Some Structural Properties of Product Automata

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Abstract

Some properties of product and self-product automata are analyzed by the product of state transition matrices.

It can be shown that the self-product automata of cyclic type automata with $2m+1$ ($m \geqq 1$) states are again cyclic, and the relationships bitween self-product automata are made clear. As the most important result, we show that a set of self-product automata with some states forms a group with the product operation of automata. Moreover, concerning the number of states of automata, the necessary and sufficient condition to form the group can be derived. Furthermore, it can be shown that the sequence of self-product automata with 2^m ($m \geqq 1$) states converges to the unity type automata.

1. Introduction

The automata model, whose aspects vary with the lapse of time, can be found in the theories of nonhomogeneous markov chains and L-systems. In a recent paper, A. Paz[1),2),3)], J. Hajnal[4)], and J. Wolfowitz[5)] gave some properties of stochastic matrices and discussed ergodicity of the nonhomogeneous markov chains, and H. Jürgensen[6)] investigated some limiting properties of the system introducing the probabilistic factors into L-systems. From the antomata-theoretical point of view[7),8)], it will be interesting to study the transition structures of those time-variant systems.

In this paper, we introduce a product automaton defined by the product of the matrices which represent the state transitions, and investigate the transition structures of the product automata. In order to simplify the problem, we deal with a cyclic type of DAOA, where "DAOA" denotes the deterministic autonomous outputless automaton. Furthermore, we introduce a new concept of self-product automata, and derive some algebraic properties of them.

In Section 2, we give some definitions and notations concerning DAOA's, and give several fundamental propositions derived from those definitions. In Section 3, we investigate the transition structures of product automata, and in Section 4, we investigate some algebraic properties of a sequence of self-product automata.

2. Preliminaries

In this section, we give some definitions concerning autonomous outputless automata, and give some fundamental propositions derived from those definitions.

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Notation 2.1

(2.1.1) Let A be a square matrix of order n for some $n \geqq 1$. Then A_{ij} denotes the ij entry of A ($1 \leqq i, j \leqq n$).

(2.1.2) For any set U, $|U|$ denotes the number of elements of U.

Definition 2.1. An autonomous outputless automaton (AOA) is a couple $S=[Z, A]$, where "Z" is a finite set of states, and "A" is a square matrix of order $|Z|$ whose each entry A_{ij} ($1 \leqq i, j \leqq |Z|$) is 1 or 0.

Definition 2.2. Let $S=[Z, A]$ be an AOA. The directed graph $G=[V,E]$ is called the state transition graph of S, where $V=Z$, $E=\{(z_i, z_j) \in Z \times Z \mid Aij=1\}$ and (z_i, z_j) denotes the edge directed from z_i to z_j.

Let $Z=\{z_1, z_2, \cdots, z_n\}$ ($n=|Z|$). For each i, j ($1 \leqq i, j \leqq n$), "$A_{ij}=1$" means that when the present state of S is z_i, one of the next states of S is z_j. Also, "autonomous" means that the behaviour of S is independent of the input signal.

Difinition 2.3. Let A be a square matrix of order n for some $n \geqq 1$. A is called "deterministic" if $\sum_{j=1}^{n} A_{ij}=1$ for all $i \in \{1,2,\cdots,n\}$.

Definition 2.4. An AOA $S=[Z, A]$ is called "deterministic" if A is deterministic. We denote a deterministic AOA by "DAOA".

For a deterministic matrix, a proposition is given as follows.

Proposition 2.1. If P and Q are deterministic matrices, then the product matrix PQ is also a deterministic matrix.

The proof of this proposition and subsequent propositions are omitted, because they are obvious.

Definition 2.5. $S^{\circ}(Z) \triangleq [Z, I]$ is called a "unity" DAOA with the state set Z, where I is an unit matrix of order $|Z|$.

A DAOA can be considered as a kind of directed graph. Therefore, this paper may play a special role of connecting automata theory with graph theory.

Definition 2.6. Let $S_1=[Z_1, A_1]$ and $S_2=[Z_2, A_2]$ be DAOA's. Then $S_1 \triangleq S_2$ if and only if $Z_1=Z_2$ and $A_1=A_2$.

Definition 2.7. Let $S_1=[Z, A_1]$, $S_2=[Z, A_2], \cdots$, and $S_n=[Z, A_n]$ ($n \geqq 2$) be DAOA's with the same set of states, Z. Then, the DAOA $[Z, A_1A_2 \cdots A_n]$ is called the product automaton of $S_1, S_2, \cdots$, and S_n, and is denoted by "$S_1S_2 \cdots S_n$".

Definition 2.8. Let $S=[Z, A]$ be a DAOA. Then, S^m ($m \geqq 1$) is called as self-product automaton (where $S^m \triangleq \underbrace{S \cdots S}_{m} \triangleq [Z, A^m]$).

Definition 2.9. Let $S=[Z, A]$ be a DAOA, and $Z=\{z_1, z_2, \ldots, z_n\}$ $(n=|Z|)$.

(2.9.1) For any two states $z_i, z_j \in Z$, we call that the state z_j is accessible from the state z_i if there exists a natural number m such that $A^m_{ij}=1$.

(2.9.2) S is called "cyclic" or "connected" if, for any two states $z_i, z_j \in Z$, z_j is accessible from z_i and vice versa. We denote a cyclic DAOA by "CDAOA". And, for each CDAOA $S=[Z, A]$, A is called a cyclic matrix.

We note that, for any CDAOA $S=[Z, A]$, there exists exactly one "1" on each row and each column of A, respectively.

Definition 2.10.

(2.10.1) Let Z be the set of n states $z_1, z_2, \ldots, z_n$. Then, a mapping $\tau: Z \to Z$ satisfying the following condition (C) is called a state renaming function for Z.

(C)$\forall i, j$ $(1 \leqq i, j \leqq n)$ $[i \neq j \to \tau(z_i) \neq \tau(z_j)]$.

A state renaming function τ for Z is simply denoted by "SRF(Z)" or "SRF(Z)τ".

(2.10.2) Let $S=[Z, A]$ be a DAOA and τ be a SRF(Z). Then we define $\tau(A)$ to be the deterministic matrix of order $|Z|$ as follows. For each i, j $(1 \leqq i, j \leqq |Z|)$ $\tau(A)_{ij}=1$ if and only if $z_i=\tau(z_p)$ and $z_j=\tau(z_q)$ and $A_{pq}=1$.

Definition 2.11. For any $n \geqq 1$, the square matrix of order n

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & O & 0 \\ & & & \ddots & \\ 0 & & O & & 1 \\ 1 & 0 & & & 0 \end{pmatrix}$$

is called "canonical".

Proposition 2.2.

(2.2.1) If $S=[Z, A]$ is a cyclic (non-cyclic) DAOA, then $S'=[Z, \tau(A)]$ is again a cyclic (non-cyclic) DAOA for any SRF(Z) τ.

(2.2.2) For any CDAOA $S=[Z, A]$, there exists a SRF(Z)τ such that $\tau(A)$ is canonical.

(2.2.3) For any DAOA $S=[Z, A]$ and any SRF(Z)τ, there exists a SRF(Z) τ' such that $\tau'(\tau(A))=A$.

(2.2.4) Let $S_1=[Z, A_1]$ and $S_2=[Z, A_2]$ be any DAOA's. Then, for any SRF(Z) τ, $\tau(A_1A_2)=\tau(A_1)\tau(A_2)$.

(2.2.5) Let $S=[Z, A]$ be a DAOA. Then, for any SRF(Z)'s τ' and τ'', there exists a SRF(Z) τ such that $\tau'(\tau''(A))=\tau(A)$.

(2.2.6) For any $S^\circ(Z)=[Z, I]$ and any SRF(Z)τ, $\tau(I)=I$.

Definition 2.12. Let A and B be two deterministic matrices of order $r \geqq 1$ and $s \geqq 1$, respectively. Then the matrix

$$A \oplus B \triangleq \begin{pmatrix} (A) & O \\ O & (B) \end{pmatrix}$$

of order r+s is called the direct sum of A and B.

Definition 2.13. Let R denote a form

$$R = \begin{pmatrix} i_1 & i_2 & \cdots i_n \\ j_1 & j_2 & \cdots j_n \end{pmatrix} \quad (n \geqq 1)$$

where, (1) for each $p(1 \leqq p \leqq n)$, i_p and j_p are natural number, and (2) for each p, $p(1 \leqq p < q \leqq n)$ $i_p \neq i_q$. If $\{j_1, j_2, \ldots, j_n\} \subseteq \{i_1, i_2, \ldots i_n\}$, then R is called "pseudo permutation". Especially, if $\{j_1, j_2, \ldots, j_n\} = \{i_1, i_2, \ldots, i_n\}$, then R is called "permutation".

Definition 2.14. Let C and D are pseudo permutations as follows.

$$C = \begin{pmatrix} i_1 \, i_2 \cdots j_n \\ \\ j_1 \, j_2 \cdots j_n \end{pmatrix}, \qquad D = \begin{pmatrix} r_1 \, r_2 \cdots r_n \\ \\ s_1 \, s_2 \cdots s_n \end{pmatrix} \quad (n \geqq 1)$$

where $\{i_1, i_2, \ldots, i_n\} = \{r_1, r_2, \ldots, r_n\}$. Then

$$CD \triangleq \begin{pmatrix} i_1 \, i_2 \cdots i_n \\ \\ u_1 \, u_2 \cdots u_n \end{pmatrix},$$

where, for each $i(1 \leqq i \leqq n)$, if $j_i = r_k (1 \leqq k \leqq n)$, then $u_i = s_k$.

Convention 2.1 Let C and D be the pseudo permutations described in the Definition 2.13. Then, we identify D with C, if D is obtained from C by rearranging each column of C, appropriately. For example.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 2 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 \\ 3 & 2 & 1 & 2 & 5 \end{pmatrix}$$

Definition 2.15. A permutation

$$\begin{pmatrix} i_1 \, i_2 \cdots i_n \\ \\ i_2 \, i_3 \cdots i_1 \end{pmatrix}$$

is called "cyclic", and is denoted by $(i_1 i_2 \ldots i_n)$.

Notation 2.2 Let $A_1, A_2, \ldots,$ and $A_k (k \geqq 1)$ be cyclic permutations such that

(1) $A_i=(j_1^i j_2^i \cdots j_{n_i}^i)$ $(n_i \geqq 1,\ 1 \leqq i \leqq k)$, and

(2) $\{j_1^p, \ldots, j_{n_p}^p\} \cap \{j_1^q, \ldots, j_{n_q}^q\} = \phi$ for each p, q $(1 \leqq p < q \leqq k)$.

(ϕ denotes the null set.)

Then, $A_1 \odot A_2 \odot \cdots \odot A_k$ denotes the permutation

$$\begin{pmatrix} j_1^1 j_2^1 \cdots j_{n1}^1 j_1^2 j_2^2 \cdots j_{n_2}^2 \cdots\cdots j_1^k j_2^k \cdots j_{n_k}^k \\ \\ j_2^1 j_3^1 \cdots j_1^1 j_2^2 j_3^2 \cdots j_1^2 \cdots\cdots j_2^k j_3^k \cdots j_1^k \end{pmatrix}.$$

From convention 2.1, note that $A_1 \odot A_2 = A_2 \odot A_1$.

Definition 2.16. For any deterministic matrix A of order $n \geqq 1$, $\varphi(A)$ is defind as follows.

$$\varphi(A) \triangleq \begin{pmatrix} 1 & 2 & \cdots i & \cdots n \\ \\ j_1 & j_2 & \cdots j_i & \cdots j_n \end{pmatrix}, \qquad j_1, j_2, \cdots, j_n \in \{1, 2, \cdots, n\}$$

where, for each i $(1 \leqq i \leqq n)$, the ij_i entry of A is "1". (Note that there exists exactly one "1" on each row of the deterministic matrix.)

Proposition 2.3.

(2.3.1) For any deterministic matrices A_1 and A_2 of order $n \geqq 1$, $\varphi(A_1)\varphi(A_2) = \varphi(A_1 A_2)$.

(2.3.2) For any cyclic matrix A of order $n \geqq 1$ and any SRF(Z) τ, if $\varphi(\tau(A))^n = \varphi(I)$, then $\varphi(A)^n = \varphi(I)$.

Example 1. Let A_1 and A_2 be two deterministic matrices as follows.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

Then

$$\varphi(A_1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}, \quad \varphi(A_2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \text{ and}$$

$$\varphi(A_1)\varphi(A_2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix} = (1\ 2) \odot (3\ 4\ 5)$$

Notation 2.3.

(2.3.1) The set of all DAOA's (CDAO's) is denoted by $\&$ ($\&_c$).

(2.3.2) $\mathcal{E} \triangleq \mathcal{E}^{(e)} \cup \mathcal{E}^{(o)}$, where $\mathcal{E}^{(e)}$ ($\mathcal{E}^{(o)}$) is the sets of all DAOA's with an even (odd) number of states.

(2.3.3) $\mathcal{E}_c \triangleq \mathcal{E}_c^{(e)} \cup \mathcal{E}_c^{(o)}$, where $\mathcal{E}_c^{(e)}$ ($\mathcal{E}_c^{(o)}$) is the sets of all CDAOA's with an even (odd) number of states.

(2.3.4) $\mathcal{E}^{(e)} \triangleq \mathcal{E}^{[2]} \cup \mathcal{E}^{[4]} \cup \cdots$, $\mathcal{E}^{(o)} \triangleq \mathcal{E}^{[1]} \cup \mathcal{E}^{[3]} \cup \cdots$.

$$\mathcal{E}_c^{(e)} \triangleq \mathcal{E}_c^{[2]} \cup \mathcal{E}_c^{[4]} \cup \cdots, \quad \mathcal{E}_c^{(o)} \triangleq \mathcal{E}_c^{[1]} \cup \mathcal{E}_c^{[3]} \cup \cdots.$$

where, for any natural number i, $\mathcal{E}^{[i]}$ ($\mathcal{E}_c^{[i]}$) is the ste of all DAOA's (CDAOA's) with just i states ($\mathcal{E}^{[o]} \triangleq \phi$).

3. The Transition Structures of Product Automata

In this section, we investigate the transition structures of product automata. First, we give a lemma whose proof is obvious from Definition 2.9.2.

Lemma 3.1. Let S=[Z, A] be in $\mathcal{E}^{[n]}$ ($n \geqq 1$) and let

$$\varphi\ (A) = \begin{pmatrix} 1 & 2 & \cdots n \\ i_1 & i_2 & \cdots i_n \end{pmatrix}.$$

If $i_r = i_s$ for some r, s$\in$\{1, 2,..., n\} ($r \neq s$), then S is not in $\mathcal{E}^{[n]}$

Theorem 3.1. Let $S_1=[Z, A_1]$, $S_2=[Z, A_2]$ be DAOA's, and let

$$\varphi\ (A_1) = \begin{pmatrix} 1 & 2 & \cdots n \\ i_1 & i_2 & \cdots i_n \end{pmatrix}, \quad \varphi\ [A_2) = \begin{pmatrix} 1 & 2 & \cdots n \\ j_1 & j_2 & \cdots j_n \end{pmatrix}.$$

Then, if $i_r = i_s$ or $j_p = j_q$ for some r, s, p, q$\in$\{1, 2, $\cdots$, n\} ($r \neq s$, $p \neq q$), then the product automaton $S_1S_2 = S_3 = [Z, A_3]$ is not in $\mathcal{E}_c^{[n]}$.

Proof. Suppose that $= i_r = i_s$. Then

$$\varphi\ (A_3) = \varphi\ (A_1A_2) = \varphi\ (A_1)\ \varphi\ (A_2)$$

$$= \begin{pmatrix} 1 & \cdots r & \cdots s & \cdots n \\ i_1 & \cdots i_r & \cdots i_r & \cdots i_n \end{pmatrix} \begin{pmatrix} 1 & \cdots i_r \cdots n \\ j_1 & \cdots j_{i_r} \cdots j_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots r & \cdots s & \cdots n \\ k_a & \cdots j_{i_r} & \cdots j_{i_r} & \cdots k_b \end{pmatrix}$$

It follows from Lemma 3.1 that S_3 is not in $\mathcal{E}_c^{[n]}$. The similar argument is used for the case of $j_p = j_q$. Q.E.D.

The following theorem implies that the self-product automata of cyclic DAOA's with an old number of states are again cyclic or unity.

Theorem 3.2. Let $S=[Z, A]\in \mathcal{E}_c^{[2m-1]}$ $(m\geqq 1)$.

(1) $S^n=S^\circ(Z)$, if $n=r(2m-1)$ for some natural number $r\geqq 1$,

(2) $S^n\in \mathcal{E}_c^{[2m-1]}$, otherwise.

Proof. (1) Let $S=[Z, A]\in \mathcal{E}_c^{[2m-1]}$. Then, for a SRE(Z) τ, the matrix A can be transformed into the canonical matrix $\tau(A)$ of order 2m-1 (from Proposition 2.2.2), and $\varphi(\tau(A))$ can be denoted as follows.

$$\varphi(\tau(A))=\begin{pmatrix} 1 & 2 \cdots 2m-2 & 2m-1 \\ 2 & 3 \cdots 2m-1 & 1 \end{pmatrix}.$$

Easily, we have

$$\varphi(\tau(A))^{r(2m-1)}=\begin{pmatrix} 1 & 2 \cdots 2m-1 \\ 1 & 2 \cdots 2m-1 \end{pmatrix},$$

for any natural number $r\geqq 1$. Since $\varphi(\tau(A))^{r(2m-1)}=\varphi(\tau(A)^{r(2m-1)})$ (from Proposition 2.2.4), it follows that

$$\varphi(\tau(A)^{r(2m-1)})=\begin{pmatrix} 1 & 2 \cdots 2m-1 \\ 1 & 2 \cdots 2m-1 \end{pmatrix}$$

Therefore $\tau(A)^{r(2m-1)}$ is the unit matrix of order 2m-1. From Propositions 2.2.4 and 2.2.6, it follows that $A^{r(2m-1)}$ is the unit matrix of order 2m-1, which implies that $S^{r(2m-1)}=S^\circ(Z)$.

(2) To prove (2), from part (1) of the theorem, it suffices to show that, for each n $(1\leqq n\leqq 2m-2)$, S^n is cyclic, Let $\tau(A)$ be the canonical matrix as the mentioned above. Then

$$\varphi(\tau(A)^n)=\varphi(\tau(A))^n$$
$$=\begin{pmatrix} 1 & 2 & \cdots & 2m-n-1 & 2m-n\cdots 2m-1 \\ n+1 & n+2 & \cdots & 2m-1 & 1 \quad\cdots\quad n \end{pmatrix}.$$

From the state transition graph of $S'^n=[Z, \tau(A)^n]$, it is easily shown that S'^n is cyclic. Therefore, from Propositions 2.2.1 and 2.2.3, it is shown that $S^n=[Z, A^n]$ is cyclic. Q. E. D.

Corollary 3.1. Let $S\in \mathcal{E}_c^{[2m-1]}$ $(m\geqq 1)$. Then, for each $n\geqq 1$, S^{2n} is in $\mathcal{E}_c^{[2m-1]}$.

When the number of states is even, the state transition structures of product automata differ from the odd cace, as follows.

Theorem 3.3. Let $S=[Z, A]\in \mathcal{E}_c^{[2m]}$ $(m\geqq 1)$, and let $S^2=[Z, A^2]$. Then, there exists a SRF(Z) τ such that $\tau(A^2)=A_1\oplus A_2$, for some cyclic matrices A_1 and A_2 of order m.

Proof. For each $S=[Z, A]\in\mathcal{S}_c^{[2m]}$, there exists a SRF(Z) τ' such that the matrix $\tau'(A)$ (of order 2m) is canonical, and

$$\tau'(A)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & & O & 0 \\ 0 & 0 & 0 & 1 & 0 & & 0 \\ 0 & & & & & \cdot & \\ & & & O & & \cdot & \\ 0 & 0 & & & & 0 & 1 \\ 1 & 0 & & & & & 0 \\ 0 & 1 & 0 & & & & 0 \end{pmatrix}.$$

From the state transition graph of $S'=[Z, \tau'(A)^2]$, it is easily shown that there exists a SRF(Z) τ'' such that $\tau''(\tau'(A)^2)=A_1\oplus A_2$, for some cyclic matrices A_1 and A_2 of order m. Therefore, from Propositions 2.2.3 and 2.2.4, it follows that, for some SRF(Z) τ, $\tau(A^2)=A_1\oplus A_2$. This completes the proof of the theorem. Q. E. D.

Theorem 3.4. For each $S_1\in\mathcal{S}_c^{[2m]}$ $(m\geqq 1)$, there exists some S_2 in $\mathcal{S}_c^{[2m]}$ such that $S_2\notin\mathcal{S}_c^{[2m]}$ and $S_1S_2\in\mathcal{S}_c^{[2m]}$.

Proof. Let $S_1=[Z, A_1]\in\mathcal{S}_c^{[2m]}$. From Proposition 2.2.2, there exists a SRF(Z) τ such that $\tau(A_1)$ is the canonical matrix of order 2m.

Now we consider a non-cyclic matrix A of order 2m as follows.

$$A = \begin{pmatrix} 0 & & & & 0 & 1 & 0 \\ 0 & & & & & 0 & 1 \\ 1 & & & & O & & 0 \\ 0 & 1 & 0 & & & & \\ & O & & \cdot & & & \\ & & & & \cdot & & \\ 0 & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\tau(A_1)A = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ & 0 & 1 & 0 & O & 0 \\ & & & \cdot & & \\ & & & & \cdot & \\ & O & & & \cdot & 0 \\ 0 & & & & 0 & 1 \\ 1 & 0 & & & & 0 \end{pmatrix} \begin{pmatrix} 0 & & & 0 & 1 & 0 \\ 0 & & & & 0 & 1 \\ 1 & 0 & & O & & 0 \\ 0 & \cdot & & & & \\ & & \cdot & & & \\ & O & & \cdot & & \\ 0 & & & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & & 0 & 1 \\ 1 & 0 & & & & 0 \\ 0 & 1 & 0 & O & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & O & & & \cdot & \\ 0 & & & 0 & 1 & 0 \end{pmatrix}.$$

From the state transition graph of $S'=[Z, \tau(A_1)A]$, it is easily shown that S' is cyclic. From Proposition 2.2.3, there exists a SRF(Z) τ' such that $\tau'(\tau(A_1))=A_1$, and thus $\tau'(\tau(A_1)A)=\tau'(\tau(A_1))\tau'(A)=A_1\tau'(A)$.

Noting that $\tau(A_1)A$ is cyclic, from Proposition 2.2.1, it follows that $A_1\tau'(A)$ is cyclic. Furthermore, since A is non-cyclic, $\tau'(A)$ is also non-cyclic (from Proposition 2.2.1). Thus $S_2=[Z, \tau'(A)]$ is not cyclic, but $S_3=S_1S_2=[Z, A_1\tau'(A)]$ is cyclic. Q. E. D.

The next theorem implies that the product automaton S_1S_2 with an odd number of states is not always cyclic, even if both S_1 and S_2 are cyclic.

Theorem 3.5. There exists $S_1, S_2 \in \&_c^{(o)}$ $(S_1 \neq S_2)$ such that $S_1S_2 \in \&_c^{(o)}$

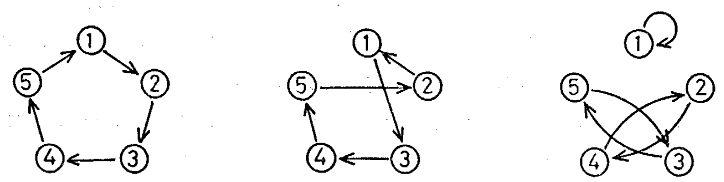
Proof. Let $S_1 = [Z, A_1]$, $S_2 = [Z, A_2] \in \&_c^{[5]}$, where

$$A_1 = \begin{pmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \end{pmatrix}.$$

Then

$$A_3 = A_1A_2 = \begin{pmatrix} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \end{pmatrix}$$

It is obvious that $S_3 = [Z, A_3]$ is not in $\&_c^{[5]}$ (See Fig.1).



S_1=[{1,2,3,4,5},A_1] S_2=[{1,2,3,4,5},A_2] S_3=[{1,2,3,4,5},A_1A_2]

Fig. 1 State transition graphs of S_1, S_2 and S_3 in Theorem 3.5

4. Self-product Automata

In Section 3, we considered the transition structures of product automata of DAOA's. In this section, we investigate some algebraic properties concerning an infinite sequence of self-product automata.

Notation 4.1. For some natural number n and m, $[n]_m$ denotes the remainder when n is divided by m.

Notation 4.2.

(4.2.1) Let S be a DAOA. Then S* denoted the infinite sequence of self-product automata S, S^2, S^4, ..., S^{2^n}, ...

(4.2.2) For each DAOA S = [Z, A], $\Omega(S) \triangleq \{S^\circ(Z)\} \cup \{S^{2^n} \mid n \geqq 0\}$.

Definition 4.1. Let S = [Z, A] be a DAOA. The least natural number n such that

$A^{2^n}=A$ is called a "cycle" of S^*, and is denoted by $C(S^*)$.

Proposition 4.1. Let $S=[Z, A]\in \mathcal{E}^{[n]}$ $(n\geqq 1)$. If $C(S^*)=n$, then

(4.1.1) $S^{2^{mn}}=S$, for each natural number, m,

(4.1.2) For any SRF(Z) τ, let $S_\tau=[Z, \tau(A)]$, then $C(S^*)=C(S_\tau^*)$.

In a sense, the concept of a infinite esquence of self-product automata has a physical and biological image such that the present aspects of a system depends upon its past aspects.

In the following theorem, it is shown that a set $\Omega(S)$ for a CDAOA S consists a group with the product operation of automata.

First, we give the wellknown lemma, as follows.

Lemma 4.1. The order of a cyclic group generated from a cyclic permutation (1 2 ... n) is n.

Theorem 4.1. If $S\in \mathcal{E}_c^{[2m+1]}$ and $C(S^*)=2m$, then the set $\in(S)$ forms a group with the product operation of automata.

Proof. Let $S=[Z, A]\in \mathcal{E}_c^{[2m+1]}$ and let a set U be

$$U \triangleq \{S^\circ(Z)\}\cup\{S, S^2, \ldots, S^{2^{2m}-1}\}.$$

If $C(S^*)=2m$, then $|U|=2m+1$, and it is obvious that U is isomorphic to a cyclic group generated from the cyclic permutation $(1\ 2\ \cdots\ 2m+1)$. Thus, it follyws from the Lemma 4.1 that the $\Omega(S)$ forms a group. Q. E. D.

For the number of states of automata, the necessary and sufficient condition to form the group can be derived, as follows.

Theorem 4.2. Let $S=[Z, A]\in \mathcal{E}_c^{[2m+1]}$ $(m\geqq 1)$. Then, $C(S^*)=2m$ if and only if (1) $[2^{2m}]_{2m+1}=1$ (2) for each $i(1\leqq i\leqq 2m-1)$ $[2^i]_{2m+1}\neq 1$.

Proof. Let $S=[Z, A]\in \mathcal{E}_c^{[2m+1]}$ From proposition 4.1.2, cycle of S^* is independent of permutation of their states. Therefore, without loss of generality, we can assume $\varphi(A)=(1\ 2\ \cdots\ 2m+1)$. Then, for each $k\geqq 1$, we have

$$\varphi(A)^{2^k}=(1\ f(k, 2)\ f(k, 3)\ \cdots\ f(k, 2m+1)),$$

where $f(k, r)=[1+(r-1)2^k]_{2m+1}$ for each $r(2\leqq r\leqq 2m+1)$. It is obvious that for each k, j $(k, j\geqq 1)$, if $f(k, 2)=f(j, 2)$, then $\varphi(A)^{2^k}=\varphi(A)^{2^j}$. It follows from $C(S^*)=2m$ and Proposition 4.2.1 that

$$\varphi(A)^{2^{2m}}=\varphi(A^{2^{2m}})=\varphi(A)$$

Thus, the condition can be derived as follows.

$$[1+2^{2m}]_{2m+1}=2, \text{ and}$$
$$[1+2^i]_{2m+1}\neq 2, \text{ for each } i=1, 2, \cdots, 2m-1.$$

Namely,

$$[2^{2m}]_{2m+1}=1, \text{ and}$$
$$[2^i]_{2m+1}\neq 1, \text{ for each } i=1, 2, \cdots, 2m-1.$$

Converse is easily shown. Q. E. D.

As an example of Theorem 4.2, in Figure 2, we show one cycle of S* of a CDAOA with five states.

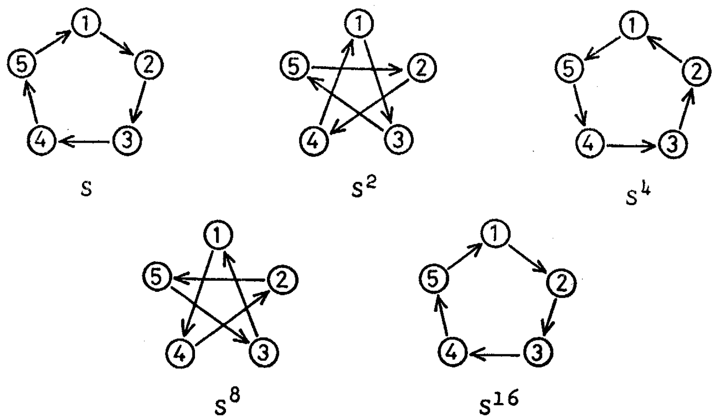


Fig. 2 One cycle of S* (S=[{1,2,3,4,5}, A])

Concerning a S* of CDAOA S with 2^m ($m\geqq 1$) states, we obtain a theorem as follows.

Theorem 4.3. If $S=[Z, A]\in \&_c^{[2m]}$ ($m\geqq 1$), then $S^{2^m}=S^\circ(Z)$, and $\lim_{n\to\infty} S^{2^n}=S^\circ(Z)$.

Proof. Let $S=[Z, A]\in \&_c^{[2m]}$. From Proposition 2.2.2, there exists a SRF(Z) τ such that τ (A) is the canonical matrix, and

$\varphi(\tau(A))=(1\ \ 2\ \ldots\ 2^m)$.

The proof is derived by the induction on the number m.

The initial step is trivial. $((1\ \ 2)(1\ \ 2)=(1)\odot(2))$

The induction step is taken from k to k+1. For $\varphi(\tau(A))=(1\ \ 2\ \ 3\ \ldots\ 2^{k+1})$, we have

$$\varphi(\tau(A))^2=(1\ 3\ 5\ \cdots 2^{k+1}-1)\odot(2\ 4\ 6\ \cdots 2^{k+1})\quad \text{(by Theorem 3.3)}$$

LetB$=(1\ \ 3\ \ 5\ \ldots\ 2^{k+1}-1)$,C$=(2\ \ 4\ \ 6\ \ldots\ 2^{k+1})$. Nothing that the set $\{1, 3,\ldots, 2^{k+1}-1\}$ and $\{2, 4, 6,\ldots,2^{k+1}\}$ are disjoint, we have

$$\varphi(\tau(A))^{2^{k+1}}=\{\varphi(\tau(A))^2\}^{2^k}$$
$$=(B\odot C)^{2^k}=B^{2^k}\odot C^{2^k}$$

Since $|\{1, 3, 5,\ldots,2^{k+1}-1\}|=|\{2, 4, 6,\ldots,2^{k+1}\}|=2^k$, by the inductionhyposesis,

$$B^{2^k}=(1)\odot(3)\odot\cdots\odot(2^{k+1}-1)$$
$$C^{2^{k+1}}=(2)\odot(4)\odot\cdots\odot(2^{k+1}).$$

Therefore,

$$\varphi(\tau(A))2^{k+1}=(1)\odot(2)\odot(3)\odot\cdots\odot(2^{k+1}).$$

From Proposition 2.3.2, we obtain $A^{2^{R+1}}=I$, namely $S^{2^{k+1}}=S^\circ(Z)$.

Then $\lim_{n\to\infty} S^{2^n}=S^\circ(Z)$. Q. E. D.

Fig. 3 illustrates an example of Theorem 4.3.

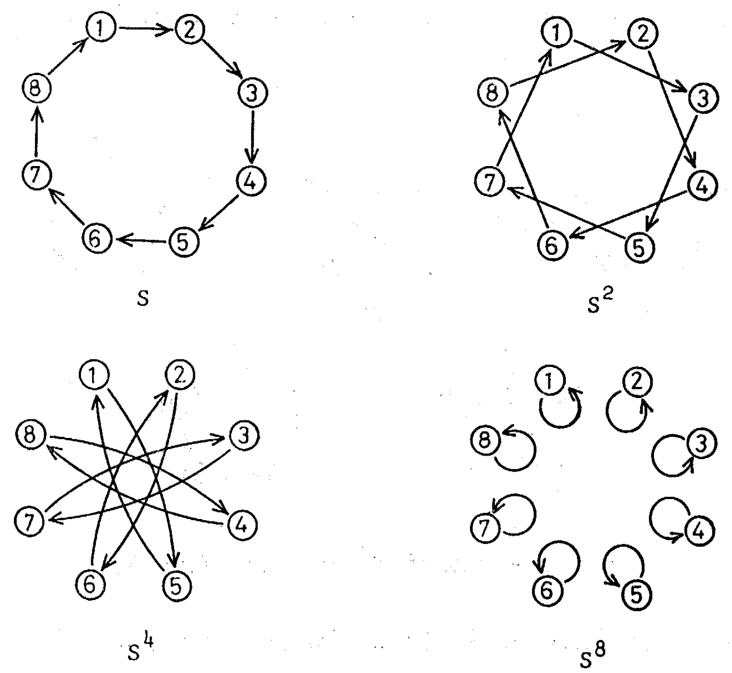


Fig. 3 Example of Theorem 4.3. (S=[Z, A], |Z|=8.)

5. Conclusion

As the automata model, whose aspects vary with the lapse of time, we introduce product automata and self-product automata. And some structural properties of product and self-product automata are analyzed by the product of state transition matrices.

In this paper, in order to simplify the problem, we deal with cyclic type of deterministic autonomous outputless automaton, however, it will be of interest to study the structural properties of general type of non-deterministic one.

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References

1) A. Paz: "Graph-theoretic and algebraic characterizations of some markov processes", Israel J. Math. **1**, No. 3, 169-180 (1963).
2) A. Paz: "Definite and quasidefinite sets of stochastic matrix", Proc. Amer. Math. Soc. **16**, No. 4, 634-641 (1965).
3) A. Paz: "Ergodic theorems for infinite probabilistic tables", Ann. Math. Statistics **41**, 2, 539-550 (1969).
4) J. Hajnal: "Weak ergodicity in nonhomogeneous markog chains", Proc. Camb. phil. Soc. **54**, 233-246 (1958).
5) J. Wolfowitz: "Products of indecomposable, aperiodic, stochastic matrices", Proc. Amer. Math. Soc.

14, No. 5, 733-737 (1963).
6) H. Jürgensen: "Probilistic L systems", in: Automata, languages, development, eds. A. Lindenmayer and G. Rosenberg, North Holland, Amsterdam (1976).
7) ,A. Paz: "Introduction to probabilistic automata", Academic Press (1971).
8) P. H. Starke: "Abstract Automata", North Holland (1972).