}{2}$$
$$\times\log_\varepsilon\frac{T+r+l-\sqrt{r^2+m^2}}{T+r+l+\sqrt{r^2+m^2}}\Big]_{\eta_2}^{\eta_1} \tag{10}$$

where

$$\eta_1=\sqrt{2r(r+l)(1-\cos\alpha)+m^2+l^2}$$
$$\eta_2=\sqrt{m^2+l^2}$$

Consequently M_1 is followed

$$M_1=\frac{\mu_0 r}{4\pi}\Big\{(1-\cos\alpha)\log_\varepsilon(l+r-r\cos\alpha+\eta_1)-(1-\cos\alpha)+\frac{\eta_1-\eta_2}{r}$$

$$-\log_\varepsilon\frac{r(1-\cos\alpha)+l+\eta_1}{l+\eta_2}+\frac{\eta_3}{r}\log_\varepsilon\frac{r+l+\eta_1-\eta_3}{r+l+\eta_1+\eta_3}-\frac{\eta_3}{r}\log_\varepsilon\frac{r+l+\eta_2-\eta_3}{r+l+\eta_2+\eta_3}\Big\} \tag{11}$$

where

$$\eta_3=\sqrt{r^2+m^2}$$

Next we calculate M_2. Substituting similarly in the case M_1.

$$M_2=\frac{\mu_0 r}{4\pi}\int_0^{1-\cos\alpha}\log_\varepsilon(rt+\sqrt{2r^2t+m^2})\,\mathrm{d}t$$

$$=\frac{\mu_0 r}{4\pi}\Big[t\log_\varepsilon(rt+\sqrt{2r^2t+m^2})\Big]_0^{1-\cos\alpha}$$

$$-\frac{\mu_0 r}{4\pi}\int_0^{1-\cos\alpha}\frac{rt+\dfrac{r^2t}{\sqrt{2r^2t+m^2}}}{rt+\sqrt{2r^2t+m^2}}\,\mathrm{d}t$$

$$=\frac{\mu_0 r}{4\pi}(1-\cos\alpha)\log_\varepsilon\{r-r\cos\alpha+\sqrt{2r^2(1-\cos\alpha)+m^2}\}-I_c \tag{12}$$

where I_c is the second term of Eq.(12).

Integrating I_c by repacing $\sqrt{2r^2t+m^2}$ with U

$$I_c=\frac{\mu_0}{4\pi r}\int_m^{\sqrt{2r^2(1-\cos\alpha)+m^2}}\frac{U^3+rU^2-m^2U-rm^2}{U^2+2rU-m^2}\,\mathrm{d}U$$

$$=\frac{\mu_0 r}{4\pi}\Big\{(1-\cos\alpha)-\frac{\eta_4-m}{r}+\log_\varepsilon\frac{r(1-\cos\alpha)+\eta_4}{m}$$

$$-\frac{\eta_3}{r}\log_\varepsilon\frac{r+\eta_4-\eta_3}{r+\eta_4+\eta_3}+\frac{\eta_3}{r}\log_\varepsilon\frac{r+m-\eta_3}{r+m+\eta_3}\Big\}$$

where

$$\eta_4=\sqrt{2r^2(1-\cos\alpha)+m^2}$$

Consequently M_2 is followed

$$M_2=\frac{\mu_0 r}{4\pi}\Big\{(1-\cos\alpha)\log_\varepsilon(r-r\cos\alpha+\eta_4)-(1-\cos\alpha)+\frac{\eta_4-m}{r}$$

$$-\log_\varepsilon\frac{r(1-\cos\alpha)+\eta_4}{m}+\frac{\eta_3}{r}\log_\varepsilon\frac{r+\eta_4-\eta_3}{r+\eta_4+\eta_3}-\frac{\eta_3}{r}\log_\varepsilon\frac{r+m-\eta_3}{r+m+\eta_3}\Big\} \tag{13}$$

Finaly inserting Eq.(11) and Eq.(13) into Eq.(6)

$$\mathrm{M}=\frac{\mu_0 r}{4\pi}\Big[(1-\cos\alpha)\log_\varepsilon\frac{l+r(1-\cos\alpha)+\eta_1}{r(1-\cos\alpha)+\eta_4}+\frac{\eta_1-\eta_2-\eta_4+m}{r}$$

$$-\log_\varepsilon\frac{m\{r(1-\cos\alpha)+l+\eta_1\}}{(l+\eta_2)\{r(1-\cos\alpha)+\eta_4\}}-\frac{\eta_3}{r}\log_\varepsilon\frac{(r+l+\eta_2-\eta_3)(r+m+\eta_3)}{(r+l+\eta_2+\eta_3)(r+m-\eta_3)}$$

$$+\frac{\eta_3}{r}\log_\varepsilon\frac{(r+l+\eta_1-\eta_3)(r+\eta_4+\eta_3)}{(r+l+\eta_1+\eta_3)(r+\eta_4-\eta_3)}\Big] \tag{14}$$

Using Eq.(14) we show M for following various cases.

2.1 $\alpha=\pi$ (See Fig.2)

$$\eta_1=\sqrt{4r(r+l)+l^2+m^2}$$

$$\eta_2=\sqrt{l^2+m^2}$$

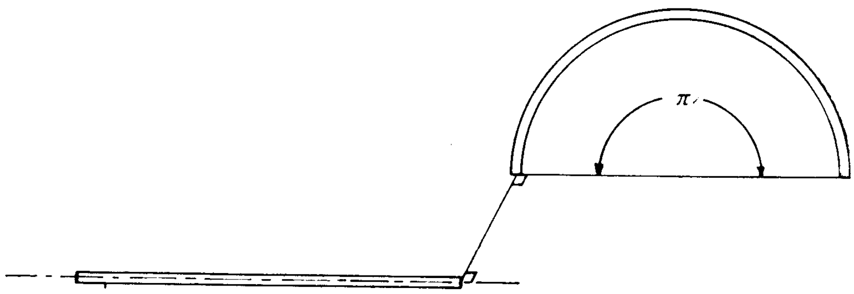

Fig.2 Arrangement of the conductors $(\alpha=\pi)$

$$\eta_3=\sqrt{r^2+m^2}$$

$$\eta_4=\sqrt{4r^2+m^2}$$

$$\mathrm{M}=\frac{\mu_0 r}{4\pi}\Big[\log_\varepsilon\frac{(l+\eta_2)(l+2r+\eta_1)}{m(2r+\eta_4)}$$

$$+\frac{\eta_3}{r}\log_\varepsilon\frac{(r+l+\eta_2+\eta_3)(r+m-\eta_3)(r+l+\eta_1-\eta_3)(r+\eta_4+\eta_3)}{(r+l+\eta_2-\eta_3)(r+m+\eta_3)(r+l+\eta_1+\eta_3)(r+\eta_4-\eta_3)}$$

$$+\frac{\eta_1-\eta_2-\eta_4+m}{r}\Big] \qquad (15)$$

2.2 $\alpha=\frac{\pi}{2}$ (See Fig. 3)

$$\eta_1=\sqrt{2r(r+l)+l^2+m^2}$$

$$\eta_2=\sqrt{l^2+m^2}$$

$$\eta_3=\sqrt{r^2+m^2}$$

$$\eta_4=\sqrt{4r^2+m^2}$$

$$M=\frac{\mu_0 r}{4\pi}\Big[\log_\varepsilon\frac{l+\eta_2}{m}$$

$$+\frac{\eta_3}{r}\log_\varepsilon\frac{(r+l+\eta_2+\eta_3)(r+m-\eta_3)(r+l+\eta_1-\eta_3)(r+\eta_4+\eta_3)}{(r+l+\eta_2-\eta_3)(r+m+\eta_3)(r+l+\eta_1+\eta_3)(r+\eta_4-\eta_3)}+\frac{\eta_1-\eta_2-\eta_4+m}{r}\Big] \qquad (16)$$

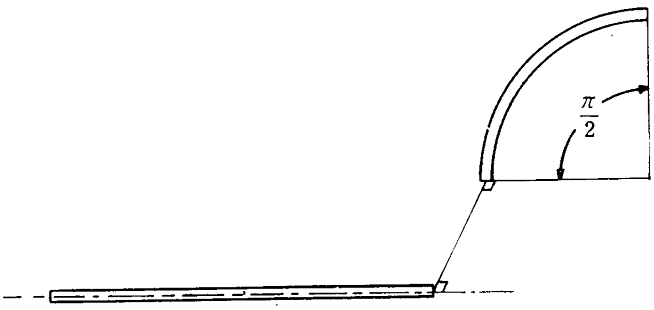

Fig.3 Arrangement of the conductors ($\alpha=\pi/2$)

2.3 $m=0$ (See Fig. 4)

$$\eta_1=\sqrt{2r(r+l)(1-\cos\alpha)+l^2}$$

$$\eta_2=l$$

$$\eta_3=r$$

$$\eta_4=r\sqrt{2(1-\cos\alpha)}$$

$$\mathrm{M}=\frac{\mu_0 r}{4\pi}\Big[(1-\cos\alpha)\log_\varepsilon\frac{l+r(1-\cos\alpha)+\eta_1}{r(1-\cos\alpha)+\eta_4}+\frac{\eta_1-l-\eta_4}{r}$$

$$-\log_\varepsilon\frac{\{r(1-\cos\alpha)+l+\eta_1\}r}{\{r(1-\cos\alpha)+\eta_4\}(r+l)}+\log_\varepsilon\frac{(l+\eta_1)(2r+\eta_4)}{(2r+l+\eta_1)\eta_4}\Big] \qquad (17)$$

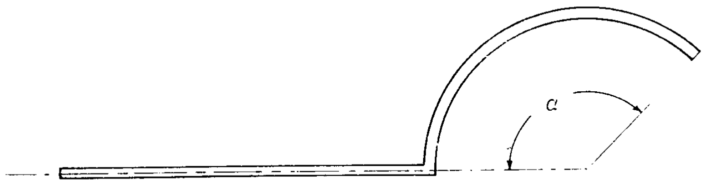

Fig.4 Arrangement of the conductors ($m=0$)

2.4 $m=0, \alpha=\pi$ (See Fig. 5)

$$\eta_1=l+2r$$

$\eta_2 = l$

$\eta_3 = r$

$\eta_4 = 2r$

$$M = \frac{\mu_0 r}{2\pi} \log_\varepsilon \frac{r+l}{r} \tag{18}$$

which agrees with the formula given by authors.[1)]

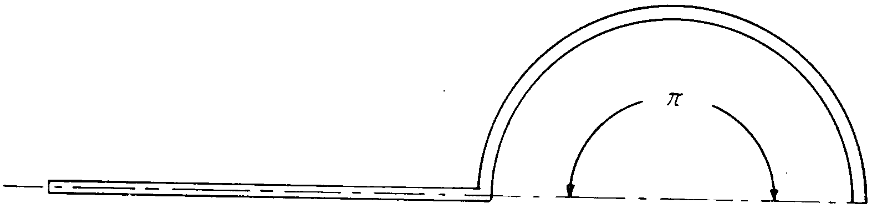

Fig.5　Arrangement of the conductors ($m=0$, $\alpha=\pi$)

2.5　$m=0$, $\alpha=\pi/2$ (See Fig. 6)

$\eta_1 = \sqrt{2r(r+l)+l^2}$

$\eta_2 = l$

$\eta_3 = r$

$\eta_4 = \sqrt{2}\, r$

$$M = \frac{\mu_0 r}{4\pi}\left[\log_\varepsilon \frac{(1+\sqrt{2})(r+l)\{l+\sqrt{2r(r+l)+l^2}\}}{r\{2r+l+\sqrt{2r(r+l)+l^2}\}} + \frac{\sqrt{2r(r+l)+l^2}-l-\sqrt{2}\,r}{r}\right] \tag{19}$$

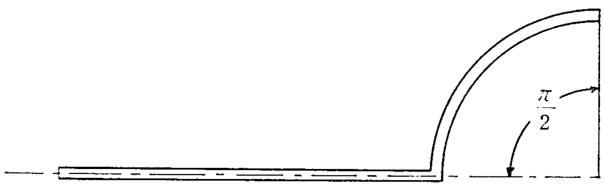

Fig.6 Arrangement of the conductors ($m=0$, $\alpha=\pi/2$)

Reference

1) Y. Koide, M. Kotani and N. Takehira : Memoirs of the Faculty of Engineering Yamaguchi University 20, No. 2,1 (1969)

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