Analysis of Stresses around a Circular Shaft in an Elastic Ground.

——In Case the Directions of the Initial Stresses are Vertical and Horizontal.

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Abstract

The author treated of the analysis of stresses around a circular shaft without lining sunk in an elastic ground, in accordance with the results of investigation elaborated by H. Suzuki, and obtained the stress distribution by numerical calculation for some concrete examples. He then compared the results of the calculation with the results obtained by simple equation, supposed to be derived by treatment as a two dimensional problem, and found that the both results coincide very well.

Being based on the fact mentioned above, he described the solution of the stresses around a circular shaft, in case one of the principal stresses in the virgine ground is vertical, and the two horizontal principal stresses have different magnitudes.

1. Introduction

The solution of the stresses in an ground around a circular shaft has been obtained by the theory of elasticity, assuming that the ground in which the shaft is sunk consists of homogeneous elastic body, and the surface of the ground is horizontal. This solution was obtained in 1931 by T. Sugihara under the assumptions that the stresses are produced only under the effect of own weight of the ground and that the Poisson's ratio of the ground is equal to 2[1]. The latter assumption means, in other words, that the value of the initial stresses increase proportionally with the depth, and have uniform magnitudes in all directions. It is one of the important subjects of research in the field of the rock mechanics, whether above mentioned assumtion is right or wrong. Anyway his solution was obtained assuming a special state of the initial stresses.

Later on, in 1949, another solution for the same problem was obtained by H. Suzuki, under more generally acceptable assumptions that the rock pressure is produced only under the effect of the own weight of the ground and that the Poisson's ratio of the ground can take any value[2].

The solution of the stresses in a ground around a circular shaft with no lining, however, has not been definitely shown in his paper. Moreover, this solution was obtained utilizing a boundary condition that, in case the shaft wall is covered by the lining, the stresses on the wall surface are equal to the stresses on the outer wall of the lining. This assumption, however, seems to be inappropriate, because the lining is built many days after the shaft has been excavated. Consequently, it is considered that among the solutions treated by him, the solution relating to the shaft without the lining would be significant.

The author therefore tried to calculate the stresses around a circular shaft without the lining, and to give the stress distribution concretely.

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2. Analysis of stresses around a circular shaft.

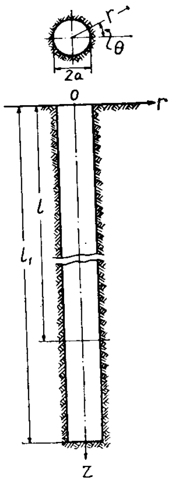

Fig. 1 A circular shaft without lining sunk in an elastic ground.

Consider now a ground with horizontal top surface forming a semi-infinite elastic body, in which a circular shaft with diameter $2a$ is sunk. Let us take a cylindrical co-ordinate (r, θ, z) as shown in Fig. 1, having its origine at the center of the top end of the shaft, and z-axis coinciding to the center line of the shaft. Then the displacements as well as the stresses at any point in the ground being symmetrical in respect to the z-axis, the equations of equilibrium are given by

$$\left.\begin{aligned}(\lambda+2\mu)\frac{\partial e}{\partial r}+2\mu\frac{\partial \omega}{\partial z}=0,\\(\lambda+2\mu)\frac{\partial e}{\partial z}-2\mu\frac{1}{r}\frac{\partial}{\partial r}(r\omega)+\gamma=0,\end{aligned}\right\}\quad\cdots\cdots\cdots(1)$$

in which λ and μ mean the Lamé's constants, e the cubical dilatation, ω rotation component around a normal line of rz-plane, and γ specific weight of the ground[3]. Denoting by u and w the components of displacement in r- and z- direction, we have

$$\left.\begin{aligned}e=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z},\\2\omega=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r},\end{aligned}\right\}\quad\cdots\cdots\cdots(2)$$

and from Eqs. (1) we find

$$\frac{\partial^2 e}{\partial r^2}+\frac{1}{r}\frac{\partial e}{\partial r}+\frac{\partial^2 e}{\partial z^2}=0,\quad\cdots\cdots\cdots(3)$$

$$\frac{\partial^2 \omega}{\partial r^2}+\frac{1}{r}\frac{\partial \omega}{\partial r}-\frac{\omega}{r^2}+\frac{\partial^2 \omega}{\partial z^2}=0.\quad\cdots\cdots\cdots(4)$$

Substituting Eqs. (1), the solutions of Eqs. (3) and (4) have the form

$$e=\sum_{n=1}^{\infty}\left\{A_nK_0(k_nr)+B_nI_0(k_nr)\right\}\begin{matrix}\sin\\\cos\end{matrix}\Bigg\}k_nz-\frac{\gamma}{\lambda+2\mu}\cdot z,\quad\cdots\cdots\cdots(5)$$

$$2\omega=\sum_{n=1}^{\infty}\frac{\lambda+2\mu}{\mu}\left\{A_nK_1(k_nr)-B_nI_1(k_nr)\right\}\begin{matrix}-\cos\\\sin\end{matrix}\Bigg\}k_nz,\quad\cdots\cdots\cdots(6)$$

in which $I_0(k_nr)$, $I_1(k_nr)$ are the modified Bessel functions of the first kind, $K_0(k_nr)$, $K_1(k_nr)$ the modified Bessel functions of the second kind, and A_n, B_n, k_n are the arbitrary constants.

From Eqs. (2) we find the following equations:

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial e}{\partial r}+2\frac{\partial \omega}{\partial z},\\\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}=\frac{\partial e}{\partial z}-2\left(\frac{\partial \omega}{\partial r}+\frac{\omega}{r}\right).\end{aligned}\right\}\quad\cdots\cdots\cdots(7)$$

Solving these equations we obtain

$$u=-\sum_{n=1}^{\infty}\left\{\frac{\lambda+\mu}{2\mu}A_n r K_0(k_n r)+C_n K_1(k_n r)+\frac{\lambda+\mu}{2\mu}B_n r I_0(k_n r)\right.$$
$$\left.+D_n I_1(k_n r)\right\}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}k_n z, \quad\cdots\cdots (8)$$

$$w=\sum_{n=1}^{\infty}\left\{\frac{\lambda+\mu}{2\mu}A_n r K_1(k_n r)-\frac{\lambda+2\mu}{\mu}A_n\frac{K_0(k_n r)}{k_n}+C_n K_0(k_n r)\right.$$
$$\left.-\frac{\lambda+\mu}{2\mu}B_n r I_1(k_n r)+\frac{\lambda+2\mu}{\mu}B_n\frac{I_0(k_n r)}{k_n}-D_n I_0(k_n r)\right\}\left\{\begin{matrix}\cos\\ -\sin\end{matrix}\right\}k_n z$$
$$-\frac{1}{2}\frac{\gamma z^2}{\lambda+2\mu}, \quad\cdots\cdots (9)$$

where C_n and D_n are the arbiturary constants.

Denoting l_1 and l the depth of the shaft and a depth a little smaller than l_1, in which l means the depth of points on the shaft wall where the influence of shaft bottom becomes negligibly small. We shall carry on the analysis within the range of $0\leqq z\leqq l$. Practically, the length of l are considered to be several times of the shaft diameter shorter than l_1.

The boundary conditions for the shaft wall, for the infinite distance and for the ground surface are given as follows:

for $r=a$,

$$\frac{\tau_{rz}}{\mu}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}=0, \quad\cdots\cdots (10)$$

$$\sigma_r=\lambda e+2\mu\frac{\partial u}{\partial r}=0, \quad\cdots\cdots (11)$$

for $r=\infty$,

$$u=0,\quad w=-\frac{\gamma}{2(\lambda+2\mu)}z^2 \quad\cdots\cdots (12)$$

and for $z=0$,

$$\left.\begin{aligned}\frac{\tau_{rz}}{\mu}&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}=0,\\ \sigma_z&=\lambda e+2\mu\frac{\partial w}{\partial z}=0,\end{aligned}\right\} \quad\cdots\cdots (13)$$

Substituting Eqs. (8) and (9) into Eqs. (12), we have

$$B_n=D_n=0, \quad\cdots\cdots (14)$$

and we find from Eqs. (5), (8), (9) and (13)

$$\cos k_n z=0. \quad\cdots\cdots (15)$$

Whences from Eqs. (8) and (9), u and w are expressed by the following equations:

$$u = -\sum_{n=1}^{\infty}\left\{\frac{\lambda+\mu}{2\mu}A_n r K_0(k_n r) + C_n K_1(k_n r)\right\}\sin k_n z, \qquad (16)$$

$$w = -\sum_{n=1}^{\infty}\left\{\frac{\lambda+\mu}{2\mu}A_n r K_1(k_n r) - \frac{\lambda+2\mu}{\mu}A_n\frac{K_0(k_n r)}{k_n r}\right.$$

$$\left. + C_n K_0(k_n r)\right\}\sin k_n z - \frac{1}{2}\frac{\gamma}{\lambda+2\mu}z^2. \qquad (17)$$

Substituting Eqs. (16) and (17) into Eq. (10), it is found that

$$\frac{\lambda+\mu}{\mu}A_n k_n a K_0(k_n a) - \frac{\lambda+2\mu}{\mu}A_n K_1(k_n a) + 2C_n k_n K_1(k_n a) = 0. \qquad (18)$$

Also from Eqs. (5), (16) and the boundary condition Eq. (11)

$$\sum_{n=1}^{\infty}\left\{-\mu A_n K_0(k_n a) + (\lambda+\mu)A_n a k_n K_1(k_n a) + 2\mu C_n k_n K_0(k_n a)\right.$$

$$\left. + \frac{2\mu}{a}C_n K_1(\mathrm{k}_n a)\right\}\sin k_n z - \frac{\gamma\lambda z}{\lambda+2\mu} = 0. \qquad (19)$$

Eliminating C_n from Eqs. (18) and (19), we have

$$z = \frac{\lambda+2\mu}{\gamma\lambda}\sum_{n=1}^{\infty}A_n\left[(\lambda+\mu)k_n a K_1(k_n a) - (\lambda+\mu)k_n a\frac{\{K_0(k_n a)\}^2}{K_1(k_n a)}\right.$$

$$\left. + (\lambda+2\mu)\frac{K_1(k_n a)}{k_n a}\right]\sin k_n z. \qquad (20)$$

Developing z in a Fourier series over $-l < z < l$,

$$z = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(k_n z),$$

in which $\qquad (21)$

$$k_n = \frac{n\pi}{l}.$$

We obtain, from Eqs. (20) and (21)

$$A_n = \frac{(-1)^{n-1}}{n}\frac{2\gamma l}{\pi}\frac{\lambda}{\lambda+2\mu}\left[\left\{(\lambda+\mu)k_n a + (\lambda+2\mu)\frac{1}{k_n a}\right\}K_1(k_n a)\right.$$

$$\left. - (\lambda+\mu)k_n a\frac{\{K_0(k_n a)\}^2}{K_1(k_n a)}\right]^{-1}. \qquad (22)$$

Substituting Eq. (22) into Eq. (18), C_n is derived in the following form:

$$C_n=\frac{(-1)^{n-1}}{nk_n}\frac{\gamma l}{\pi}\frac{\lambda}{\mu(\lambda+2\mu)}\Big\{(\lambda+2\mu)K_1(k_na)$$
$$-(\lambda+\mu)k_aaK_0(k_na)\Big\}\Big[\Big\{(\lambda+\mu)k_na+(\lambda+2\mu)\frac{1}{k_na}\Big\}\Big\{K_1(k_na)\Big\}^2$$
$$-(\lambda+\mu)k_na\Big\{K_0(k_na)\Big\}^2\Big]^{-1} \quad\cdots\cdots\cdots (23)$$

Denote by σ_r, σ_θ, σ_z, $\tau_{r\theta}$, $\tau_{\theta z}$ and τ_{rz} the stress components in the ground, then we have

$$\left.\begin{aligned}&\sigma_r=\lambda e+2\mu\frac{\partial u}{\partial r}, \quad \sigma_\theta=\lambda e+2\mu\frac{u}{r},\\&\sigma_z=\lambda e+2\mu\frac{\partial w}{\partial z}, \quad \tau_{rz}=\mu\Big(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\Big),\\&\tau_{r\theta}=\tau_{\theta z}=0.\end{aligned}\right\}\quad\cdots\cdots\cdots (24)$$

Substituing Eqs (5), (16) and (17) into Eqs. (24), we obtain

$$\left.\begin{aligned}\sigma_r=&\sum_{n=1}^{\infty}\Big[-\mu A_nK_0(k_nr)+(\lambda+\mu)k_nrA_nK_1(k_nr)\\&+2\mu C_n\Big\{k_nK_0(k_nr)+\frac{K_1(k_nr)}{r}\Big\}\Big]\sin k_nz-\frac{\gamma\lambda}{\lambda+2\mu}z,\\\sigma_\theta=&-\sum_{n=1}^{\infty}\Big\{\mu A_nK_0(k_nr)+C_n\frac{2\mu}{r}K_1(k_nr)\Big\}\sin k_nz-\frac{\gamma\lambda}{\lambda+2\mu}z,\\\sigma_z=&-\sum_{n=1}^{\infty}\{-(3\lambda+4\mu)A_nK_0(k_nr)+(\lambda+\mu)k_nrA_nK_1(k_nr)\\&+2\mu k_nC_nK_0(k_nr)\}\sin k_nz-\gamma z,\\\tau_{rz}=&-\sum_{n=1}^{\infty}\{(\lambda+\mu)k_nrA_nK_0(k_nr)-(\lambda+2\mu)A_nK_1(k_nr)\\&+2\mu k_nC_nK_1(k_nr)\}\cos k_nz.\end{aligned}\right\}\quad\cdots\cdots\cdots (25)$$

Denote by ν the Poisson's ratio of the ground, then, substituting Eqs. (22) and (23) into Eq. (25), each stress components are given by the following equations:

$$\sigma_r=-\frac{\nu}{1-\nu}\gamma z+\frac{2}{\pi}\frac{\nu}{1-\nu}\gamma l\sum_{n=1}^{\infty}\Big[K_1(k_na)K_0(k_nr)+\Big\{k_nr$$
$$+\frac{2(1-\nu)}{k_nr}\Big\}K_1(k_na)K_1(k_nr)-k_naK_0(k_na)\Big\{K_0(k_nr)$$
$$+\frac{1}{k_nr}K_1(k_nr)\Big\}\Big]\Big[(-1)^{n-1}n\Big\{k_na+\frac{2(1-\nu)}{k_na}\Big\}\Big\{K_1(k_na)\Big\}^2$$
$$-k_na\Big\{K_0(k_na)\Big\}^2\Big]^{-1}\sin k_nz,$$

$$\left.\begin{aligned}
\sigma_\theta &= -\frac{\nu}{1-\nu}\gamma z - \frac{2}{\pi}\frac{\nu}{1-\nu}\gamma l \sum_{n=1}^{\infty}\Big[(1-2\nu)K_1(k_na)K_0(k_nr) \\
&\quad + 2\{(1-\nu)K_1(k_na) - k_naK_0(k_na)\}\frac{K_1(k_nr)}{k_nr}\Big] \\
&\quad \times\Big[(-1)^{n-1}n\Big\{k_na + \frac{2(1-\nu)}{k_na}\Big\}\{K_1(k_na)\}^2 \\
&\quad - k_na\{K_0(k_na)\}^2\Big]^{-1}\sin k_nz, \\
\sigma_z &= -\gamma z + \frac{2}{\pi}\frac{\nu}{1-\nu}\gamma l\sum_{n=1}^{\infty}[\{k_naK_0(k_na) \\
&\quad + 2K_1(k_na)\}K_0(k_nr) - K_1(k_na)k_nrK_1(k_nr)] \\
&\quad \times\Big[(-1)^{n-1}n\Big\{k_na + \frac{2(1-\nu)}{k_na}\Big\}\{K_1(k_na)\}^2 \\
&\quad - k_na\{K_0(k_na)\}^2\Big]^{-1}\sin k_nz, \\
\tau_{rz} &= -\frac{2}{\pi}\frac{\nu}{1-\nu}\gamma l\sum_{n=1}^{\infty}\{K_1(k_na)k_nrK_0(k_nr) \\
&\quad - k_naK_0(k_na)K_1(k_nr)\}\Big[(-1)^{n-1}n\Big\{k_na + \frac{2(1-\nu)}{k_na}\Big\} \\
&\quad \times\{K_1(k_na)\}^2 - k_na\{K_0(k_na)\}^2\Big]^{-1}\cos k_nz, \\
\tau_{r\theta} &= \tau_{\theta z} = 0.
\end{aligned}\right\} \quad \cdots\cdots\cdots (26)$$

3. Numerical example for the stress distribution around a circular shaft.

The stress distribution in an elastic ground around a circular shaft is given by Eqs. (26). The author then tried to calculate the stresses numerically giving an example, in which $\pi a/l = 0.05$ and $l/2a = 31.4$. The results of caluculation for σ_r, σ_θ and σ_z are shown in Figs. 2, 3 and 4, where the ratios of the stresses to $(-\gamma l)$ and r/a are taken on the ordinate and abscissa respectively. As the value of ν 1/3 and 1/5 were taken.

The relation between the ratios of each stress components at any point to the initial principal stresses at this point and r/a are shown in Fig. 5 by three full lines. Those ratios are nearly independent of either the Poisson's ratio or the depth.

Stress ratios σ_z/p_1, σ_θ/p_2 and σ_r/p_2 are taken on the ordinate of this figure, where p_1 and p_2 denotes the vertical initial stress $(-\gamma z)$ and the horizontal initial stress $\left(-\frac{\nu}{1-\nu}\gamma z\right)$ respectively.

It is found from Fig. 5 that the stresses around a vertical shaft with circular cross section are

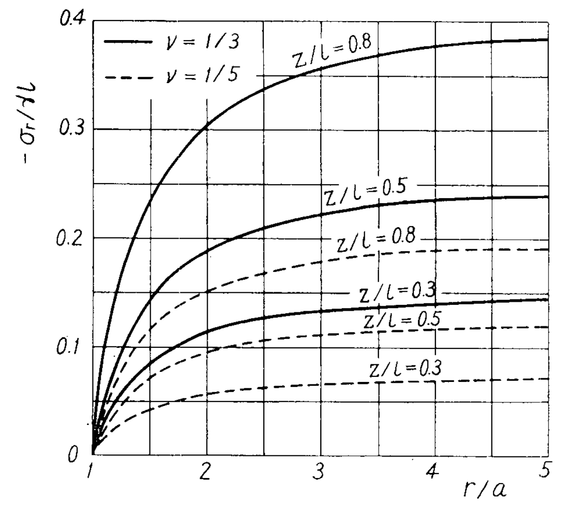

Fig. 2 Distribution of σ_r around a circular shaft in a ground where the state of the initial principal stresses is symmetrical in respect to the vertical axis.

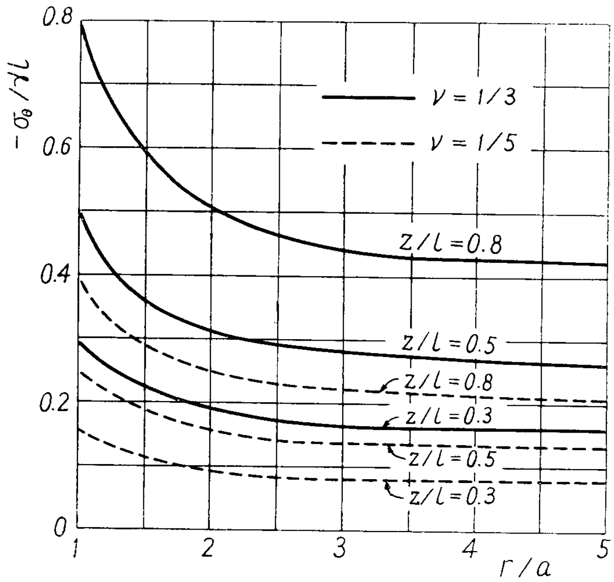

Fig. 3 Distribution of σ_θ around a circular shaft in a ground where the state of the initial principal stresses is symmetrical in respect to the vertical axis.

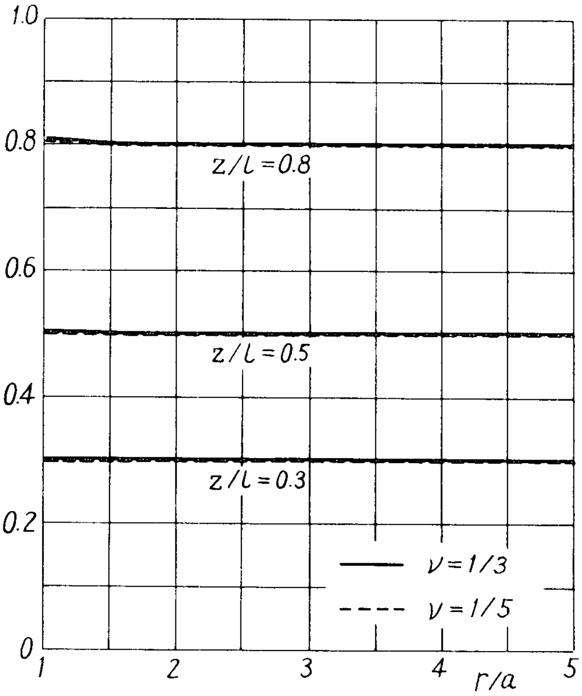

Fig. 4 Distribution of σ_z around a circular shaft in a ground where the state of the initial principal stresses is symmetrical in respect to the vertical axis.

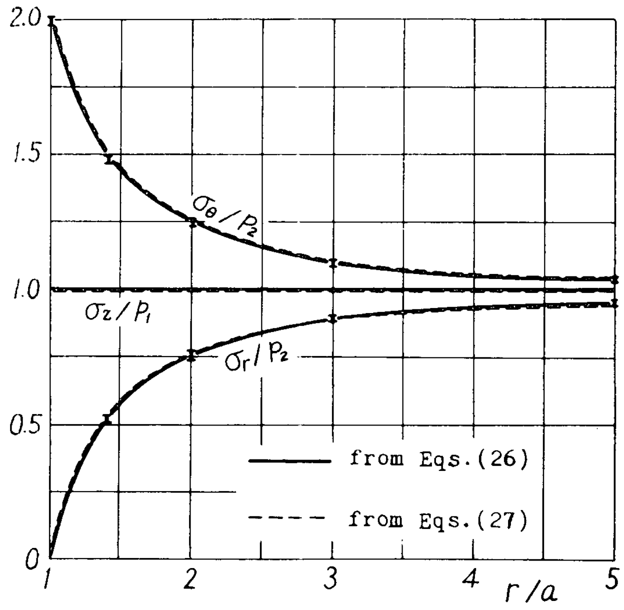

Fig. 5 The ratios of each stress components to the initial principal stresses around a circular shaft in a ground where the state of the initial principal stresses is symmetrical in respect to the vertical axis.

determined by r/a, the ratio of distance from the center line of the shaft to the radius of cross section of the shaft, and are nearly independent of the values of ν or z/l. The shearing stress vanishes on the shaft wall, and is also very small in the groud as compared with the other stress components.

4. Stress distribution obtained by two dimensional treatment.

The following simple equations giving the stresses around a circular shaft have been proposed by Refnetsky[4),]

$$\left.\begin{aligned}\sigma_r &= -\frac{\nu}{1-\nu}\left\{1-\left(\frac{a}{r}\right)^2\right\}\gamma z,\\ \sigma_\theta &= -\frac{\nu}{1-\nu}\left\{1+\left(\frac{a}{r}\right)^2\right\}\gamma z,\\ \sigma_z &= -\gamma z.\end{aligned}\right\} \qquad \cdots\cdots (27)$$

It is infered that these equations were derived by putting σ_r and σ_θ equal to the stresses which are produced around the opening in a slice by cutting the ground around the shaft by many horizontal planes, when this slice is loaded on the periphery at the infinite distance by compressive load of the intensity $\left(-\frac{\nu}{1-\nu}\gamma z\right)$, and putting σ_z equal to the stress which is produced when each slice is compressed in the direction of z-axis by compressive load of the intensity $(-\gamma z)$.

That is to say, taking a rectangular co-ordinate x, y, in a plane perpendicular to the z-axis, if the principal stresses p_2 and p_3 exist in the directions of x- and y-axis at the infinite distance, and p_3 is equal to zero, the stress components at any point in the ground around a shaft can be expressed by the following equations:

$$\left.\begin{aligned}\sigma_r &= \left[\left\{1-\left(\frac{a}{r}\right)^2\right\}+\left\{1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]\frac{p_2}{2},\\ \sigma_\theta &= \left[\left\{1+\left(\frac{a}{r}\right)^2\right\}-\left\{1+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]\frac{p_2}{2},\\ \tau_{r\theta} &= -\left[\left\{1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right\}\sin 2\theta\right]\frac{p_2}{2},\end{aligned}\right\} \qquad \cdots\cdots (28)$$

where r and θ denote the distance of this point from the origin and the angle between a straight line connecting the point with the origin and x-axis, respectively.

In the same manner those stress components for $p_2=0$ are given by

$$\left.\begin{aligned}\sigma_r &= \left[\left\{1-\left(\frac{a}{r}\right)^2\right\}-\left\{1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]\frac{p_3}{2},\\ \sigma_\theta &= \left[\left\{1+\left(\frac{a}{r}\right)^2\right\}+\left\{1+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta\right]\frac{p_3}{2},\\ \tau_{r\theta} &= -\left[\left\{1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right\}\sin 2\theta\right]\frac{p_3}{2}.\end{aligned}\right\} \qquad \cdots\cdots (29)$$

Consequently, if p_2 and p_3 exist simultaneously and $p_2=p_3=-\frac{\nu}{1-\nu}\gamma z$, the stress components σ_r, σ_θ and $\tau_{r\theta}$ take the form shown in Eqs. (27). These equations have very simple form as compared with Eqs. (26).

If we illustrate the ratios of the stress components to the initial stress against r/a in order to compare the stresses obtained by Eqs. (26) and (27), however, we can find the stresses obtained by both equations coincide very well. Accordingly, when we calculate the stresses around a circular shaft without the lining, nearly equal results are obtained using either Eqs. (26) or Eqs. (27).

For the wall surface of the shaft, follwing expressions for the stress components are obtained:

$$(\sigma_r)_{r=a}=0,\quad (\sigma_\theta)_{r=a}=-\frac{2\nu}{1-\nu}\gamma z,\quad \sigma_z=-\gamma z. \qquad \cdots\cdots (30)$$

5. Stresses around a circular shaft in an elastic ground where the directions of the three initial principal stresses are vertical and horizontal, and the two horizontal principal stresses have different magnitudes.

Denote by p_1, p_2 and p_3 the three initial principal stresses, then, if the direction of p_1 is vertical, and that of p_2 and p_3, the magnitudes of which are not equal, are horizontal, the stresses around a circular shaft are obtained as described below.

Let us now consider a rectangular co-ordinate x, y, z, in which z-axis coincides with the center line of the shaft, and the directions of x- and y-axis coincide with the directions of p_2 and p_3. The treatments described in **1**, **2** and **3** are the analysis relating to the case that the principal stress p_1 is equal to $(-\gamma z)$ and p_2, p_3 are equal to $\left(-\frac{\nu}{1-\nu}\gamma z\right)$.

As described in **3**, the stresses around a shaft in an elastic ground under such state of initial stresses can be obtained by either Eqs. (26) or (27), both of which give nearly equal results. Eqs. (27) mean, however, that, among the three stress components, σ_z is equal to the vertical intial stress p_1 at any point in the ground and that horizontal stresses σ_r, σ_θ are equal to the stresses which are produced around the shaft when p_2 and p_3, the magnitudes of which are equal, act at the infinite distance in a plane perpendicular to z-axis.

It is admitted without difficulty, even when the magnitudes of p_2 and p_3 are not equal, that we can obtain the magnitudes of σr and σ_θ as the sum of the stresses around the shaft, when the principal stress p_2 acts at the infinite distance in the direction of x-axis and the principal stress p_3 acts at the infinite distance in the direction of y-axis, utilizing the principle of superposition.

Let us now take a point on a straight line in a plane perpendicular to z-axis and denote by r and θ the distance from the center line of the shaft to this point and the angle between x-axis and the straight line. Then, the stresses which are produced around the shaft when the principal stresses p_2 and p_3 act in a plane perpendicular to the z-axis at the infinite diststance, in the direction of x-axis and y-axis respectively, can be obtained, utilizing the principle of superposition as sum of stresses which are produced by the action of p_2 only and the stresses which are produced by the action of p_3 only. Hence, from Eqs. (28) and (29), we find

$$\left.\begin{aligned}
\sigma_r &= \frac{1}{2}\left\{1-\left(\frac{a}{r}\right)^2\right\}(p_2+p_3)+\frac{1}{2}\left\{1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right\}(p_2-p_3)\cos 2\theta,\\
\sigma_\theta &= \frac{1}{2}\left\{1+\left(\frac{a}{r}\right)^2\right\}(p_2+p_3)-\frac{1}{2}\left\{1+3\left(\frac{a}{r}\right)^4\right\}(p_2-p_3)\cos 2\theta,\\
\sigma_z &= p_1,
\end{aligned}\right\}\cdots (31)$$

$$\tau_{r\theta} = -\frac{1}{2}\left\{1+2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right\}(p_2-p_3)\sin 2\theta,$$

$$\tau_{\theta z} = \tau_{zr} = 0.$$

6. Summary

The author treated the stress analysis for a circular shaft without lining sunk in an elastic ground in accordance with the results of investigation published by H. Suzuki, and gave the stress distribution obtained by numerical calculation for some concrete examples. He then compared the results of the calculation with the results obtained by simple equations, supposed to be derived by treatment as a two dimensional problem, and found that the both results coincide very well,

Being based on the fact mentioned above, he described the solution of the stresses around a circular shaft, in case the principal stresses in the ground are vertical and horizontal and the two horizontal principal stresses have different magnitudes.

The work described in this paper was carried out at the Department of Mining, Kyoto University, as a part of research on the rock mechanics relating to the mine shaft. Acknowledgements are due to Professor Hiramatsu and Assistant Professor Oka, who took the leadership throughout this investigation.

References

1) T. Sugihara: Jour. Min. Met. Inst. Jap., 47, 1176 (1931)
2) H. Suzuki: Jour. Min. Met. Inst. Jap., 65, 118 (1949)
3) A. E. H. Love: "Mathematical Theory of Elasticity", Cambridge University Press (1959) p. 143
4) I. J. Bjaler: Ygol, 14 (1956)