A Construction of Cartesian Closed Categories

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Abstract

In this note we present a simple proof of cartesian closedness of the category of topological spaces and k-continuous maps with the universal property of evaluation maps.

Throughout of this note we assume that $\mathscr{C}$ is a category with finite products, $\mathscr{C}$ is a subcategory of a category $\mathscr{A}$ with $Obj\,\mathscr{C}=Obj\,\mathscr{A}$, and the inclusion functor of $\mathscr{C}$ into $\mathscr{A}$ preserves finite products. Moreover, let $\mathscr{S}$ be a full subcategory of $\mathscr{C}$. We define $\mathscr{S}\mathscr{C}$ as the subcategory of $\mathscr{A}$ with objects all objects of $\mathscr{A}$ and with arrows f: $x\to y$ all those arrows $f: x\to y$ in $\mathscr{A}$ for which the composite $f\alpha: s\to y$ lies in $\mathscr{C}$ for any arrow $\alpha: s\to x$ in $\mathscr{C}$ with $s\in Obj\,\mathscr{S}$. We can easily verify that $\mathscr{C}$ is a subcategory of $\mathscr{S}\mathscr{C}$ and the inclusion functor of $\mathscr{C}$ into $\mathscr{S}\mathscr{C}$ preserves finite products. We say an object x in $\mathscr{A}$ is an *$\mathscr{S}$-generated* object if any arrow $f: x\to y$ in $\mathscr{S}\mathscr{C}$ with domain x lies in $\mathscr{C}$. Let $\mathscr{S}\mathscr{G}$ denote the full subcategory of $\mathscr{C}$ with objects all $\mathscr{S}$-generated objects.

Let $k: \mathscr{S}\mathscr{C}^{op}\times\mathscr{S}\mathscr{C}\to\mathscr{A}$ be a functor with a dinatural transformation $\varepsilon_{<y,z>}$: $k(y, z)\times y\to z$ in $\mathscr{A}$ (precisely, natural in z and dinatural in y), that is, for any arrows $g: z\to z'$ and $h: y'\to y$ in $\mathscr{S}\mathscr{C}$, the following two diagrams

$$(1)\qquad \begin{array}{ccc} k(y, z)\times y & \xrightarrow{\varepsilon_{<y,z>}} & z \\ {\scriptstyle k(y,g)\times y}\downarrow & & \downarrow{\scriptstyle g} \\ k(y, z')\times y & \xrightarrow[\varepsilon_{<y,z'>}]{} & z' \end{array}$$

$$(2)\qquad \begin{array}{ccc} k(y, z)\times y' & \xrightarrow{k(y,z)\times h} & k(y, z)\times y \\ {\scriptstyle k(h,z)\times y'}\downarrow & & \downarrow{\scriptstyle \varepsilon_{<y,z>}} \\ k(y', z)\times y' & \xrightarrow[\varepsilon_{<y',z>}]{} & z \end{array}$$

are commutative. (Note that $a\times b$ denotes the product of a and b in $\mathscr{C}$.)

The dinatural transformation $\varepsilon_{<y,z>}: k(y, z)\times y\to z$ is called *quasi-universal* if it satisfies the following universal property: For any arrow $f: x\times y\to z$ in $\mathscr{S}\mathscr{C}$, there is a unique arrow $\hat{f}: x\to k(y, z)$ in $\mathscr{A}$ such that $f=\varepsilon_{<y,z>}(\hat{f}\times y)$.

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(3)
$$\begin{array}{ccc} x & \qquad & x\times y \\ \hat{f}\downarrow & & \hat{f}\times y\downarrow \quad \searrow f \\ k(y,z) & & k(y,z)\times y \xrightarrow[\varepsilon_{<y,z>}]{} z \end{array}$$

An object y in $\mathscr{A}$ is $(\mathscr{S}\text{-})$ *admissible* if $\varepsilon_{<y,z>}$ is an arrow in $(\mathscr{S})\mathscr{C}$ for any object z in $\mathscr{A}$. An object y in $\mathscr{A}$ is $(\mathscr{S}\text{-})$ *proper* if, for any arrow $f\colon x\times y\to z$ in $(\mathscr{S})\mathscr{C}$, the unique arrow $\hat{f}\colon x\to k(y, z)$ is an arrow in $(\mathscr{S})\mathscr{C}$, that is, if, whenever $\varepsilon_{<y,z>}(h\times y)$ lies in $(\mathscr{S})\mathscr{C}$, so does h.

Throughout the rest of this note we assume that there are given a functor $k\colon \mathscr{S}\mathscr{C}^{op}\times\mathscr{S}\mathscr{C}\to\mathscr{A}$ with a quasi-universal dinatural transformation $\varepsilon_{<y,z>}\colon k(y, z)\times y\to z$ and a full subcategory $\mathscr{U}$ of $\mathscr{C}$ with $\mathscr{S}\subset\mathscr{U}\subset\mathscr{S}\mathscr{G}$, satisfying the following four axioms:

(A) If s and t are two objects in $\mathscr{S}$, then $s\times t$ is an $\mathscr{S}$-generated object.

(B) Every object s in $\mathscr{S}$ is admissible.

(C) Every object u in $\mathscr{U}$ is proper.

(D) An arrow $h\colon x\to k(y, z)$ in $\mathscr{A}$ is an arrow in $\mathscr{C}$ if and only if, for any arrow $n\colon u\to y$ in $\mathscr{C}$ with $u\in Obj\,\mathscr{U}$, the composite

$$x \xrightarrow{h} k(y, z) \xrightarrow{k(n,z)} k(u, z)$$

lies in $\mathscr{C}$.

Lemma 1. *For any arrow $n\colon u\to y$ in $\mathscr{C}$ with $u\in Obj\,\mathscr{U}$, the arrow $k(n, z)$: $k(y, z)\to k(u, z)$ lies in $\mathscr{C}$.*

Proof. It is immediate from the axiom (D).

Proposition 2. *If $g\colon y'\to y$ is an arrow in $\mathscr{S}\mathscr{C}$, then $k(g, z)\colon k(y, z)\to k(y', z)$ is an arrow in $\mathscr{C}$.*

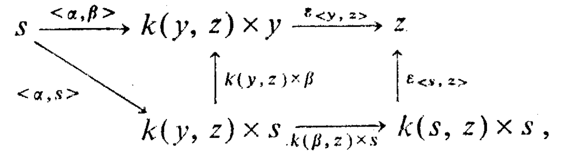
Proof. By virtue of the axiom (D), we have only to prove that the composite $k(n, z)k(g, z)$ lies in $\mathscr{C}$ for any arrow $n\colon u\to y'$ in $\mathscr{C}$ with $u\in Obj\,\mathscr{U}$. But $k(n, z)k(g, z)=k(gn, z)$, that is, the triangle

$$\begin{array}{ccc} k(y,z) & \xrightarrow{k(g,z)} & k(y',z) \\ & {\scriptstyle k(gn,z)}\searrow & \downarrow{\scriptstyle k(n,z)} \\ & & k(u,z) \end{array}$$

is commutative and gn lies in $\mathscr{C}$, because u is an $\mathscr{S}$-generated object. Hence, by Lemma 1, $k(gn, z)$ lies in $\mathscr{C}$.

Theorem 3. *Every object y in $\mathscr{A}$ is $\mathscr{S}$-admissible.*

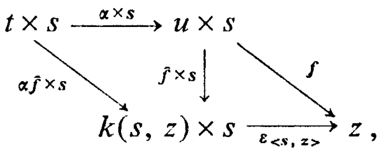
Proof. We have to show that $\varepsilon_{<y,z>}<\alpha, \beta>$ lies in $\mathscr{C}$ for any arrow $<\alpha, \beta>$: $s\to k(y, z)\times y$ in $\mathscr{C}$ with $s\in Obj\,\mathscr{S}$. In the commutative diagram



three arrows $<\alpha, s>$, $k(\beta, z)\times s$ and $\varepsilon_{<s,z>}$ lie in $\mathscr{C}$, by Lemma 1 ($s\in\mathscr{S}\subset\mathscr{U}$) and the axiom (B). Therefore $\varepsilon_{<y,z>}<\alpha, \beta>$ is an arrow in $\mathscr{C}$, as desired.

Theorem 4. *If u is an $\mathscr{S}$-generated object and s is an object in $\mathscr{S}$, then $u\times s$ is an $\mathscr{S}$-generated object.*

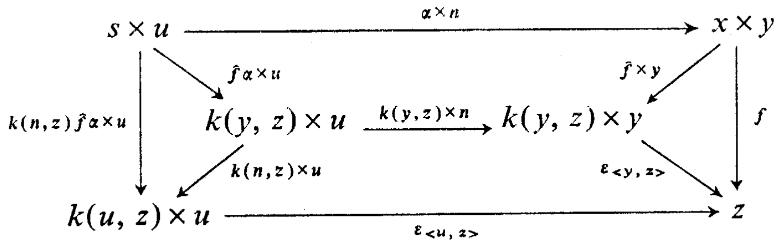
Proof. To prove this theorem, it suffices to show that every arrow $f: u\times s\to z$ in $\mathscr{S}\mathscr{C}$ lies in $\mathscr{C}$. Let $\alpha: t\to u$ be an arrow in $\mathscr{C}$ with $t\in Obj\,\mathscr{S}$. In the commutative diagram in $\mathscr{A}$



$f(\alpha\times s)$ lies in $\mathscr{S}\mathscr{C}$ ($\alpha\times s\in\mathscr{C}, f\in\mathscr{S}\mathscr{C}$) and hence $f(\alpha\times s)$ lies in $\mathscr{C}$ since $t\times s$ is an $\mathscr{S}$-generated object from the axiom (A). Applying the axiom (C) to $s\in\mathscr{S}\subset\mathscr{U}$, we have $\alpha\hat{f}$ is an arrow in $\mathscr{C}$. This states that $\hat{f}$ lies in $\mathscr{S}\mathscr{C}$. But u is an $\mathscr{S}$-generated object, so $\hat{f}$ lies in $\mathscr{C}$. From the axiom (B), $\varepsilon_{<s,z>}$ lies in $\mathscr{C}$ and consequently so does $f=\varepsilon_{<s,z>}(\hat{f}\times s)$. This proves the theorem.

Theorem 5. *Every object y in $\mathscr{A}$ is $\mathscr{S}$-proper.*

Proof. To prove this theorem we will show that, for any arrow $f: x\times y\to z$ in $\mathscr{S}\mathscr{C}$, the unique arrow $\hat{f}: x\to k(y, z)$ lies in $\mathscr{S}\mathscr{C}$. Consider the commutative diagram



for all arrows $\alpha: s\to x$ in $\mathscr{C}$ with $s\in Obj\,\mathscr{S}$ and $n: u\to y$ in $\mathscr{C}$ with $u\in Obj\,\mathscr{U}$. Since $s\times u$ is an $\mathscr{S}$-generated object by Theorem 4, $f(\alpha\times n)$ lies in $\mathscr{C}$. On the other hand, u is proper from the axiom (C). Hence $k(n, z)\hat{f}\alpha$ lies in $\mathscr{C}$ and $\hat{f}\alpha$ lies in $\mathscr{C}$ from the axiom (D). This proves that $\hat{f}$ lies in $\mathscr{S}\mathscr{C}$.

Theorem 6. *If $g: z\to z'$ is an arrow in $\mathscr{S}\mathscr{C}$, then $k(y, g)$: $k(y, z)\to k(y, z')$ is an arrow in $\mathscr{S}\mathscr{C}$.*

Proof. Consider the commutative square

$$\begin{array}{ccc} k(y,z)\times y & \xrightarrow{\varepsilon_{<y,z>}} & z \\ {\scriptstyle k(y,g)\times y}\downarrow & & \downarrow{\scriptstyle g} \\ k(y,z')\times y & \xrightarrow[\varepsilon_{<y,z'>}]{} & z'. \end{array}$$

Since g is an arrow in $\mathscr{S}\mathscr{C}$ from the hypothesis and $\varepsilon_{<y,z>}$ is an arrow in $\mathscr{S}\mathscr{C}$ by Theorem 3, the composite $g\varepsilon_{<y,z>}$ lies in $\mathscr{S}\mathscr{C}$ and so the arrow $k(y, g)=\{g\varepsilon_{<y,z>}\}^\wedge$ lies in $\mathscr{S}\mathscr{C}$ by Theorem 5, as desired.

As a consequence of Proposition 2, Theorem 6 and Theorem 3, we can conclude the following

Theorem 7. *The category $\mathscr{S}\mathscr{C}$ is cartesian closed.*

From the standard arguments in cartesian closed categories, we have the exponential laws: For all objects x, y, z in $\mathscr{C}$, there exist natural isos

$$k(x\times y, z)\simeq k(x, k(y, z))$$

and

$$k(x, y\times z)\simeq k(x, y)\times k(x, z)$$

in $\mathscr{S}\mathscr{C}$.

Example 8. Let $\mathscr{C}$ be the category of topological spaces and continuous maps, $\mathscr{A}$ the category of topological spaces and set maps, and $\mathscr{S}$ the full subcategory of $\mathscr{C}$ consisting of compact Hausdorff spaces. Then $\mathscr{S}\mathscr{G}$ is the category of compactly generated spaces [5]. Further, let $\mathscr{U}$ be some full subcategory of $\mathscr{C}$ with $\mathscr{S}\subset\mathscr{U}\subset\mathscr{S}\mathscr{G}$. Denote by $F(X, Y)$ the function space of all continuous maps $X\rightarrow Y$ with compact open topology, and by $k(Y, Z)$ the function space of all maps $Y\rightarrow Z$ in $\mathscr{S}\mathscr{C}$ with the initial (or smallest) topology determined by the family of set maps

$$k(Y, Z)\xrightarrow{n^*} F(U, Z): f\mapsto fn,$$

where $U\in Obj\,\mathscr{U}$ and $n: U\rightarrow Y$ is a continuous map. (Note that n^* is well defined because $U\in Obj\,\mathscr{S}\mathscr{G}$.) It is trivial that $k(U, Z)=F(U, Z)$ for $U\in Obj\,\mathscr{U}$, $k: \mathscr{S}\mathscr{C}^{op}\times\mathscr{S}\mathscr{C}\rightarrow\mathscr{A}$ is a bifunctor and the axiom (D) is satisfied. Next, define $\varepsilon_{<Y,Z>}: k(Y, Z)\times Y\rightarrow Z$ to be the usual evaluation map. Then it can be checked without difficulty that $\varepsilon_{<Y,Z>}$ is a quasi-universal dinatural transformation and that the axioms (A), (B) and (C) follows from the familiar properties of compact open topology. Hence, the category $\mathscr{S}\mathscr{C}$ is a cartesian closed category by Theorem 7.

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