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On an Inequality Related to Capacity of $MA(1)$ Gaussian Channel with Feedback

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There are several inequalities related to capacity of Gaussian channel with feedback. We give an answer for unsolved problem under some condition. And also we give a new inequality in the case of $MA(1)$ Gaussian noise.

Keywords: Gaussian channel, capacity, feedback.

1 Gaussian Channels

The following model for a discrete time Gaussian channel with feedback is considered:

$$Y_n = S_n + Z_n, \ \ n = 1, 2, \dots,$$

where $Z = \{Z_n; n = 1, 2, \dots\}$ is a non-degenerate, zero mean Gaussian process representing the noise and $S = \{S_n; n = 1, 2, \dots\}$ and $Y = \{Y_n; n = 1, 2, \dots\}$ are stochastic processes representing input signals and output signals, respectively. The channel is with noiseless feedback, so S_n is a function of a message to be transmitted and the output signals $Y_1, \dots, Y_{n-1}$. For a code of rate R and length n, with code words $x^n(W, Y^{n-1})$, $W \in \{1, \dots, 2^{nR}\}$, and a decoding function $g_n : \mathbb{R}^n \to \{1, \dots, 2^{nR}\}$, the probability of error is

$$Pe^{(n)} = Pr\{g_n(Y^n) \neq W; Y^n = x^n(W, Y^{n-1}) + Z^n\},$$

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where W is uniformly distributed over $\{1,\dots,2^{nR}\}$ and independent of Z^n. The signal is subject to an expected power constraint

$$\frac{1}{n}\sum_{i=1}^{n} E[S_i^2] \le P,$$

and the feedback is causal, i.e., S_i is dependent of $Z_1,\dots,Z_{i-1}$ for $i=1,2,\dots,n$. Similarly, when there is no feedback, S_i is independent of Z^n. We denote by $R_X^{(n)}$ and $R_Z^{(n)}$ the covariance matrices of X and Z, respectively. It is well known that a finite block length capacity is given by

$$C_{n,FB,Z}(P) = \max \frac{1}{2n} \log \frac{|R_X^{(n)} + R_Z^{(n)}|}{|R_Z^{(n)}|},$$

where the maximum is taken over all symmetric, nonnegative definite matrix $R_X^{(n)}$ and strictly lower triangular matrix B, such that

$$\mathrm{Tr}[(I+B)R_X^{(n)}(I+B^t) + BR_Z^{(n)}B^t] \le nP.$$

Similarly, let $C_{n,Z}(P)$ be the maximal value when $B=0$, i.e. when there is no feedback. Under these conditions, Cover and Pombra proved the following.

Proposition 1.1 (Cover and Pombra [6]). *For every $\epsilon > 0$ there exist codes, with block length n and $2^{n(C_{n,FB,Z}(P)-\epsilon)}$ codewords, $n=1,2,\dots$, such that $Pe^{(n)} \to 0$, as $n\to\infty$. Conversely, for every $\epsilon>0$ and any sequence of codes with $2^{n(C_{n,FB,Z}(P)+\epsilon)}$ codewords and block length n, $Pe^{(n)}$ is bounded away from zero for all n. The same theorem holds in the special case without feedback upon replacing $C_{n,FB,Z}(P)$ by $C_{n,Z}(P)$.*

When block length n is fixed, $C_{n,Z}(P)$ is given exactly.

Proposition 1.2 (Gallager [10]).

$$C_{n,Z}(P) = \frac{1}{2n}\sum_{i=1}^{k} \log \frac{nP + r_1 + \cdots + r_k}{kr_i},$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$ are eigenvalues of $R_Z^{(n)}$, and k $(\le n)$ is the largest integer satisfying $nP + r_1 + r_2 + \cdots + r_k > kr_k$.

2 Mixed Gaussian Channels with Feedback

Let Z_1, Z_2 be Gaussian processes with mean 0 and covariance matrices $R_{Z_1}^{(n)}, R_{Z_2}^{(n)}$, respectively. A mixed Gaussian channel is defined by an additive Gaussian channel with noise $\tilde{Z}$ whose mean is 0 and whose covariance matrix is

$$R_{\tilde{Z}}^{(n)} = \alpha R_{Z_1}^{(n)} + \beta R_{Z_2}^{(n)},$$

where $\alpha, \beta \geq 0$ $(\alpha + \beta = 1)$. Let $C_{n,\tilde{Z}}(P)$ be the capacity of mixed Gaussian channel and $C_{n,FB,\tilde{Z}}(P)$ the capacity of mixed Gaussian channel with feedback.

Theorem 2.1 (Y-C-Y [20], Y-Y-C [21], C-Y [4])**.** *For any $P > 0$,*

$$C_{n,\tilde{Z}}(P) \leq \alpha C_{n,Z_1}(P) + \beta C_{n,Z_2}(P).$$

Theorem 2.2 (Y-C-Y [20], Y-Y-C [21], C-Y [4])**.** *For any $P > 0$, there exist $P_1, P_2 \geq 0$ $(P = \alpha P_1 + \beta P_2)$ such that*

$$C_{n,FB,\tilde{Z}}(P) \leq \alpha C_{n,FB,Z_1}(P_1) + \beta C_{n,FB,Z_2}(P_2).$$

These theorems are proved by the property that $\log(1 + t^{-1})$ is an operator convex function. But we have the following conjecture.

Conjecture 2.1. For any $P > 0$,

$$C_{n,FB,\tilde{Z}}(P) \leq \alpha C_{n,FB,Z_1}(P) + \beta C_{n,FB,Z_2}(P).$$

We solved the above conjecture partially.

Theorem 2.3 (Yanagi, Yu, and Chao [21])**.** *If one of the following conditions is satisfied, then the conjecture holds.*

(1) $R_{Z_1}^{(n-1)} = R_{Z_2}^{(n-1)}$.
(2) $R_{\tilde{Z}}$ *is white.*

3 Kim's Result

Let $Z = \{Z_i; i = 1, 2, \ldots\}$ be a discrete time first order moving average Gaussian process that we denote by $MA(1)$. $MA(1)$ can be characterized in the following three properties.

(1) $Z_i = \alpha U_{i-1} + U_i,\ i = 1, 2, \ldots$, where $U_i \sim N(0, 1)$ are i.i.d.

(2) Spectral density function (SDF) is given by

$$f(\lambda) = \frac{1}{2\pi}\left|1 + \alpha e^{-i\lambda}\right|^2 = \frac{1}{2\pi}(1 + \alpha^2 + 2\alpha \cos \lambda).$$

(3) $Z_n = (Z_i, \ldots, Z_n) \sim N_n(0, K_Z)$ for each n, where covariance matrix K_Z is given by the following:

$$K_Z = \begin{pmatrix} 1+\alpha^2 & \alpha & 0 & \cdots & 0 \\ \alpha & 1+\alpha^2 & \alpha & \cdots & 0 \\ 0 & \alpha & 1+\alpha^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \alpha \\ 0 & 0 & 0 & \cdots & 1+\alpha^2 \end{pmatrix}.$$

We define the capacity of Gaussian channel with the $MA(1)$ Gaussian noise by the following:

$$C_{FB,Z}(P) = \lim_{n\to\infty} C_{n,FB,Z}(P)$$

Resently Kim obtained $C_{FB,Z}(P)$ in above conditions, which is the first result of feedback capacity.

Theorem 3.1 (Kim [13])**.**

$$C_{FB,Z}(P) = -\log x_0,$$

where x_0 *is a unique positive root of*

$$Px^2 = (1-x^2)(1-|\alpha|x)^2. \tag{3.1}$$

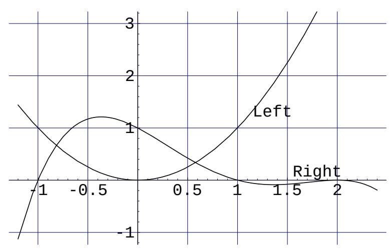

Figure 3.1: Graph of $Px^2 = (1-x^2)(1-|\alpha|x)^2$, where $P = 1, \alpha = 0.5$

4 An Inequality Related to Conjecture 2.1

The following inequality holds:

$$R_{\alpha Z+\beta W} \le \alpha R_Z + \beta R_W \le R_{\sqrt{\alpha}Z+\sqrt{\beta}W},$$

where

$$Z \sim MA(1,p),\ Z_i = U_i + pU_{i-1},\ \ 0 < p \le 1,$$

$$W \sim MA(1,q),\ W_i = U_i + qU_{i-1},\ \ 0 < q \le 1.$$

Since

$$\alpha R_Z + \beta R_W = R_{\alpha Z+\beta W} + \alpha\beta R_{Z-W},$$

we have

$$R_{\alpha Z+\beta W} \le \alpha R_Z + \beta R_W.$$

On the other hand we have

$$\alpha R_Z + \beta R_W + \sqrt{\alpha\beta}(R_{ZW} + R_{WZ}) = R_{\sqrt{\alpha}Z+\sqrt{\beta}W},$$

where

$$R_{ZW} + R_{WZ} = \begin{pmatrix} 2+2pq & p+q & 0 & \dots & 0 \\ p+q & 2+2pq & p+q & \dots & 0 \\ 0 & p+q & 2+2pq & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & p+q \\ 0 & 0 & 0 & \dots & 2+2pq \end{pmatrix}.$$

The eigenvalues r_i of this covariance matrix are represented as follows.

$$\begin{aligned} r_i &= 2+2pq-2(p+q)\cos\frac{i\pi}{n+1} \quad (i=1,2,\dots,n) \\ &\geq 2+2pq-2(p+q) = 2(1-p)(1-q) \geq 0. \end{aligned}$$

Since $R_{ZW} + R_{WZ} \geq 0$, we have $\alpha R_Z + \beta R_W \leq R_{\sqrt{\alpha}Z+\sqrt{\beta}W}$.

Proposition 4.1. *The following inequality holds.*

$$C_{FB,\sqrt{\alpha}Z+\sqrt{\beta}W}(P) \leq C_{FB,\tilde{Z}}(P) \leq C_{FB,\alpha Z+\beta W}(P),$$

where $R_{\tilde{Z}} = \alpha R_Z + \beta R_W$.

We put $V = \sqrt{\alpha}Z + \sqrt{\beta}W$. Then

$$V_i = (\sqrt{\alpha}+\sqrt{\beta})U_i + (\sqrt{\alpha}p + \sqrt{\beta}q)U_{i-1}.$$

And we also put

$$Y_i = U_i + \frac{\sqrt{\alpha}p+\sqrt{\beta}q}{\sqrt{\alpha}+\sqrt{\beta}}U_{i-1}.$$

Then

$$Y = \frac{\sqrt{\alpha}Z+\sqrt{\beta}W}{\sqrt{\alpha}+\sqrt{\beta}} \sim MA\left(1, \frac{\sqrt{\alpha}p+\sqrt{\beta}q}{\sqrt{\alpha}+\sqrt{\beta}}\right).$$

$$\begin{aligned} C_{n,FB,V}(P) &= \max\left\{\frac{1}{2n}\log\frac{|R_{S+V}|}{|R_V|}; Tr[R_S] \leq nP\right\} \\ &= \max\left\{\frac{1}{2n}\log\frac{|R_{S+(\sqrt{\alpha}+\sqrt{\beta})Y}|}{|R_{(\sqrt{\alpha}+\sqrt{\beta})Y}|}; Tr[R_S] \leq nP\right\} \\ &= \max\left\{\frac{1}{2n}\log\frac{|R_{S/(\sqrt{\alpha}+\sqrt{\beta})+Y}|}{|R_Y|}; Tr[R_{S/(\sqrt{\alpha}+\sqrt{\beta})}] \leq \frac{nP}{(\sqrt{\alpha}+\sqrt{\beta})^2}\right\} \\ &= C_{n,FB,Y}\left(\frac{P}{(\sqrt{\alpha}+\sqrt{\beta})^2}\right). \end{aligned}$$

We propose Conjecture 4.1 which is weaker than Conjecture 2.1.

Conjecture 4.1. For any $P > 0$,

$$C_{FB,\sqrt{\alpha}Z+\sqrt{\beta}W}(P) \le \alpha C_{FB,Z}(P) + \beta C_{FB,W}(P).$$

In particular we prove the Conjecture in the case of $\alpha = \beta = 1/2$.

Since we can represent (3.1) as

$$|\alpha| = \frac{1}{x} - \frac{\sqrt{P}}{\sqrt{1-x^2}},$$

we put the function

$$f(t,P) = \frac{1}{t} - \frac{\sqrt{P}}{\sqrt{1-t^2}}$$

in order to prove the Conjecture. Then there uniquely exist $0 < a < b < 1$ such that $f(a,P) = 1,\ f(b,P) = 0$. That is

$$1 = \frac{1}{a} - \frac{\sqrt{P}}{\sqrt{1-a^2}}, \quad 0 = \frac{1}{b} - \frac{\sqrt{P}}{\sqrt{1-b^2}}. \tag{4.1}$$

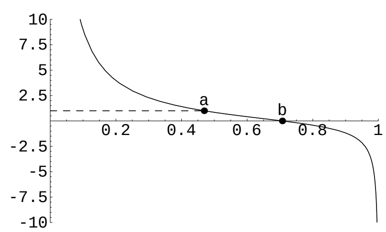

Figure 4.1: Graph of $f(t,P) = 1/t - \sqrt{P}/\sqrt{1-t^2}$, where $P = 1$.

However, since $f(t,P)$ is not convex function of $t (a \le t \le b)$, we put the following concave function

$$g(t,P) = t\left(1 - \frac{\sqrt{P}}{\sqrt{t^2-1}}\right), \quad \frac{1}{b} \le t \le \frac{1}{a}.$$

Now we put $L = \sqrt{(1-a)^2(1-a^2)+a^2}$. Then b and P can be represented as the following functions of a:

$$b = \frac{a}{L}, \quad P = \frac{L^2}{a^2} - 1.$$

Lemma 4.1. *For any* $P > 0$,

$$\frac{\sqrt{P}}{\sqrt{1-a^2}} \ge \frac{1}{2-\sqrt{2}} \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}.$$

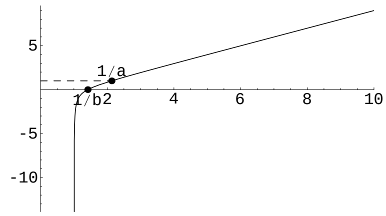

Figure 4.2: Graph of $g(t,P) = t\left(1 - \sqrt{P}/\sqrt{t^2-1}\right)$, where $P = 1$.

Proof. Since $2(2-\sqrt{2}) > 1$ and $L > a$,

$$2(2-\sqrt{2})\left(\frac{1}{a} - 1\right) > \frac{1}{a} - 1 > \frac{1}{\sqrt{a}} - 1 > \frac{1}{\sqrt{L}} - 1.$$

And since $L < 1$,

$$\frac{1-a}{a} > \frac{1}{2(2-\sqrt{2})}\left(\frac{1}{\sqrt{L}} - 1\right) = \frac{1}{2-\sqrt{2}}\frac{1-\sqrt{L}}{2\sqrt{L}} > \frac{1}{2-\sqrt{2}}\frac{1-\sqrt{L}}{1+\sqrt{L}}.$$

By (4.1),

$$\frac{1-a}{a} = \frac{\sqrt{P}}{\sqrt{1-a^2}}.$$

The inequality is proved by putting $L = a/b$. □

Lemma 4.2. *For any* $t, s \ \ (1/b \le t \le s \le 1/a)$*,*

$$\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} \ge \frac{\sqrt{s}-\sqrt{t}}{\sqrt{s}+\sqrt{t}}.$$

Proof. Since

$$\sqrt{\frac{a}{b}} = \min_{1/b \le t \le s \le 1/a} \sqrt{\frac{t}{s}},$$

the following inequality is obtained.

$$\begin{aligned}\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} = 2\left(\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \frac{1}{2}\right) &= 2\left(\frac{1}{\sqrt{a/b}+1} - \frac{1}{2}\right) \\ &\ge 2\left(\frac{1}{\sqrt{t/s}+1} - \frac{1}{2}\right) \\ &= 2\left(\frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}} - \frac{1}{2}\right) \\ &= \frac{\sqrt{s}-\sqrt{t}}{\sqrt{s}+\sqrt{t}}.\end{aligned}$$

□

Lemma 4.3. *For any* $t, s\ (1/b \le t \le s \le 1/a)$

$$\frac{1}{2}g(t,P) + \frac{1}{2}g(s,P) \le g\Big(\sqrt{ts}, \frac{P}{2}\Big).$$

Proof. Since $g(t,P)$ is concave function of t,

$$\frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}g\Big(t, \frac{P}{2}\Big) + \frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}g\Big(s, \frac{P}{2}\Big) \le g\Big(\sqrt{ts}, \frac{P}{2}\Big).$$

Then we have to show the following inequality:

$$\frac{1}{2}g(t,P) + \frac{1}{2}g(s,P) \le \frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}g\Big(t, \frac{P}{2}\Big) + \frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}g\Big(s, \frac{P}{2}\Big).$$

By Lemma 4.1 and Lemma 4.2

$$\frac{\sqrt{P}}{\sqrt{1-a^2}} \ge \frac{1}{2-\sqrt{2}}\frac{\sqrt{s}-\sqrt{t}}{\sqrt{s}+\sqrt{t}} = \frac{2}{2-\sqrt{2}}\left(\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}} - \frac{1}{2}\right).$$

Since, for any $t, s(1/b \le t \le s \le 1/a)$,

$$0 \le s\Big(1 - \frac{\sqrt{P}}{\sqrt{s^2-1}}\Big) - t\Big(1 - \frac{\sqrt{P}}{\sqrt{t^2-1}}\Big) \le 1,$$

we have the following inequality:

$$\begin{aligned}\left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{P}}{\sqrt{1-a^2}} &\ge \frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}} - \frac{1}{2}\\ &\ge \left(\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}} - \frac{1}{2}\right)\Big\{s\Big(1 - \frac{\sqrt{P}}{\sqrt{s^2-1}}\Big) - t\Big(1 - \frac{\sqrt{P}}{\sqrt{t^2-1}}\Big)\Big\}.\end{aligned}$$

Since

$$\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}\frac{\sqrt{P}}{\sqrt{1-1/t^2}} + \left(1 - \frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}\right)\frac{\sqrt{P}}{\sqrt{1-1/s^2}} \ge \frac{\sqrt{P}}{\sqrt{1-a^2}},$$

we have

$$\begin{aligned}&\left(\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}} - \frac{1}{2}\right)\Big\{t\Big(1 - \frac{\sqrt{P}}{\sqrt{t^2-1}}\Big) - s\Big(1 - \frac{\sqrt{P}}{\sqrt{s^2-1}}\Big)\Big\}\\ &+ \left(1-\frac{1}{\sqrt{2}}\right)\left\{\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}\frac{\sqrt{P}}{\sqrt{1-1/t^2}} + \left(1 - \frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}\right)\frac{\sqrt{P}}{\sqrt{1-1/s^2}}\right\} \ge 0.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}} - \frac{1}{2}\right)t\Big(1 - \frac{\sqrt{P}}{\sqrt{t^2-1}}\Big) + \left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}\frac{\sqrt{P}t}{\sqrt{t^2-1}}\\ &+ \left(\frac{1}{2} - \frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}\right)s\Big(1 - \frac{\sqrt{P}}{\sqrt{s^2-1}}\Big) + \left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}\frac{\sqrt{P}s}{\sqrt{s^2-1}} \ge 0.\end{aligned}$$

Then

$$\left(\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}-\frac{1}{2}\right)t\left(1-\frac{\sqrt{P}}{\sqrt{t^2-1}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}\frac{\sqrt{P}t}{\sqrt{t^2-1}}$$
$$+\left(\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}-\frac{1}{2}\right)s\left(1-\frac{\sqrt{P}}{\sqrt{s^2-1}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}\frac{\sqrt{P}s}{\sqrt{s^2-1}}\geq 0.$$

Thus

$$\begin{aligned}&\frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}\left\{t\left(1-\frac{\sqrt{P}}{\sqrt{t^2-1}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{P}t}{\sqrt{t^2-1}}\right\}\\&+\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}\left\{s\left(1-\frac{\sqrt{P}}{\sqrt{s^2-1}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\frac{\sqrt{P}s}{\sqrt{s^2-1}}\right\}\\&\geq\frac{1}{2}t\left(1-\frac{\sqrt{P}}{\sqrt{t^2-1}}\right)+\frac{1}{2}s\left(1-\frac{\sqrt{P}}{\sqrt{s^2-1}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\sqrt{s}}{\sqrt{s}+\sqrt{t}}t\left(1-\frac{\sqrt{P/2}}{\sqrt{t^2-1}}\right)+\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}s\left(1-\frac{\sqrt{P/2}}{\sqrt{s^2-1}}\right)\\&\quad\geq\frac{1}{2}t\left(1-\frac{\sqrt{P}}{\sqrt{t^2-1}}\right)+\frac{1}{2}s\left(1-\frac{\sqrt{P}}{\sqrt{s^2-1}}\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{2}g(t,P)+\frac{1}{2}g(s,P)&=\frac{1}{2}t\left(1-\frac{\sqrt{P}}{\sqrt{t^2-1}}\right)+\frac{1}{2}s\left(1-\frac{\sqrt{P}}{\sqrt{s^2-1}}\right)\\&\leq\frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}t\left(1-\frac{\sqrt{P/2}}{\sqrt{t^2-1}}\right)+\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}s\left(1-\frac{\sqrt{P/2}}{\sqrt{s^2-1}}\right)\\&=\frac{\sqrt{s}}{\sqrt{t}+\sqrt{s}}g\left(t,\frac{P}{2}\right)+\frac{\sqrt{t}}{\sqrt{t}+\sqrt{s}}g\left(s,\frac{P}{2}\right).\end{aligned}$$ □

Now we have the following theorem.

Theorem 4.1. *For any* $P > 0$*,*

$$C_{FB,(Z+W)/\sqrt{2}}(P)\leq\frac{1}{2}C_{FB,Z}(P)+\frac{1}{2}C_{FB,W}(P).$$

Proof. Let $C_{FB,Z}(P) = -\log x$ and $C_{FB,W}(P) = -\log y$. By putting $s = 1/x$ and $t = 1/y$ in Lemma 4.3, we have

$$\frac{1}{2}f(x,P)+\frac{1}{2}f(y,P)\leq f\left(\sqrt{xy},\frac{P}{2}\right). \tag{4.2}$$

Since $Z \sim MA(1,p),\ 0 < p \leq 1$ and $W \sim MA(1,q),\ 0 < q \leq 1$,

$$p=\frac{1}{x}-\frac{\sqrt{P}}{\sqrt{1-x^2}}=f(x,P),$$

$$q = \frac{1}{y} - \frac{\sqrt{P}}{\sqrt{1-y^2}} = f(y, P).$$

We take z such that

$$\frac{p+q}{2} = f\Big(z, \frac{P}{2}\Big).$$

Then by (4.2)

$$f\Big(z, \frac{P}{2}\Big) \le f\Big(\sqrt{xy}, \frac{P}{2}\Big).$$

Since $f(t, P/2)$ is a decreasing function of t, we have $z \ge \sqrt{xy}$. Then we have the following:

$$\begin{aligned} C_{FB,(Z+W)/\sqrt{2}}(P) &= C_{FB,(Z+W)/2}\Big(\frac{P}{2}\Big) \\ &= -\log z \\ &\le \frac{1}{2}\Big(-\log x\Big) + \frac{1}{2}\Big(-\log y\Big) \\ &= \frac{1}{2}C_{FB,Z}(P) + \frac{1}{2}C_{FB,W}(P). \end{aligned}$$

□

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