Pair Connections on Homogeneous spaces which are Invariant under Tangential Transformations

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Abstract

Two types of invariant pair connections are considered, one has an invariant horizontal subspace and parallel translations with respect to some curves in tangent space are represented by left translations and the otner has an invariant horizontally horizontal subspace and projec tions of one parameter subgroups are paths of a base manifold.

Introdnction

In the formulation of Finsler geometry, M. Matsumoto[1] and T. Okada[2] have introduced a notion of a pair connection on a tangent vector bundle of a differentiable manifold. The purpose of this paper is to study a pair connection on a homogeneous manifold invariant under a suitable differentiable transformation group which was introduced by S. Kobayashi[3] as a tangential group. Pair connections of a manifold M are originally considered in a bundle Q on a tangent vector bundle B which is induced by a projection $\tau : B \rightarrow M$. But in this paper we give our definition in a bundle $T(P)$ for our purpose. K. Nomizu[4] gave the definition of ordinary invariant connection on a homogeneous space as the connection invariant under transformations of a base manifold. This seems too general for our case, because we treat a bundle on B. Therefore, here we consider transformations on B induced from transformations of base manifold M, represented with multiplication of tangent vectors to the transformation grop. Two types of invariant connections are defined in this paper, one has an invariant horizontal subspace and the other has an invariant horizontally horizontal subspace. These connections correspond to invariant connections of the first kind and the second kind of K. Nomizu. The author wishes to express his sincere gratitude to Prof. M. Matsumoto for his encouragement and advice during the preparation of this paper.

1. **Preliminaries**.[5),6)] Let P be a Lie group, $\mathfrak{p}$ its Lie algebra, and $T(P)$ a tangent bundle space of P, that is, a space of all tangent vectors to P. $T(P)$ can be identified with $P\times \mathfrak{p}$ by an isomorphism which maps a vector X_p to a pair (p, A) where $A=L_{p^{-1}}X_p$.

Let $\varphi : P\times P \rightarrow P$ be a map of multiplication of group P, then the induced map $\delta\varphi : T(P)\times T(P) \rightarrow T(P)$ defines multiplication in the space $T(P)$, that is, in $P\times\mathfrak{p}$.

For two pairs (p_1, X_1), (p_2, X_2) in $P\times\mathfrak{p}$ their product is given by

$$(p_1, X_1)\cdot(p_2, X_2)=(p_1 p_2,\ \mathrm{ad}(p_2^{-1})X_1+X_2)$$

and this multiplication makes $T(P)$ into a Lie group, called a tangential group

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over P.

Let P be a Lie grup, acting transitively and effectively on a manifold M. Let G be an isotropic subgroup of P at a point $X_0 \in M$. Then the manifold M can be identified with a quotient space P/G. If we let G act on P on the right, we can get a principal fibre bundle $P(M, G)$ over M with a structural group G.

Now let $\pi: P \to M$ be a bundle projection, and let $r: P \times G \to P$ be a right translation. Then their differential maps $\partial\pi: T(P) \to T(M)$ and $\partial r: T(P) \times T(G) \to T(P)$ together define a principal bundle structure $T(P)(T(M), T(G))$, called tangential bundle of $P(M, G)$. We donote π' for $\partial\pi$.

A tangent bundle (or tangent bundle space) of M is denoted by $T(M)$ or B, and an induced bundle of $P(M, G)$ by the mapping $\tau: B \to M$ is denoted by $Q(B, G)$.

A principal bundle $T(P)(B, T(G))$ is uniquely determined as an extension of $Q(B, G)$.

Let $\bar{e}$ be the class eG of P/G. A tangent space $T_{\bar{e}}(M)$ is isomorphic to a vector space $F = \mathfrak{p}/\mathfrak{g}$. Let $\tilde{B}$ be an associated bundle of Q, with fibre F, then $\tilde{B}$ is a product of B. The operation of $T(G) = G \times \mathfrak{g}$ on F is defined by $(g, U) \cdot f = \text{is}(g) \cdot f$ for $g \in G$, $U \in \mathfrak{g}$, $f \in F$, then a bundle space of an associated bundle of $T(P)$ with fibre F coincides with a bundle space of B. So we denote it by $\tilde{B}$.

For the convenience, we abbreviate $T(T(M))$ as $T^2(M)$. We have isomorphic relations $T(M) \cong T(P)/T(G)$ and $T^2(M) \cong T^2(P)/T^2(G)$.

To identify $T^2(P)$ with $(P \times \mathfrak{p}) \times (\mathfrak{p}^{(1)} + \mathfrak{p}^{(2)})$ we denote a left translation of $T(P)$ by an element $(p, X) \in T(P)$ by $L_{(p,X)}$. Then by this translation $T^2(P)$ can be identified with $T(P) \times T^2_{(e,o)}(P)$, and $T^2_{(e,o)}(P)$ with $\mathfrak{p}^{(1)} + \mathfrak{p}^{(2)}$, where (e, O) is a unit of $T(P)$. Therefore, elements of $T^2(P)$ can be denoted in a form $(p, X; A, B)$, $p \in P$, $X \in \mathfrak{p}$, $A \in \mathfrak{p}^{(1)}$, $B \in \mathfrak{p}^{(2)}$ and an adjoint transformation by an element of $T(G)$, operated on $(A, B) \in T^2_{(e,o)}(P)$ is

$\text{ad}(g, U) \cdot (A, B) = (\text{ad}(g) \cdot A,\ \text{ad}(g) B - \text{ad}(g) \cdot [A, U])$ for $(g, U) \in T(G)$.

2. Pair connections Let Q be a bundle over B as above. If at each point q in Q a tangent space $T_q(Q)$ is decomposed into a direct sum satisfying the following conditions $(a) \sim (f)$, then we say that a pair connection is defined in the bundle Q:

(a) $T_q(Q) = Q^v_q + \Gamma'^v_q$, where Q^v_q is a tangent space to a fibre through q in Q.

(b) $R_g \Gamma'_q = \Gamma'_{qg}$ for any $q \in Q$, $g \in G$.

(c) Γ'_q depends differentiably on q.

(d) $\Gamma'_q = \Gamma'^v_q + \Gamma'^h_q$, where Γ'^v_q covers a vertical space of $T_b(B)$. $b = \pi q$.

(e) $R_g \Gamma'^h_q = \Gamma'^h_{qg}$

(f) Γ'^h_q depends differentiably on q.

Obviously an ordinary connection is induced on $Q(B, G)$ by the above conditions $(a) \sim (c)$. Once a pair connection is defined on Q, the injection $Q \to T(P)$ induces the pair connection on $T(P)$, so we shall confine ourselves to the study of a pair connection on $T(P)$ and denote by the same symbols $\Gamma'_{(p,X)}$, $\Gamma'^h_{(p,X)}$, $\Gamma'^v_{(p,X)}$ on $T(P)$.

3. Invariant pair connections of the first kind on $T(P)(B, T(G))$ Assume that a pair connection is defined on $T(P)$. The condition that a vector $(p, X; A, B)$ in $T(P)$ belongs to the class O in $T^2(P)/T^2(G)$ i, e. the condition that a vector $(p, X; A, B)$ is a vertical vector is

Proposition $(p, X; A, B)$ is vertical if and only if A and B are in $\mathfrak{g}$

proof. $(p, X; A, B)$ is vertical if there exist $g \in G$, and $U, V, W \in \mathfrak{g}$ such that ad $(g^{-1})A + V = 0$, $[\text{ad}(g^{-1})A, U] + \text{ad}(g^{-1})B + W = 0$, and conversely.

Similarly, we have

$(p, X; A, B) \in \Gamma^{v}_{(p.X)}$ if and only if

$\begin{cases} (1) & (p, X; A, B) \text{ is horizontal} \\ (2) & A \in \mathfrak{g} \end{cases}$

Now we shall study a pair connection on $T(P)$ which is invariant under the transformation of B induced from the transformation of M. We begin with the following definition.

Definition If a pair connection on $T(P)(B, T(G))$ satisfies the following condition, we call it (C1)—connection.

(C1) $L_{(p_1.X_1)}\Gamma'_{(p_2.X_2)} = \Gamma'_{(p_1.X_1)(p_2.X_2)}$ for any $p_1, p_2 \in P$ and any $X_1, X_2 \in \mathfrak{p}$

From this definition we have

Theorem If there exists a (C,1)-connection on $T(P)$, the Lie algebra of $T(P)$, i, e. $\mathfrak{p}^{(1)} + \mathfrak{p}^{(2)}$, admits a decomposition into a direct sum (D1)

(D1) $\mathfrak{p}^{(1)} + \mathfrak{p}^{(2)} = \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)} + \mathfrak{M}$, $\mathfrak{p}^{(i)} \supset \mathfrak{g}^{(i)}$

where a vector subspace $\mathfrak{M}$ satisfies $ad(T(G))\,\mathfrak{M} = \mathfrak{M}$

Remark. A pair connection is determined by (D1) if we give nonlinear conneciion on $B(M,G)$

Proof. The decomposition at unit element (e,O) and invariance under a left translation $L_{(p.X)}$ lead us to the result (D1).

If P/G is reductive we can define (C1)-connection as follows.

From reductivity of P/G we have $\mathfrak{p} = \mathfrak{g} + \mathfrak{m}$, $ad(G)\mathfrak{m} = \mathfrak{m}$ thus as for $\mathfrak{M}$ we put $\mathfrak{M} = \mathfrak{m}^{(1)} + \mathfrak{m}^{(2)}$ where $\mathfrak{m}^{(1)} \subset \mathfrak{p}^{(1)}$, $\mathfrak{m}^{(2)} \subset \mathfrak{p}^{(2)}$ and for Γ^h we take a lift of invariant horizontal subspace of B in the sense of K. Nomizu.

The pair connection obtained in this way is called a canonical (C1)-connection.

Proposition On a reductive homogeueous space P/G let ω be a connection form of an ordinary invariant connection defined by this decomposition. Then the conection form $\tilde{\omega}$ of the canonical (C1) connection induced by this ordinary connection is given by

$\tilde{\omega}(p, X; A, B) = (\omega(e, A), \omega(e, B))$

Proof. For a horizontal vector $(p, X; A, B)$ $A, B \in \mathfrak{m}$, $\tilde{\omega}(p, X; A, B)$ is equal to (O, O) and for a vertical vector (p, X, U, V) $U, V \in \mathfrak{G}$ (p, X, U, V) is equal to (U, V). We have, by a formula of adjoint transformation, $R_{(g.U)}\,\tilde{\omega} = ad((g, U)^{-1}) \cdot \tilde{\omega}$, so this proposition is valid.

Similar to the case of ordinary connections we have

Propostion On a reductive homogeneous space P/G the horizontal space of a (C1)-connection is uniquely determined by the following pair of R-linear mappings.

$\rho_1 : \mathfrak{M} \to \mathfrak{g}, \quad \rho_1 \cdot ad(g, U)(A, B) = ad(g) \cdot \rho_1(A, B)$

$\rho_2 : \mathfrak{M} \to \mathfrak{g}, \quad \rho_2 \cdot ad(g, U)(A, B) = ad(g) \cdot \rho_2(A, B) - ad(g) \cdot [\mathfrak{g}_1(A, B), U]$

If (C1)—connection is given on $T(P)$, then we can define a pair connection on B, this is done by mapping R_{f_0}

in[2], and there arises the notion of parallel translations.

The parallelism is defined as follows; A lift $\tilde{b}(t)=(b(t), d(t))$ on $\tilde{B}$ of a curve $b(t)$ on B is uniquely determined if we give $d(O)$, and we call $d(t)$ the parallel translation of $d(O)$ with respect to $b(t)$.

Theorem Consider $T(P)$ on which (C1)—connection is defined. Let (A,B) be an arbitrary element in $\mathfrak{M}$. The one-parameter subgroup $(p(t).X(t))=\exp t(A, B)$ generated by (A, B) induces a curve $b(t)=\pi'(p(t), X(t))$ on B. Then, a parallel translation of a vector (e,f_0) tangent to P/G at $\bar{e}$ with respect to $b(t)$ is the same as the left translation of (e,f_0) by the element $p(t)$, that is $L_{p(t)}(e,f_0)$.

Proof. The result follows immediately if we show that the horizontal curve $\tilde{b}(t)$ on $\tilde{B}$ through (e, f_0) which covers $b(t)$ is given by

$$\tilde{b}(t)=(\pi'(p(t), X(t)), (p(t), f_0))$$

4. Invariant pair connection of the second kind on $T(P)$ $(B,T(G))$ we shall define (C2) – connection by

Definition If a pair connection on $T(P)$ satisfies a condition

(C2) $L_{(p_1.X_1)}\Gamma^h_{(p_2.X_2)}=\Gamma^h_{(p_1.X_1)(p_2.X_2)}$ for any $p_1,p_2\in P$ and any $X_1,X_2\in\mathfrak{p}$, we call it (C2) –connection.

Theorem For the (C2) – connection on $T(P)$, we have the following decomposition of the Lie algebra

(D2) $\mathfrak{p}^{(1)}+\mathfrak{p}^{(2)}=\mathfrak{g}^{(1)}+\mathfrak{g}^{(2)}+\mathfrak{M}_1+\mathfrak{M}_2$

where $\mathfrak{p}^{(i)}\supset\mathfrak{g}^{(i)}$, $\gamma\mathfrak{M}_2\subset\mathfrak{g}^{(1)}$ $\mathfrak{M}_1\simeq\mathfrak{M}_2\simeq F$

$ad(T(G))\mathfrak{M}_1=\mathfrak{M}_1$, γ is a projection to the $\mathfrak{p}^{(1)}$ – component.

Proof. This is easily shown if we take as $\mathfrak{M}_2$ the vertically horizontal subspace at unit (e, O) and for $\mathfrak{M}_1$ the horizontally horizontal subspace at (e,O).

As for the property of a homogeneous space which admits a (C2) – connection, we have

Theorem If $P\supset G$ admits on its tangential bundle, the decomposition shown in the above proposition, then G is an invariant subgroup of P.

Proof. Spaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both isomorphic to F. Therefore, there exists isomorphisms $j_i: F\to\mathfrak{M}_i$. Let β be a projection to $\mathfrak{p}^{(2)}$ – component and we denote two mappings $\gamma\circ j_i$ and $\beta\circ j_i$ with $j_i^{(1)}$ ajd $j_i^{(2)}$ respectively.

These are mappings onto linear spaces. If we project both sides of decomposition (D2) by projection γ, then their images are $\mathfrak{p}^{(1)}=\mathfrak{g}^{(1)}+\gamma\mathfrak{M}_1$

On the right hand side of this relation the sum is a direct sum. Denote $\gamma\mathfrak{M}_1$ by $\mathfrak{m}_1$, then dim $\mathfrak{m}_1$ is equal to dim F, and so $j_1^{(1)}$ is regular. Let j be a mapping $j_1^{(2)}\circ j_1^{(1)^{-1}}$. An element of $\mathfrak{M}_1$ can be represented uniquely in the form (X, jX) where $X\in\mathfrak{m}_1$.

The condition $ad(T(G))\mathfrak{M}_1=\mathfrak{M}_1$ shows

$$ad(g, U)(X, jX) = (ad(g)X, ad(g)jX - ad(g)\cdot[X, U])$$

for any $g\in G$, and any $U\in\mathfrak{g}$.

Therefore, we have $(X, jX\ [X, U])\in\mathfrak{m}_1$ for any $U\in\mathfrak{g}$.

Because of uniqueness of this representation we have

$[X. U]=0$ for any $U\in\mathfrak{g}$, any $X\in\mathfrak{m}_1$.

This implies $[\mathfrak{m}_1,\mathfrak{g}]=0$ and hence $[\mathfrak{p},\mathfrak{g}]\subset[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}$

The last relation shows that $\mathfrak{g}$ is an ideal of $\mathfrak{p}$, and that G is an invariant subgroup of P.[6]

(C 2)−connection is called (SC 2)−connection if the following condition (SC 2) is satisfied.

(SC 2) $\beta\mathfrak{M}_1=0$

Theorem Assume that on $T(P)$ (SC 2)−connection is defined. Let (A, O) be an arbitrary element in $\mathfrak{M}_1$, and let expt (A, O) be an one-parameter subgroup generated by (A, O). Then $x(t) \equiv \tau\circ\pi'$ expt (A, O) is a path on P/G.

Remark. If a lift $b(t)$ on B of a curve $x(t)$ of M, coincides with a tangent vector field $x'(t)$, then $x(t)$ is called a path on M.

Proof. Let expt (A, O) be $(p(t), O)$. The tangent vector of this curve is $(p(t), O)'=L_{(p(t),O)}(e, O; A, O)$. Let $b(t)$ be a curve $\pi'(p(t), O)$, $b(t)$ is a curve on B and its tangent vector at $b(t)$ is

$b'(t) = \pi'(p(t), O)' = L_{p(t),O)}\pi'(e, O; A, O)$.

Let $x(t)$ be a curve $\tau\cdot b(t)$. Then its tangent vector at $x(t)$ is given by

$x'(t)=\pi\cdot b'(t)=\tau\cdot L_{(p(t).O)}\pi'(e, O; A, O)=L_{p(t)}\tau\cdot\pi'(e, O; A, O)=L_{p(t)}\tau\cdot\gamma(e, O; A, O)=L_{p(t)}\pi(e, A)=\pi(p(t), A)$.

The curve $x'(t)$ in B covers $x(t)$, that is $\tau\cdot x'(t)=x(t)$, and its tangent vector at $x'(t)$ is

$x''(t)=\pi'\cdot L_{(p(t).O)}(p(o), A)'=L_{(p(t).O)}\pi'(e, A; A, O)=L_{(p(t).A)}\pi'(e, O; A, O)$.

$(e, O; A, O)$ is a horizontal vector, and (SC 2) means that $x''(t)$ is also horizontal vector. Therefore, $x'(t)$ is a horizontal curve. And $x(t)$ is a path of M.

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