1. Introduction

Computers have been used mainly for numerical computation in science and engineering, for example, stress analysis in solid mechanics, simulations of crack propagation in fracture mechanics, computations of band structure in condensed matter physics, and so on.

But recently the use of computers for non-numerical problems has rapidly increased. Pattern recognition of hand-written letters or pictures by computers is typical of such use.

For the recongnition of these it is important to make computers distinguish "connected components" of figure. For instance, in case of the pattern recognition of a portrait we must make computers recognize what part of the picture is the nose of the person. The nose, in short, is a "connected component" of face. There is another example. When a picture of a configulation of the ground is shown, how does a computer understand which part of the picture is a mountain?

By interpreting the distribution of normatized degree of brightness of a portrait as a fuzzy subset, fuzzy set theory plays an important role of pattern recongnition. From this point of view, Rosenfeld introduced the concept of connetedness for a fuzzy subset in a rectangular array of integer-coordinate points and studied the properties for picture analysis and description [2]. The purpose of this paper is to extend the concept of "fuzzy connectedness" and the properties to topological spaces. In section 2 we shall define strongly arcwise-connectedness and weakly arcwise-connectedness and discuss the properties. In section 3 we shall study the properties of tops in metric space and show that "arcwise-connected components" of tops with respect to a fuzzy subset have the properties simliar to standard ones.

2. Fuzzy Arcwise-Connectedness

Let X be a topological space. By a fuzzy subset of X, we mean a map from X to the interval [0, 1]. A continuous map ϕ from [0, 1] to X is called a continuous curve from ϕ (0) to ϕ (1) in X. Throughout this section let X be a topological space, σ a fuzzy subset and ϕ a continuous curve in X.

For a continuous curve ϕ in X we define the strength $S\sigma(\phi)$ of ϕ with respect to σ as $\inf\{\sigma(\phi(t)) : 0 \leqq t \leqq 1\}$.

When P and Q are arcwise-connected in the standard sense, that is, there is a continuous curve in X from P to Q, we define the degree of connectedness $C\sigma(P, Q)$ of P and Q with respect to σ as $\sup\{S\sigma(\phi)\}$ where the supremum is taken over all continuous curves from P to Q. We clearly have the inequality $0 \leqq C\sigma(P, Q) \leqq \mathrm{Min}\{\sigma(P), \sigma(Q)\}$.

DEFENITION 2. 1. Let P and Q be in X. If the equality $C\sigma(P, Q)=\mathrm{Min}\{\sigma(P), \sigma(Q)\}$ holds, we say that P and Q are weakly arcwise-connected in σ. By w-$C\sigma$ we denote the set of all pairs (P, Q) such that P and Q are weakly arcwise-connected in σ.

PROPOSITION 2. 2. The relation w-$C\sigma$ is reflexive and symmetric, but not necessarily transitive.

Proof. It is evident that w-$C\sigma$ is reflexive and symmetric. Let X be the set of all real numbers R and $\sigma(X)=|\cos x|$. Two pairs $(0, \pi/2)$ and $(\pi/2, \pi)$ are weakly arcwise-connected in σ, but 0 and π are not.

PROPOSITION 2. 3. Let P and Q be arcwise-connected in the standard sense. Then $r = C\sigma(P, Q)$ if and only if

(1) $r \geqq S\sigma(\phi)$ for all continuous curve ϕ from P to Q in X ;

(2) for each $\varepsilon > 0$ there is a continuous curve ϕ from P to Q in X such that $\sigma(\phi(t)) > r - \varepsilon$ for all $t \in [0, 1]$.

Proof. Necessity. There is a continuous curve ϕ from P to Q in X such that $S\sigma(\phi) > r - \varepsilon$. Thus we have

$$\sigma(\phi(t)) \geqq S\sigma(\phi) > r - \varepsilon \quad \text{for all } t \in [0, 1].$$

Sufficiency. From (1) we have $r \geqq C\sigma(P, Q)$. For a continuous curve ϕ from P to Q satisfying $\sigma(\phi(t)) > r - \varepsilon$ for all $t \in [0, 1]$, we get the inequality $S\sigma(\phi) \geqq r - \varepsilon$. Thus it follow $C\sigma(P, Q) \geqq r$.

From Proposition 2. 3 we get easily the following:

PROPOSITION 2. 4. Let P and Q be arcwise-connected in the standard sense. Then P and Q are weakly arcwise-connected in σ if and only if for each $\varepsilon > 0$ there is a continuous curve ϕ from P to Q such that $\sigma(\phi(t)) > \mathrm{Min}\{\sigma(P), \sigma(Q)\} - \varepsilon$ for all $t \in [0, 1]$.

DEFENITION 2. 5. Let P and Q be in X. If there is a continuous curve ϕ from P to Q in X such that $\sigma(\phi(t)) \geqq \mathrm{Min}\{\sigma(P), \sigma(Q)\}$ for all $t \in [0, 1]$, we say that P and Q are strongly arcwise-connected in σ. By s-Cσ we denote the set of all pairs (P, Q) such that P and Q are strongly arcwise-connented in σ. It is obvious that (P, Q) is in w-Cσ if (P, Q) is in s-Cσ.

REMARK. A pair (P, Q) in w-Cσ is not necessarily in s-Cσ. In fact, let X be the two-dimensional Euclidean space R^2. We set a fuzzy subset σ of X as follows:

$$\sigma(x, y) = \begin{cases} 1 - |y| & \text{if } y \neq 0 \text{ and } |y| \leqq 1 \\ 0 & \text{if } |y| > 1 \\ 1 & \text{if } (x, y) = (0, 0) \text{ or } (1, 0) \\ 0 & \text{if } y = 0, x \neq 0 \text{ and } x \neq 1. \end{cases}$$

If we set P = (0, 0) and Q = (1, 0), P and Q are weakly arcwise-connented in σ but not strongly arcwise-connented in σ.

As in Proposition 2. 2, we see that the relation s-Cσ is reflexive and symmetric, but not necessarily transitive.

REMARK. If $\sigma(P)=0$, then (P, Q) is in s-Cσ for all $Q \in X$.

PROPOSITION 2. 6. Let (P, Q) and (Q, R) be in s-Cσ (respectively w-Cσ). If $\sigma(Q) \geqq \mathrm{Min}\{\sigma(P), \sigma(R)\}$, (P, R) is in s-C$\sigma$. If $\sigma(Q) \geqq \mathrm{Min}\{\sigma(P), \sigma(R)\}$, (P, R) is in s-C$\sigma$ (respectively w-Cσ).

Proof. Let (P, Q), (Q, R) be in s-Cσ and $\sigma(Q) \geqq \mathrm{Min}\{\sigma(P), \sigma(R)\}$. There is a continuous curve ϕ_1 (respectively ϕ_2) from P to Q (respectively from Q to R) such that $\sigma(\phi_1(t)) \geqq \mathrm{Min}\{\sigma(P), \sigma(R)\} = \sigma(P)$ (respectively $\sigma(\phi_2(t)) \geqq \mathrm{Min}\{\sigma(Q), \sigma(R)\}$

$= \sigma$ (R)) for all $t \in [0, 1]$. It is obvious that the composition ϕ of ϕ_1 and ϕ_2 is a continuous curve from P to Q and $\sigma(\phi(t)) \geqq \mathrm{Min}\{\sigma(Q), \sigma(R)\}$. Thus (P, R) is in s-C$\sigma$. The other cases are established in a similar fashion.

DEFENTION 2. 7. A crisp subset S of X is said to be strongly (respectively weakly) arcwise-connected in σ if P and Q are strongly (respectively weakly) arcwise-connected in σ for all P, Q $\in$ S. A crisp subset S of X is said to be arcwise-connected in the standard sense if P and Q are arcwise-connected in the standard sense for all P, Q $\in$ S.

We easily have the following:

PROPOSITION 2. 8. If a crisp subset S on which σ is constant is arcwise-connected in the standard sense, S is strongly (therefore weakly) arcwise-connected in σ.

REMARK. A strongly arcwise-connected crisp subset on which σ is constant is not necessarily arcwise-connected in the standard sense.

In fact, let X be the two-dimensional Euclidean space R^2,

$S = \{(x, y) : x^2 + y^2 \leqq 1\} \cup \{(x, y) : (x - 3)^2 + y^2 \leqq 1\}$ and

$$\sigma(x, y) = \begin{cases} 1/2 & \text{if } (x, y) \in S \\ 1 & \text{if } (x, y) \notin S. \end{cases}$$

Then S is strongly arcwise-connected in σ but not arcwise-connected in the standard sense.

3. Tops and Connected Components

Throughout this section let X be a metric space with metric d and σ a fuzzy subset of X.

For two crisp subsets S, T of a metric space X we define d (S, T) as $d(S, T) = \inf\{d(P, Q) : P \in S, Q \in T\}$ and denote by $U(S; \varepsilon)$ the set of all P in X such that $d(P, S) < \varepsilon$.

DEFENITION 3. 1. Let X be a metric space and σ a fuzzy subset of X. We say that a crisp subset Π of X is a plateau in σ if the following conditions are satisfied:

(1) $\sigma(P) = \sigma(Q)$ for all P, Q $\in \Pi$,

(2) Π is arcwise-connected in the standard sense,

(3) there exists a $\varepsilon > 0$ such that

$$\sigma(P) \neq \sigma(Q) \text{ for all } P \in \Pi, Q \in U(\Pi; \varepsilon) \setminus \Pi.$$

We call a plateau Π a top if $\sigma(P) > \sigma(Q)$ for all $P \in \Pi$, $Q \in U(\Pi; \varepsilon) \setminus \Pi$. Similarly we call a plateau Π a bottom if $\sigma(P) < \sigma(Q)$ for all $P \in \Pi$, $Q \in U(\Pi; \varepsilon) \setminus \Pi$.

REMARK. From Proposition 2. 8, tops are strongly arcwise-connected in σ.

DEFENITION 3. 2. Let $\Pi \subset X$ be a top in σ. We define three crisp subsets as follows:

$A(\Pi) = \{P \in X$; there is a continuous curve ϕ from P to some point Q in Π such that $\sigma(\phi(s)) \leqq \sigma(\phi(t))$ for $0 \leqq s \leqq t \leqq 1\}$.

$B(\Pi) = \{P \in X$; there is a continuous curve ϕ from P to some point Q in Π such that $\sigma(P) \leqq \sigma(\phi(t)) \leqq \sigma(Q)$ for $0 \leqq t \leqq 1\}$.

$C(\Pi) = \{P \in X$; there is a continuous curve ϕ from P to some point Q in Π such that $\sigma(P) \leqq \sigma(\phi(t))$ for $0 \leqq t \leqq 1\}$.

It is clear that $\Pi \subset A(\Pi) \subset B(\Pi) \subset C(\Pi)$ if Π is a top in σ.

PROPOSITION 3. 3. Let $\Pi \subset X$ be a top in σ. If $P \in \Pi$ and $Q \in \Pi$ are strongly arcwise-connected in σ, we have $\sigma(Q) < \sigma(P)$.

Proof. Since Π is a top in σ, there is a $\varepsilon > 0$ such that $\sigma(P') < \sigma(Q')$ for $P' \in \Pi$, $Q' \in U(\Pi; \varepsilon) \setminus \Pi$. Therefore it follows that $\sigma(P) > \sigma(Q)$ if Q is in $U(\Pi; \varepsilon) \setminus \Pi$. Assume that Q is not in $U(\Pi; \varepsilon)$. There is a continuous curve ϕ from Q to P such that $\sigma(\phi(t)) \geqq \mathrm{Min}\{\sigma(P), \sigma(Q)\}$ for all $t \in [0, 1]$

Since the functions $d(\ \cdot\ , \Pi) : X \in P \rightarrow d(P, \Pi) \in R$ and ϕ are continuous, the composition of these two functions

$f : [0, 1] \ni t \rightarrow d(\phi(t), \Pi) \in R$

is continuous.

It follows from $f(0) = d(Q, \Pi) > \varepsilon$, $f(1) = d(p, \Pi) = 0$ and the continuity of the function f that there exists a $t_0 \in (0, 1)$ such that $0 < f(t_0) = d(\phi(t_0), \Pi) < \varepsilon$. This inequality menas that the point $\phi(t_0)$ belongs to $U(\Pi ; \varepsilon) \setminus \Pi$. Therefore we get

$$\mathrm{Min}\,\{\sigma(P), \sigma(Q)\} \leqq \sigma(\phi(t)) < \sigma(P).$$

Thus we obtain $\sigma(Q) < \sigma(P)$. This completes the proof.

REMARK. It follows from Proposition 3. 3 that if Π , Π' are tops in σ , then either $\Pi = \Pi'$ or $\Pi \cap \Pi' = \phi$.

PROPOSITION 3. 4. Let $\Pi \subset X$ be a top in σ and P in X. Then the following conditions are equivalent:

(1) P is in $C(\Pi)$,

(2) for some point Q in Π , P and Q are strongly arcwise-connected in σ ,

(3) P and Q are strongly arcwise-connected in σ for all Q in Π .

Proof. It is trivial that condition (1) implies (2) and condition (3) implies (2). Assuming thtat condition (2) holds, there exists a continuou scurve ϕ from P to Q such that $\sigma(\phi(t)) \geqq \mathrm{Min}\,\{\sigma(P), \sigma(Q)\}$ for all $t \in [0, 1]$

Since $\sigma(Q) > \sigma(P)$ from Proposition 3. 3, condition (1) follows. Again assume that condition (2) hokds. Then for each Q' in Π , Q and Q' are strongly arcwise-connected in σ from Proposition 2. 8 since Q and Q' are arcwise-connected in the standard sense.

Therefore (3) follows from Proposition 2. 6 and the fact that $\sigma(Q) > \sigma(P)$. This completes the proof.

The following is easily obtained from Proposition 2. 6, 3. 3 and 3. 4.

PROPOSITION 3. 5 . Let $\Pi \subset X$ be a top in σ . If P, Q be in $C(\Pi)$, then P and Q are strongly arcwise-connected in σ .

THEOREM 3. 6. Let $\Pi \subset X$ and $\Pi' \subset X$ be tops in σ . If $\Pi \neq \Pi'$, then the following holds.

(1) $d(\Pi, \Pi') > 0$, in particular $\Pi \cap \Pi' = \phi$,

(2) $C(\Pi) \cap \Pi' = \phi$,

Proof. (1) We assume that $\sigma(P) = r$ for all $P \in \Pi$, $\sigma(Q) = r'$ for all $Q \in \Pi'$ and $r' \geqq r$. Since Π is top in σ, there exists a $\varepsilon > 0$ such that $r = \sigma(P) > \sigma(Q)$ for $P \in \Pi, Q \in U(\Pi; \varepsilon) \setminus \Pi$. Thus $(U(\Pi; \varepsilon) \setminus \Pi) \cap \Pi' = \phi$.

We remarked earlier that $\Pi \cap \Pi' = \phi$. So we have $U(\Pi; \varepsilon) \cap \Pi' = \phi$; thewrfore $d(Q, \Pi) \geqq \varepsilon$ for all $Q \in \Pi'$ It follows $d(\Pi, \Pi') \geqq \varepsilon > 0$.

(2) Suppose that $C(\Pi) \cap \Pi'$ were not empty. If P is in $C(\Pi) \cap \Pi'$ and $P \notin \Pi'$ P and Q are strongly arcwise-connected in σ for some point Q in Π. From Proposition 3. 3 we have $\sigma(P) < \sigma(Q)$. On the other hand, we have $\sigma(P) > \sigma(Q)$ by using Proposition 3. 3 since P is in the top Π'. This is a contradiction. This completes the proof.

Though the relation S-Cσ is not an equivalence relation, Proposition 3. 5 and Theorem 3. 6 mean that $C(\Pi)$ is a "connected component".

EXAMPLE. Let X be R and σ be as follows:

$$\sigma(x) = \begin{cases} -\dfrac{1}{x} & \text{if } x \leqq -1 \\ 1 & \text{if } -1 \leqq x \leqq 0 \\ -x+1 & \text{if } 0 \leqq x \leqq 1 \\ -\dfrac{1}{2}(x-2)^2 + \dfrac{1}{2} & \text{if } 1 \leqq x \leqq 3 \\ 0 & \text{if } 3 \leqq x. \end{cases}$$

Then $\Pi_1 = [-1, 0]$ and $\Pi_2 = \{2\}$ are tops in σ and we get $C(\Pi_1) = (-\infty, 1] \cup [3, \infty)$ and $C(\Pi_2) = [1, \infty]$. Thus $C(\Pi_1) \cap C(\Pi_2) \neq \phi$.

The author does not know whether these results on tops in metric spaces can be extended to general topological spaces.

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Arcwise-Connectedness in Topological Spaces from the Point of View of Fuzzy Sets

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Abstract

New insights are given into arcwise-connectedness in topological spaces from the point of view of fuzzy set theory. The concept of arcwise-connectedness with respect to a fuzzy subset is introduced. In particular, "arcwise-connected components" of a crisp subset with respect to a fuzzy sbset are defined and discussed in case of metric spaces.