TRACE INEQUALITIES ON A GENERALIZED WIGNER-YANASE SKEW INFORMATION

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Abstract. We introduce a generalized Wigner-Yanase skew information and then derive the trace inequality related to the uncertainty relation. This inequality is a non-trivial generalization of the uncertainty relation derived by S.Luo for the quantum uncertainty quantity excluding the classical mixure. In addition, several trace inequalities on our generalized Wigner-Yanase skew information are argued.

1. Introduction

Wigner-Yanase skew information

$$I_\rho(H) \equiv \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2},H\right]\right)^2\right] = Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \tag{1.1}$$

was defined in [8]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X,Y] \equiv XY - YX$. This quantity was generalized by Dyson

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left(i\left[\rho^\alpha,H\right]\right)\left(i[\rho^{1-\alpha},H]\right)\right] = Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \;\; \alpha \in [0,1]$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H.Lieb in [5]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$, where $B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathfrak{S}(\mathcal{H})$ on the Hilbet space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [7]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 9]. In our previous paper [9], we defined a

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generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we introduce a new generalized Wigner-Yanase skew information. On a generalization of the original Wigner-Yanase skew information, our generalization is different from the Wigner-Yanase-Dyson skew information and a generalized skew information defined in our previous paper [9]. Moreover we define a new quantity by our generalized Wigner-Yanase skew information and then we derive the trace inequality expressing a kind of the uncertainty relation.

2. Trace inequalities on a generalized Wigner-Yanase skew information

Firstly we review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mehcanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) \equiv Tr[\rho\,(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have the Heisenberg's uncerainty relation:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2 \tag{2.1}$$

for a quantum state ρ and two observables A and B. The further strong result was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A,B)|^2 \geq \frac{1}{4}|Tr[\rho[A,B]]|^2,$$

where the covariance is defined by $Cov_\rho(A,B) \equiv Tr[\rho\,(A - Tr[\rho A]I)\,(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [7, 4, 9].)

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_\rho(H)\right)^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [6]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2. \tag{2.3}$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4).

In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left(i\left[\rho^\alpha, H_0\right]\right)\left(i\left[\rho^{1-\alpha}, H_0\right]\right)\right] \tag{2.5}$$

and we also define

$$J_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left\{\rho^\alpha, H_0\right\}\left\{\rho^{1-\alpha}, H_0\right\}\right],$$

where $H_0 \equiv H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr\left[\left(i\left[\rho^{\alpha}, H_0\right]\right)\left(i\left[\rho^{1-\alpha}, H_0\right]\right)\right] = \frac{1}{2}Tr\left[\left(i\left[\rho^{\alpha}, H\right]\right)\left(i\left[\rho^{1-\alpha}, H\right]\right)\right]$$

but we have

$$\frac{1}{2}Tr\left[\left\{\rho^{\alpha}, H_0\right\}\left\{\rho^{1-\alpha}, H_0\right\}\right] \neq \frac{1}{2}Tr\left[\left\{\rho^{\alpha}, H\right\}\left\{\rho^{1-\alpha}, H\right\}\right].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq J_{\rho}(H) \leq J_{\rho,\alpha}(H), \tag{2.6}$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leq Tr[\rho^{\alpha}H\rho^{1-\alpha}H]$. (See [1, 2] for example.) If we define

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_{\rho}(H)^2 - \left(V_{\rho}(H) - I_{\rho,\alpha}(H)\right)^2}, \tag{2.7}$$

as a direct generalization of Eq.(2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H) \tag{2.8}$$

due to the first inequality of (2.6). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}. \tag{2.9}$$

Remark 2.2. From the inequalities (2.4), (2.6) and (2.8), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).$$

Therefore our first concern is the ordering between $I_{\rho}(H)$ and $U_{\rho,\alpha}(H)$. However we have no ordering between them. Because we have the following examples. We set the density matrix ρ and the observable H such as

$$\rho = \begin{pmatrix} 0.6 & 0.48 \\ 0.48 & 0.4 \end{pmatrix}, H = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 5.0 \end{pmatrix}.$$

If $\alpha = 0.1$, then $U_{\rho,\alpha}(H) - I_{\rho}(H)$ approximately takes -0.14736. If $\alpha = 0.2$, then $U_{\rho,\alpha}(H) - I_{\rho}(H)$ approximately takes 0.4451.

Conjecture 2.3. Our second concern is to show an uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3) such that

$$U_{\rho,\alpha}(X)U_{\rho,\alpha}(Y) \geq \frac{1}{4}\left|Tr\left[\rho[X, Y]\right]\right|^2 \tag{2.10}$$

However we have not found the proof of the above inequality (2.10). In addition, we have not found any counter-examples of the inequality (2.10) yet.

In the present paper, we introduce a generalized Wigner-Yanase skew information which is a generalization of the Wigner-Yanase skew information defined in Eq.(1.1), but different from the Wigner-Yanase-Dyson skew information defined in Eq.(2.5).

Definition 2.4. For $0 \le \alpha \le 1$, a quantum state ρ and an observable H, we define a generalized Wigner-Yanase skew information by

$$K_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr\left[\left(i\left[\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H_0\right]\right)^2\right]$$

and we also define

$$L_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr\left[\left(\left\{\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right].$$

Remark 2.5. For two generalized Wigner-Yanase skew informations $I_{\rho,\alpha}(H)$ and $K_{\rho,\alpha}(H)$, we have the relation:

$$I_{\rho,\alpha}(H) \le K_{\rho,\alpha}(H).$$

Indeed, for a spertral decomposition of ρ such as $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, we have the following expressions:

$$I_{\rho,\alpha}(H) = \frac{1}{2}\sum_{m,n}\left(\lambda_m^{\alpha}-\lambda_n^{\alpha}\right)\left(\lambda_m^{1-\alpha}-\lambda_n^{1-\alpha}\right)|\langle\phi_m|H|\phi_n\rangle|^2$$

and

$$K_{\rho,\alpha}(H) = \frac{1}{2}\sum_{m,n}\left(\frac{\lambda_m^{\alpha}-\lambda_n^{\alpha}+\lambda_m^{1-\alpha}-\lambda_n^{1-\alpha}}{2}\right)^2|\langle\phi_m|H|\phi_n\rangle|^2.$$

By simple calculations, we see

$$\left(\frac{\lambda_m^{\alpha}-\lambda_n^{\alpha}+\lambda_m^{1-\alpha}-\lambda_n^{1-\alpha}}{2}\right)^2-\left(\lambda_m^{\alpha}-\lambda_n^{\alpha}\right)\left(\lambda_m^{1-\alpha}-\lambda_n^{1-\alpha}\right)\ge 0.$$

Throughout this section, we put $X_0 \equiv X - Tr[\rho X]I$ and $Y_0 \equiv Y - Tr[\rho Y]I$. Then we show the following trace inequality.

Theorem 2.6. *For a quantum state ρ and observables X, Y and $\alpha \in [0,1]$, we have*

$$W_{\rho,\alpha}(X)\,W_{\rho,\alpha}(Y) \ge \frac{1}{4}\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|^2 \tag{2.11}$$

where

$$W_{\rho,\alpha}(X) \equiv \sqrt{K_{\rho,\alpha}(X)L_{\rho,\alpha}(X)}.$$

Proof: Putting

$$M \equiv i\left[\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, X_0\right]x + \left\{\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, Y_0\right\} \tag{2.12}$$

for any $x \in \mathbb{R}$, then we have

$$\begin{aligned} 0 &\le Tr\left[M^*M\right] \\ &= \left(\frac{1}{4}Tr\left[\left(i[\rho^{\alpha},X_0]\right)^2+\left(i[\rho^{1-\alpha},X_0]\right)^2\right]+I_{\rho,\alpha}(X)\right)x^2 \\ &\quad+\frac{1}{2}Tr\left[\left(i[\rho^{\alpha},X_0]+i[\rho^{1-\alpha},X_0]\right)\left(\{\rho^{\alpha},Y_0\}+\{\rho^{1-\alpha},Y_0\}\right)\right]x \\ &\quad+\left(\frac{1}{4}Tr\left[\{\rho^{\alpha},Y_0\}^2+\{\rho^{1-\alpha},Y_0\}^2\right]+J_{\rho,\alpha}(Y)\right). \end{aligned}$$

Therefore we have

$$\begin{aligned}&\frac{1}{4}\left|Tr\left[\left(\rho^{\alpha}+\rho^{1-\alpha}\right)^{2}\left(i\left[X,Y\right]\right)\right]\right|^{2}\\ \le\ & 4\left(\frac{1}{4}Tr\left[\left(i[\rho^{\alpha},X_0]\right)^{2}+\left(i[\rho^{1-\alpha},X_0]\right)^{2}\right]+I_{\rho,\alpha}\left(X\right)\right)\\ &\times\left(\frac{1}{4}Tr\left[\left\{\rho^{\alpha},Y_0\right\}^{2}+\left\{\rho^{1-\alpha},Y_0\right\}^{2}\right]+J_{\rho,\alpha}\left(Y\right)\right),\end{aligned}$$

since we have

$$Tr\left[\left(i[\rho^{\alpha},X_0]+i[\rho^{1-\alpha},X_0]\right)\left(\{\rho^{\alpha},Y_0\}+\left\{\rho^{1-\alpha},Y_0\right\}\right)\right]=Tr\left[\left(\rho^{\alpha}+\rho^{1-\alpha}\right)^{2}\left(i\left[X,Y\right]\right)\right].$$

As similar as we have

$$\begin{aligned}&\frac{1}{4}\left|Tr\left[\left(\rho^{\alpha}+\rho^{1-\alpha}\right)^{2}\left(i\left[X,Y\right]\right)\right]\right|^{2}\\ \le\ & 4\left(\frac{1}{4}Tr\left[\left(i[\rho^{\alpha},Y_0]\right)^{2}+\left(i[\rho^{1-\alpha},Y_0]\right)^{2}\right]+I_{\rho,\alpha}\left(Y\right)\right)\\ &\times\left(\frac{1}{4}Tr\left[\left\{\rho^{\alpha},X_0\right\}^{2}+\left\{\rho^{1-\alpha},X_0\right\}^{2}\right]+J_{\rho,\alpha}\left(X\right)\right).\end{aligned}$$

By the above two inequalities, we have

$$W_{\rho,\alpha}\left(X\right)W_{\rho,\alpha}\left(Y\right)\ge\frac{1}{4}\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^{2}\left[X,Y\right]\right]\right|^{2}.$$

□

Corollary 2.7. For a quantum state ρ and observables (possibly unbounded operators) X,Y and $\alpha\in[0,1]$, if we have the relation $[X,Y]=\frac{1}{2\pi i}I$ on $\mathbf{dom}(XY)\cap\mathbf{dom}(YX)$ and ρ is expressed by $\rho=\sum_k\lambda_k|\phi_k\rangle\langle\phi_k|$, $|\phi_k\rangle\in\mathbf{dom}(XY)\cap\mathbf{dom}(YX)$, then

$$W_{\rho,\alpha}(X)W_{\rho,\alpha}(Y)\ge\frac{1}{4}\left|Tr\left[\rho\left[X,Y\right]\right]\right|^{2}.$$

Proof: It follows from Theorem 2.6 and the following inequality:

$$\frac{1}{4}\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^{2}\left[X,Y\right]\right]\right|^{2}\ge\frac{1}{4}\left|Tr\left[\rho\left[X,Y\right]\right]\right|^{2},$$

whenever we have the canonical commutation relation such as $[X,Y]=\frac{1}{2\pi i}I$.

□

Remark 2.8. Theorem 2.6 is not trivial one in the sense of the following (i) and (ii).

(i) Since the arithmetic mean is greater than the geometric mean, $Tr\left[\left(i\left[\rho^{\alpha},X_0\right]\right)^{2}\right]\ge 0$ and $Tr\left[\left(i\left[\rho^{1-\alpha},X_0\right]\right)^{2}\right]\ge 0$ imply $K_{\rho,\alpha}\left(X\right)\ge I_{\rho,\alpha}\left(X\right)$, by the use of Schwarz's inequality. Similarly, $Tr\left[\left\{\rho^{\alpha},Y_0\right\}^{2}\right]\ge 0$ and $Tr\left[\left\{\rho^{1-\alpha},Y_0\right\}^{2}\right]\ge 0$ imply $L_{\rho,\alpha}\left(Y\right)\ge J_{\rho,\alpha}\left(Y\right)$. We then have $W_{\rho,\alpha}\left(X\right)\ge U_{\rho,\alpha}\left(X\right)$.

From the inequality (2.8) and the above, our situation is that we have

$$U_{\rho,\alpha}(H)\le U_{\rho}(H)$$

and

$$U_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H).$$

Our third concern is the ordering between $U_\rho(H)$ and $W_{\rho,\alpha}(H)$. However, we have no ordering between them. Because we have the follwoing examples. We set

$$\rho = \begin{pmatrix} 0.8 & 0.0 \\ 0.0 & 0.2 \end{pmatrix}, H = \begin{pmatrix} 2.0 & 3.0 \\ 3.0 & 1.0 \end{pmatrix}.$$

If we take $\alpha = 0.8$, then $U_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes -0.0241367. If we take $\alpha = 0.9$, then $U_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes 0.404141. This example actually shows that there exists a triplet of α, ρ and H such that $W_{\rho,\alpha}(H) < V_\rho(H)$, since we have $U_\rho(H) \leq V_\rho(H)$ in general.

(ii) We have no ordering between $\left| Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2$ and $|Tr\,[\rho[X,Y]]|^2$, by the follwoing examples. If we take

$$\rho = \frac{1}{7}\begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & -i & 1-i \\ i & 1 & i \\ 1+i & -i & 3 \end{pmatrix},$$

then we have

$$\left| Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \simeq 0.348097, |Tr\,[\rho\,[X,Y]]|^2 \simeq 0.326531.$$

If we take

$$\rho = \frac{1}{7}\begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & i \\ 0 & -i & 3 \end{pmatrix},$$

then we have

$$\left| Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \simeq 0.304377, |Tr\,[\rho\,[X,Y]]|^2 \simeq 0.326531.$$

Remark 2.9. (i) If we take $M = \rho^{1/2}X_0 x + \rho^{1/2}Y_0$ for any $x \in \mathbb{R}$ presented in Eq.(2.12), we recover the Heisenberg uncertainty relation Eq.(2.1) shown in [3].

(ii) If we take $\alpha = \frac{1}{2}$, then we recover the inequality (2.3) presented in [6].

(iii) We have another inequalities which are different from the inequality (2.11), by taking different self-adjoint operators M appeared in the proof of Theorem 2.6.

Conjecture 2.10. Our fourth concern is whether the following inequality:

$$U_{\rho,\alpha}(X)U_{\rho,\alpha}(Y) \geq \frac{1}{4}\left| Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \tag{2.13}$$

holds or not. However we have not found its proof and any counter-examples yet.

$K_{\rho,\alpha}(H)$ and $L_{\rho,\alpha}(H)$ are respectively rewritten by

$$K_{\rho,\alpha}(H) = Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 H_0^2 - \left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right) H_0 \left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right) H_0\right]$$

and

$$L_{\rho,\alpha}(H) = Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 H_0^2 + \left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right) H_0 \left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right) H_0\right].$$

Thus we have

$$\frac{1}{2}Tr\left[\left(i\left[\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, H_0\right]\right)^2\right] = \frac{1}{2}Tr\left[\left(i\left[\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, H\right]\right)^2\right]$$

but we have

$$\frac{1}{2}Tr\left[\left(\left\{\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right] \neq \frac{1}{2}Tr\left[\left(\left\{\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, H\right\}\right)^2\right].$$

In addition, we have $L_{\rho,\alpha}(H) \geq K_{\rho,\alpha}(H)$ which implies

$$W_{\rho,\alpha}(H) \equiv \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)} \geq \sqrt{K_{\rho,\alpha}(H)K_{\rho,\alpha}(H)} \geq K_{\rho,\alpha}(H).$$

Therefore our fifth concern is whether the following inequality for $\alpha \in [0,1]$ holds or not:

$$K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) \geq \frac{1}{4}\left|Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2. \tag{2.14}$$

However this inequality fails, because we have a counter-example. If we set $\alpha = \frac{1}{2}$ and

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have,

$$K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) = I_\rho(X)I_\rho(Y) = \left(\frac{1-\sqrt{3}}{2}\right)^2$$

and

$$\frac{1}{4}\left|Tr\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 = \frac{1}{4}\left|Tr\left[\rho\,[X,Y]\right]\right|^2 = \frac{1}{4}.$$

Thus the inequality (2.14) does not hold in general.

Before closing this section, we reconsider the ordering $W_{\rho,\alpha}(H)$ and $V_\rho(H)$, although we have already stated an example of the triplet α, ρ and H satsfying $W_{\rho,\alpha}(H) < V_\rho(H)$ in the last line of (i) of Remark 2.8. If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.45 \\ 0.45 & 0.7 \end{pmatrix}, H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes -0.3072. If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.7 \end{pmatrix}, H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes 0.682011. Therefore we have no ordering between $W_{\rho,\alpha}(H)$ and $V_\rho(H)$. Thus it is natural for us to have an interest in the following conjecture, since we have $K_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H)$ in general.

Conjecture 2.11. Our final concern is whether the following inequality:

$$K_{\rho,\alpha}(H) \leq V_\rho(H), \ \ \alpha \in [0,1] \tag{2.15}$$

holds or not. However we have not found its proof and any counter-examples yet.

3. Concluding remarks

As we have seen, we introduced a generalized Wigner-Yanase skew information $K_{\rho,\alpha}(H)$ and then defined a new quantity $W_{\rho,\alpha}(H)$. We note that our generalied Wigner-Yanase skew information $K_{\rho,\alpha}(H)$ is different type of the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$. For the quantity $K_{\rho,\alpha}(H)$, we do not have a trace inequality related to an uncertainty relation. However, we showed that we have a trace inequality related to an uncertainty relation for the quantity $W_{\rho,\alpha}(H)$. This inequality is a non-trivial one-parameter extension of the uncertainty relation Eq.(2.3) shown by S.Luo in [6]. In addition, we studied several trace inequaities on informational quantities.

Finally, we give another generalized trace inequality of the inequality (2.3). For a quantum state ρ an observable H and $\alpha \in [0,1]$, we define

$$Z_{\rho,\alpha}(H) \equiv \frac{1}{4}\sqrt{Tr\left[(i[\rho^{\alpha},H_0])^2\right] Tr\left[(i[\rho^{1-\alpha},H_0])^2\right] Tr\left[\{\rho^{\alpha},H_0\}^2\right] Tr\left[\{\rho^{1-\alpha},H_0\}^2\right]},$$

with $H_0 \equiv H - Tr[\rho H]I$. Then we have the following inequality

$$\sqrt{Z_{\rho,\alpha}(X) Z_{\rho,\alpha}(Y)} \geq \frac{1}{4}\left| Tr\left[\rho^{2\alpha}[X,Y]\right] Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]\right|, \tag{3.1}$$

for a quantum state ρ, two observables X, Y and $\alpha \in [0,1]$. We note that the inequality (3.1) recovers the inequality (2.3) by taking $\alpha = 1/2$ and we do not have any weak-strong relation between the inequality (2.11) and the inequality (3.1).

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