Schrödinger uncertainty relation, Wigner-Yanase-Dyson skew information and metric adjusted correlation measure

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Abstract. In this paper, we give Schrödinger-type uncertainty relation using the Wigner-Yanase-Dyson skew information. In addition, we give Schrödinger-type uncertainty relation by use of a two-parameter extended correlation measure. We finally show the further generalization of Schrödinger-type uncertainty relation by use of the metric adjusted correlation measure. These results generalize our previous result in [Phys. Rev. A, Vol. 82(2010), 034101].

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1 Introduction

In quantum information theory, one of the most important results is the strong subadditivity of von Neumann entropy [22]. This important property of von Neumann entropy can be proven by the use of Lieb’s theorem [16] which gave a complete solution for the conjecture of the convexity of Wigner-Yanase-Dyson skew information. In addition, the uncertainty relation has been widely studied in quantum information theory [21, 31, 29]. In particular, the relations between skew information and uncertainty relation have been studied in [17, 4, 8, 9, 7]. Quantum Fisher information is also called monotone metric which was introduced by Petz [23] and the Wigner-Yanase-Dyson metric is connected to quantum Fisher information (monotone metric) as a special case. Recently, Hansen gave a further development of the notion of monotone metric, so-called metric adjusted skew information [12]. The Wigner-Yanase-Dyson skew information is also connected to the metric adjusted skew information as a special case. That is, the metric adjusted skew information gave a class including the Wigner-Yanase-Dyson skew information, while the monotone metric gave a class including the Wigner-Yanase-Dyson metric. In the paper [12], the metric adjusted correlation measure was also introduced as a generalization of the quantum covariance and correlation measure defined in [17]. Therefore there is a significance to give the relation among the Wigner-Yanase-Dyson skew information, metric adjusted correlation measure and uncertainty relation for the fundamental studies on quantum information theory.
We start from the Heisenberg uncertainty relation [13]:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2$$  \tag{1}

for a quantum state (density operator) $\rho$ and two observables (self-adjoint operators) $A$ and $B$. The further stronger result was given by Schrödinger in [27, 28]:

$$V_\rho(A)V_\rho(B) - |\text{Re \{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2,$$  \tag{2}

where the covariance is defined by $\text{Cov}_\rho(A, B) \equiv \text{Tr}[\rho (A - \text{Tr}[\rho A]) (B - \text{Tr}[\rho B])]$.

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state $\rho$ and an observable $H$. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture [18]:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2},$$  \tag{3}

with the Wigner-Yanase skew information [32]:

$$I_\rho(H) \equiv \frac{1}{2} \text{Tr} \left[ (i[\rho^{1/2}, H_0])^2 \right] = \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^{1/2}H_0\rho^{1/2}H_0], \quad H_0 \equiv H - \text{Tr}[\rho H]I$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(H)$ in [18]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2.$$  \tag{4}

As stated in [18], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state $\rho$, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [19, 20]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(H)$.

Recently, a one-parameter extension of the inequality (4) was given in [33]:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha (1 - \alpha) |\text{Tr}[\rho[A, B]]|^2,$$  \tag{5}

where

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2},$$

with the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$ is defined by

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr} \left[ (i[\rho^\alpha, H_0]) (i[\rho^{1-\alpha}, H_0]) \right] = \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

It is notable that the convexity of $I_{\rho,\alpha}(H)$ with respect to $\rho$ was successfully proven by Lieb in [16]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(H)$ has been given in [34] using the generalized Wigner-Yanase-Dyson skew information introduced in [3]. See also [1, 5, 7, 8] for the recent studies on skew informations and uncertainty relations.

Motivated by the fact that the Schrödinger uncertainty relation is a stronger result than the Heisenberg uncertainty relation, a new Schrödinger-type uncertainty relation for mixed states using Wigner-Yanase skew information was shown in [4]. That is, for a quantum state $\rho$ and two observables $A$ and $B$, we have

$$U_\rho(A)U_\rho(B) - |\text{Re \{Corr}_\rho(A, B)\}|^2 \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2,$$  \tag{6}
where the correlation measure [17] is defined by

\[ \text{Corr}_\rho(X, Y) \equiv \text{Tr}[\rho X^* Y] - \text{Tr}[\rho^{1/2} X^* \rho^{1/2} Y] \]

for any operators \( X \) and \( Y \). This result refined the Heisenberg-type uncertainty relation (4) shown in [18] for mixed states (general states). We easily find that the inequality (6) is equivalent to the following inequality:

\[ U_\rho(A) U_\rho(B) \geq |\text{Corr}_\rho(A, B)|^2. \]  

The main purpose of this paper is to give some extensions of the inequality (7) by using the Wigner-Yanase-Dyson skew information \( I_{\rho,\alpha}(H) \) and the metric adjusted correlation measure introduced in [12].

2 Schrödinger uncertainty relation with Wigner-Yanase-Dyson skew information

In this section, we give a generalization of the Schrödinger type uncertainty relation (7) by the use of the quantity \( U_{\rho,\alpha}(H) \) defined by the Wigner-Yanase-Dyson skew information \( I_{\rho,\alpha}(H) \).

Theorem 2.1 For \( \alpha \in [1/2, 1] \), a quantum state \( \rho \) and two observables \( A \) and \( B \), we have

\[ U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha}(A, B)|^2. \]  

where the generalized correlation measure [14, 36] is defined by

\[ \text{Corr}_{\rho,\alpha}(X, Y) \equiv \text{Tr}[\rho X^* Y] - \text{Tr}[\rho^{1-\alpha} X^* \rho^{1-\alpha} Y] \]

for any operators \( X \) and \( Y \).

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 ([33]) For a spectral decomposition of \( \rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle \langle \phi_j| \), putting \( h_{ij} \equiv \langle \phi_i|H_0|\phi_j \rangle \), we have the following relations.

(i) For the Wigner-Yanase-Dyson skew information, we have

\[ I_{\rho,\alpha}(H) = \sum_{i<j} (\lambda_i^\alpha - \lambda_j^\alpha) \left( \lambda_i^{1-\alpha} - \lambda_j^{1-\alpha} \right) |h_{ij}|^2. \]

(ii) For the quantity associated to the Wigner-Yanase-Dyson skew information:

\[ J_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr} \left[ \{ (\rho^\alpha, H_0) \} \{ (\rho^{1-\alpha}, H_0) \} \right] = \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \]

where \( \{ X, Y \} \equiv XY + YX \) is an anti-commutator, we have

\[ J_{\rho,\alpha}(H) \geq \sum_{i<j} (\lambda_i^\alpha + \lambda_j^\alpha) \left( \lambda_i^{1-\alpha} + \lambda_j^{1-\alpha} \right) |h_{ij}|^2. \]

Lemma 2.3 ([2, 33]) For any \( t > 0 \) and \( \alpha \in [0, 1] \), we have

\[ (1 - 2\alpha)^2(t-1)^2 \geq (t^\alpha - t^{1-\alpha})^2. \]
Proof of Theorem 2.1: We take a spectral decomposition $\rho = \sum_{j=1}^{\infty} \lambda_j \phi_j \langle \phi_j \rangle$. If we put $a_{ij} = \langle \phi_i | A_0 | \phi_j \rangle$ and $b_{ij} = \langle \phi_j | B_0 | \phi_i \rangle$, where $A_0 = A - Tr[\rho A] I$ and $B_0 = B - Tr[\rho B] I$, then we have

$$
Corr_{\rho,\alpha}(A, B) = Tr[\rho AB] - Tr[\rho^\alpha A\rho^{1-\alpha} B]
= Tr[\rho A_0 B_0] - Tr[\rho^\alpha A_0 \rho^{1-\alpha} B_0]
= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji}
= \sum_{i \neq j} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji}
= \sum_{i < j} \left\{ (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} + (\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}) a_{ij} b_{ji} \right\}.
$$

Thus we have

$$
|Corr_{\rho,\alpha}(A, B)| \leq \sum_{i < j} \left\{ |\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}| |a_{ij}| |b_{ji}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}| |a_{ij}| |b_{ji}| \right\}.
$$

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, taking a square of both sides and then using Schwarz inequality and Lemma 2.2, we have

$$
4\alpha(1 - \alpha)|Corr_{\rho,\alpha}(A, B)|^2
\leq 4\alpha(1 - \alpha) \left\{ \sum_{i < j} \left\{ |\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}| \right\} |a_{ij}| |b_{ji}| \right\}^2

= \left\{ \sum_{i < j} 2\sqrt{\alpha(1 - \alpha)} (\lambda_i^\alpha + \lambda_j^\alpha) |\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}| |a_{ij}| |b_{ji}| \right\}^2

\leq \left\{ \sum_{i < j} 2\sqrt{\alpha(1 - \alpha)} |\lambda_i - \lambda_j||a_{ij}||b_{ji}| \right\}^2

\leq \left\{ \sum_{i < j} \left\{ (\lambda_i^\alpha - \lambda_j^\alpha) (\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) + (\lambda_i^\alpha + \lambda_j^\alpha) (\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}) \right\}^{1/2} |a_{ij}| |b_{ji}| \right\}^2

\leq \left\{ \sum_{i < j} (\lambda_i^\alpha - \lambda_j^\alpha) (\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) |a_{ij}|^2 \right\} \left\{ \sum_{i < j} (\lambda_i^\alpha + \lambda_j^\alpha) (\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}) |b_{ij}|^2 \right\}

\leq I_{\rho,\alpha}(A) J_{\rho,\alpha}(B).
$$

In the above process, the inequality $(x^\alpha + y^\alpha)|x^{1-\alpha} - y^{1-\alpha}| \leq |x - y|$ for $x, y \geq 0$ and $\alpha \in [\frac{1}{2}, 1]$ and the inequality $4\alpha(1 - \alpha)(x - y)^2 \leq (x^\alpha - y^\alpha) (x^{1-\alpha} - y^{1-\alpha}) (x^\alpha + y^\alpha) (x^{1-\alpha} + y^{1-\alpha})$ for $x, y \geq 0$ and $\alpha \in [0, 1]$, which can be proven by Lemma 2.3, were used. By the similar way, we also have

$$
4\alpha(1 - \alpha)|Corr_{\rho,\alpha}(A, B)|^2 \leq I_{\rho,\alpha}(B) J_{\rho,\alpha}(A).
$$

Thus for $\alpha \geq \frac{1}{2}$ we have

$$
4\alpha(1 - \alpha)|Corr_{\rho,\alpha}(A, B)|^2 \leq U_{\rho,\alpha}(A) U_{\rho,\alpha}(B).
$$

Note that Theorem 2.1 recovers the inequality (7), if we take $\alpha = \frac{1}{2}$.
Remark 2.4 We take \( \alpha = 0.1 \) and 
\[
\rho = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 - i \\ 2 + i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},
\]
then we have 
\[
U_{\rho, \alpha}(A)U_{\rho, \alpha}(B) - 4\alpha(1 - \alpha)|\text{Corr}_{\rho, \alpha}(A, B)|^2 \simeq -0.28332.
\]
Therefore the inequality (8) does not hold for \( \alpha \in [0, 1/2) \) in general.

Corollary 2.5 Under the same assumptions with Theorem 2.1, we have the following inequality:
\[
U_{\rho, \alpha}(A)U_{\rho, \alpha}(B) - 4\alpha(1 - \alpha) \left( |\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 - |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2 \right) \\
\geq \alpha(1 - \alpha)|\text{Tr}[\rho[A, B]]|^2.
\]

Proof: From 
\[
|\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 = \frac{1}{2i} \text{Tr}[\rho[A, B]] - |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2,
\]
we have 
\[
\frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 \leq |\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2.
\]
Thus we have 
\[
|\text{Corr}_{\rho, \alpha}(A, B)|^2 = |\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + |\text{Im} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 \\
\geq |\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 - |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2,
\]
which proves the corollary.

Remark 2.6 The following inequality does not hold in general for \( \alpha \in [\frac{1}{2}, 1] \):
\[
|\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 \geq |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2.
\]
Because we have a counter-example as follows. We take \( \alpha = \frac{2}{3} \) and 
\[
\rho = \frac{1}{7} \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 - i \\ 2 + i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},
\]
then we have 
\[
|\text{Re} \{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 - |\text{Im} \{\text{Tr}[\rho^\alpha A^1 - \alpha B]\}|^2 \simeq -0.0548142.
\]
This shows Theorem 2.1 does not refine the inequality (5) in general.

3 Two-parameter extensions

In this section, we introduce the parametric extended correlation measure \( \text{Corr}_{\rho, \alpha, \gamma}(X, Y) \) by the convex combination between \( \text{Corr}_{\rho, \alpha}(X, Y) \) and \( \text{Corr}_{\rho, 1 - \alpha}(X, Y) \). Then we establish the parametric extended Schrödinger-type uncertainty relation applying the parametric extended correlation measure \( \text{Corr}_{\rho, \alpha, \gamma}(X, Y) \). In addition, introducing the symmetric extended correlation measure \( \text{Corr}_{\rho, \alpha, \gamma}^{(\text{sym})}(X, Y) \) by the convex combination between \( \text{Corr}_{\rho, \alpha}(X, Y) \) and \( \text{Corr}_{\rho, \alpha}(Y, X) \), we show its Schrödinger-type uncertainty relation.
**Definition 3.1** We define the parametric extended correlation measure $\text{Corr}_{\rho,\alpha,\gamma}(X,Y)$ for two parameters $\alpha, \gamma \in [0,1]$ by

$$\text{Corr}_{\rho,\alpha,\gamma}(X,Y) \equiv \gamma\text{Corr}_{\rho,\alpha}(X,Y) + (1-\gamma)\text{Corr}_{\rho,1-\alpha}(X,Y)$$

for any operators $X$ and $Y$.

Note that we have $\text{Corr}_{\rho,\alpha,\gamma}(H,H) = I_{\rho,\alpha}(H)$ for any observable $H$. Then we can prove the following inequality.

**Theorem 3.2** If $0 \leq \alpha, \gamma \leq \frac{1}{2}$ or $\frac{1}{2} \leq \alpha, \gamma \leq 1$, then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha,\gamma}(A,B)|^2$$

for two observables $A$, $B$ and a quantum state $\rho$.

**Proof:** By the similar way of the proof of Theorem 2.1, we have Eq.(9) and we also have

$$\text{Corr}_{\rho,1-\alpha}(A,B) = \text{Tr}[\rho AB] - \text{Tr}[\rho^{1-\alpha}A\rho^\alpha B]$$

$$= \sum_{i<j} \left\{ (\lambda_i - \lambda_j^{1-\alpha}\lambda_j^\alpha)a_{ij}b_{ji} + (\lambda_j - \lambda_i^{1-\alpha}\lambda_i^\alpha)a_{ji}b_{ij} \right\}.$$  

Thus we have

$$\text{Corr}_{\rho,\alpha,\gamma}(A,B) = \gamma\text{Corr}_{\rho,\alpha}(A,B) + (1-\gamma)\text{Corr}_{\rho,1-\alpha}(A,B)$$

$$= \sum_{i<j} \left\{ \gamma\lambda_i^\alpha(\lambda_j^{1-\alpha} - \lambda_j^{1-\alpha}) + (1-\gamma)\lambda_i^{1-\alpha}(\lambda_j^\alpha - \lambda_j^\alpha) \right\} a_{ij}b_{ji}$$

$$+ \sum_{i<j} \left\{ \gamma\lambda_j^\alpha(\lambda_j^{1-\alpha} - \lambda_j^{1-\alpha}) + (1-\gamma)\lambda_j^{1-\alpha}(\lambda_j^\alpha - \lambda_j^\alpha) \right\} a_{ji}b_{ij}.$$  

Since we have $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, we then have

$$|\text{Corr}_{\rho,\alpha,\gamma}(A,B)| \leq \sum_{i<j} \left\{ \gamma(\lambda_i^\alpha + \lambda_j^\alpha)|\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}| + (1-\gamma)(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha})|\lambda_i^\alpha - \lambda_j^\alpha| \right\} |a_{ij}||b_{ij}|$$

$$\leq \sum_{i<j} |\lambda_i - \lambda_j||a_{ij}||b_{ji}|,$$

thanks to the inequality

$$\gamma|x^\alpha + y^\alpha|x^{1-\alpha} - y^{1-\alpha}| + (1-\gamma)(x^{1-\alpha} + y^{1-\alpha})|x^\alpha - y^\alpha| \leq |x - y|$$

for $0 \leq \alpha, \gamma \leq \frac{1}{2}$ or $\frac{1}{2} \leq \alpha, \gamma \leq 1$, and $x, y \geq 0$. The rest of the proof goes similar way to that of Theorem 2.1.

**Corollary 3.3** For any $\alpha \in [0,1]$, two observables $A$, $B$ and a quantum state $\rho$, we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha,\frac{1}{2}}(A,B)|^2.$$  

**Proof:** If $\gamma = \frac{1}{2}$, then the equality of the inequality (15) holds for any $\alpha \in [0,1]$ and $x, y \geq 0$. Therefore we have the present corollary from Theorem 3.2.

We may define the following correlation measure instead of Definition 3.1.
Definition 3.4 We define a symmetric extended correlation measure $\text{Corr}_{\rho,\alpha,\gamma}^{(\text{sym})}(X,Y)$ for two parameters $\alpha, \gamma \in [0,1]$ by

$$\text{Corr}_{\rho,\alpha,\gamma}^{(\text{sym})}(X,Y) \equiv \gamma \text{Corr}_{\rho,\alpha}(X,Y) + (1-\gamma)\text{Corr}_{\rho,\alpha}(Y,X)$$

(16)

for any operators $X$ and $Y$.

Note that we have $\text{Corr}_{\rho,\alpha,\gamma}^{(\text{sym})}(A,B) = \text{Corr}_{\rho,\alpha,\gamma}^{(\text{sym})}(B,A)$ for self-adjoint operators $A$ and $B$.

Then we have the following theorem by the similar proof of the above using the inequality $(x^\alpha + y^\alpha)|x^{1-\alpha} - y^{1-\alpha}| \leq |x-y|$ for $x, y \geq 0$ and $\alpha \geq \frac{1}{2}$.

Theorem 3.5 For $\alpha \in [\frac{1}{2},1]$ and $\gamma \in [0,1]$, we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha,\gamma}^{(\text{sym})}(A,B)|^2$$

for two observables $A, B$ and a quantum state $\rho$.

4 A further generalization by metric adjusted correlation measure

Inspired by the recent results in [10] and the concept of metric adjusted skew information introduced by Hansen in [12], we here give a further generalization for Schrödinger-type uncertainty relation applying metric adjusted correlation measure introduced in [12]. We firstly give some notations according to those in [10]. Let $M_n(\mathbb{C})$ and $M_{n,sa}(\mathbb{C})$ be the set of all $n \times n$ complex matrices and all $n \times n$ self-adjoint matrices, equipped with the Hilbert-Schmidt scalar product $\langle A, B \rangle = Tr[A^*B]$, respectively. Let $M_{n,+}(\mathbb{C})$ be the set of all positive definite matrices of $M_{n,sa}(\mathbb{C})$ and $M_{n,+1}(\mathbb{C})$ be the set of all density matrices, that is

$$M_{n,+}(\mathbb{C}) \equiv \{ \rho \in M_{n,sa}(\mathbb{C})| Tr\rho = 1, \rho > 0 \} \subset M_{n,+}(\mathbb{C}).$$

Here $X \in M_{n,+}(\mathbb{C})$ means we have $\langle \phi | X | \phi \rangle \geq 0$ for any vector $| \phi \rangle \in \mathbb{C}^n$. In the study of quantum physics, we usually use a positive semidefinite matrix with a unit trace as a density operator $\rho$. In this section, we assume the invertibility of $\rho$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if the inequalities $0 \leq f(A) \leq f(B)$ hold for any $A, B \in M_{n,sa}(\mathbb{C})$ such that $0 \leq A \leq B$. An operator monotone function $f : (0, +\infty) \to (0, +\infty)$ is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$. We represents the set of all symmetric normalized operator monotone functions by $\mathcal{F}_{op}$. We have the following examples as elements of $\mathcal{F}_{op}$:

Example 4.1 ([12, 10, 6, 25])

$$f_{\text{RLD}}(x) = \frac{2x}{x+1}, \quad f_{\text{SLD}}(x) = \frac{x+1}{2}, \quad f_{\text{BKM}}(x) = \frac{x-1}{\log x},$$

$$f_{\text{WY}}(x) = \left( \frac{\sqrt{x+1}}{2} \right)^2, \quad f_{\text{WYD}}(x) = \alpha(1-\alpha) \frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}; \quad \alpha \in (0,1).$$
The functions $f_{BKM}(x)$ and $f_{WYD}(x)$ are normalized in the sense that $\lim_{x \to 1} f_{BKM}(x) = 1$ and $\lim_{x \to 1} f_{WYD}(x) = 1$. Note that a simple proof of the operator monotonicity of $f_{WYD}(x)$ was given in [6]. See also [30] for the proof of the operator monotonicity of $f_{WYD}(x)$ by use of majorization.

**Remark 4.2 ([10, 15, 24, 25])** For any $f \in \mathcal{F}_\text{op}$, we have the following inequalities:

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}, \quad x > 0.$$  

That is, all $f \in \mathcal{F}_\text{op}$ lies in between the harmonic mean and the arithmetic mean.

For $f \in \mathcal{F}_\text{op}$ we define $f(0) = \lim_{x \to 0} f(x)$. We also denote the sets of regular and non-regular functions by

$$\mathcal{F}_\text{op}^\circ = \{ f \in \mathcal{F}_\text{op} | f(0) \neq 0 \} \quad \text{and} \quad \mathcal{F}_\text{op}^n = \{ f \in \mathcal{F}_\text{op} | f(0) = 0 \}.$$

**Definition 4.3 ([8, 10])** For $f \in \mathcal{F}_\text{op}^\circ$, we define the function $\tilde{f}$ by

$$\tilde{f}(x) = \frac{1}{2} \left\{ (x + 1) - (x - 1)^2 f(0) \frac{1}{f(x)} \right\}, \quad (x > 0).$$

Then we have the following theorem.

**Theorem 4.4 ([8, 6, 26])** The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_\text{op}^\circ$ and $\mathcal{F}_\text{op}^n$.

We can use matrix mean theory introduced by Kubo-Ando in [15]. Then a mean $m_f$ corresponds to each operator monotone function $f \in \mathcal{F}_\text{op}$ by the following formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} BA^{-1/2}) A^{1/2},$$

for $A, B \in M_{n,+}(\mathbb{C})$. By the notion of matrix mean, we may define the set of the monotone metrics [23] by the following formula

$$\langle A, B \rangle_{\rho,f} = \text{Tr}[A m_f(L_{\rho}, R_{\rho})^{-1}(B)],$$

where $L_{\rho}(A) = \rho A$ and $R_{\rho}(A) = A \rho$.

**Definition 4.5 ([12, 8])** For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+}(\mathbb{C})$ and $f \in \mathcal{F}_\text{op}^\circ$, we define the following quantities:

$$\text{Corr}_{\rho}^f(A, B) \equiv \frac{f(0)}{2} \langle [i \rho, A], i \rho, B \rangle_{\rho,f}, \quad I_{\rho}^f(A) \equiv \text{Corr}_{\rho}^f(A, A),$$

$$C_{\rho}^f(A, B) \equiv \text{Tr}[m_f(L_{\rho}, R_{\rho})(A)B], \quad C_{\rho}^f(A) \equiv C_{\rho}^f(A, A),$$

$$U_{\rho}^f(A) \equiv \sqrt{V_{\rho}(A)^2 - (V_{\rho}(A) - I_{\rho}^f(A))^2}. $$

The quantity $I_{\rho}^f(A)$ is known as metric adjusted skew information [12]. It is notable that the metric adjusted correlation measure $\text{Corr}_{\rho}^f(A, B)$ was firstly introduced in [12] for a regular Morozova-Chentsov function $c$. Recently the notation $I_{\rho}^f(A, B)$ in [1] and the notation $I_{\rho}^f(A, B)$ in [11] were used. In addition, it is useful for the readers to be noted that the correlation $I_{\rho}^f(A, B)$ can be expressed as a difference of covariances [11]. Throughout the present paper, we use the notation $\text{Corr}_{\rho}^f(A, B)$ as the metric adjusted correlation measure, to avoid the confusion of the readers. (In the previous sections, we have already used $\text{Corr}_\rho(A, B)$, $\text{Corr}_{\rho,\alpha}(A, B)$ and $\text{Corr}_{\rho,\alpha,\gamma}(A, B)$ as correlation measures and done $I_{\rho}(H)$ and $I_{\rho,\alpha}(H)$ as skew informations.) Then we have the following proposition.
Proposition 4.6 ([8, 10]) For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+1}(\mathbb{C})$ and $f \in \mathcal{F}_\text{op}$, we have the following relations, where we put $A_0 \equiv A - \text{Tr}[\rho A]I$ and $B_0 \equiv B - \text{Tr}[\rho B]I$.

1. \[ I_\rho^f(A) = \text{Tr}[\rho A_0^2] - \text{Tr}[m_f(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) - C_\rho^f(A_0). \]

2. \[ J_\rho^f(A) = \text{Tr}[\rho A_0^2] + \text{Tr}[m_f(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) + C_\rho^f(A_0). \]

3. \[ 0 \leq I_\rho^f(A) \leq U_\rho^f(A) \leq V_\rho(A). \]

4. \[ U_\rho^f(A) = \sqrt{I_\rho^f(A) J_\rho^f(A)}. \]

5. \[ \text{Corr}_\rho^f(A, B) = \frac{1}{2} \text{Tr}[\rho A_0 B_0] + \frac{1}{2} \text{Tr}[\rho B_0 A_0] - \text{Tr}[m_f(L_\rho, R_\rho)(A_0)B_0] = \frac{1}{2} \text{Tr}[\rho A_0 B_0] + \frac{1}{2} \text{Tr}[\rho B_0 A_0] - C_\rho^f(A_0, B_0). \]

The following inequality is the further generalization of Corollary 3.3 by the use of the metric adjusted correlation measure.

Theorem 4.7 For $f \in \mathcal{F}_\text{op}$, if we have
\[
\frac{x + 1}{2} + f(x) \geq 2f(x),
\]
then we have
\[
U_\rho^f(A) U_\rho^f(B) \geq 4f(0) |\text{Corr}_\rho^f(A, B)|^2,
\]
for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+1}(\mathbb{C})$.

In order to prove Theorem 4.7, we use the following two lemmas.

Lemma 4.8 ([35]) If Eq.(17) is satisfied, then we have the following inequality:
\[
\left( \frac{x + y}{2} \right)^2 - m_f(x, y)^2 \geq f(0)(x - y)^2.
\]

Proof: By Eq.(17), we have
\[
\frac{x + y}{2} + m_f(x, y) \geq 2m_f(x, y).
\]

We also have
\[
m_f(x, y) = y \tilde{f} \left( \frac{x}{y} \right)
\]
\[
= y \left\{ \frac{x}{y} + 1 - \left( \frac{x}{y} - 1 \right) \right\}^2 \frac{f(0)}{f(x/y)}
\]
\[
= \frac{x + y}{2} - f(0)(x - y)^2 \cdot \frac{1}{2m_f(x, y)}.
\]

Therefore
\[
\left( \frac{x + y}{2} \right)^2 - m_f(x, y)^2 \geq \left\{ \frac{x + y}{2} - m_f(x, y) \right\} \left\{ \frac{x + y}{2} + m_f(x, y) \right\}
\]
\[
\geq \frac{f(0)(x - y)^2}{2m_f(x, y)} \cdot 2m_f(x, y)
\]
\[
= f(0)(x - y)^2.
\]
We have the following expressions for the quantities $I_\rho^f(A)$, $J_\rho^f(A)$, $U_\rho^f(A)$ and $Corr_\rho^f(A, B)$ by using Proposition 4.6 and a mean $m_f$.

**Lemma 4.9 ([10])** Let $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ be a basis of eigenvectors of $\rho$, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We put $a_{jk} = \langle \phi_j|A_0|\phi_k\rangle$, $b_{jk} = \langle \phi_j|B_0|\phi_k\rangle$, where $A_0 = A - \text{Tr}[\rho A]I$ and $B_0 = B - \text{Tr}[\rho B]I$ for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+1}(\mathbb{C})$. Then we have

$$I_\rho^f(A) = \frac{1}{2} \sum_{j,k} (\lambda_j + \lambda_k) a_{jk} a_{kj} - \sum_{j,k} m_f(\lambda_j, \lambda_k) a_{jk} a_{kj}$$

$$= 2 \sum_{j<k} \left\{ \frac{\lambda_j + \lambda_k}{2} - m_f(\lambda_j, \lambda_k) \right\} |a_{jk}|^2,$$

$$J_\rho^f(A) = \frac{1}{2} \sum_{j,k} (\lambda_j + \lambda_k) a_{jk} a_{kj} + \sum_{j,k} m_f(\lambda_j, \lambda_k) a_{jk} a_{kj}$$

$$\geq 2 \sum_{j<k} \left\{ \frac{\lambda_j + \lambda_k}{2} + m_f(\lambda_j, \lambda_k) \right\} |a_{jk}|^2,$$

$$U_\rho^f(A)^2 = \frac{1}{4} \left( \sum_{j,k} (\lambda_j + \lambda_k) |a_{jk}|^2 \right)^2 - \left( \sum_{j,k} m_f(\lambda_j, \lambda_k) |a_{jk}|^2 \right)^2$$

and

$$Corr_\rho^f(A, B) = \frac{1}{2} \sum_{j,k} \lambda_j a_{jk} b_{kj} + \frac{1}{2} \sum_{j,k} \lambda_k a_{jk} b_{kj} - \sum_{j,k} m_f(\lambda_j, \lambda_k) a_{jk} b_{kj}$$

$$= \sum_{j<k} \left( \frac{\lambda_j + \lambda_k}{2} - m_f(\lambda_j, \lambda_k) \right) a_{jk} b_{kj} + \sum_{j<k} \left( \frac{\lambda_k + \lambda_j}{2} - m_f(\lambda_k, \lambda_j) \right) a_{kj} b_{kj}.$$  

(19)

We are now in a position to prove Theorem 4.7.

**Proof of Theorem 4.7:** From Eq.(19), we have

$$|Corr_\rho^f(A, B)| \leq \sum_{j<k} \left| \left( \frac{\lambda_j + \lambda_k}{2} - m_f(\lambda_j, \lambda_k) \right) a_{jk} b_{kj} \right| + \sum_{j<k} \left| \left( \frac{\lambda_k + \lambda_j}{2} - m_f(\lambda_k, \lambda_j) \right) a_{kj} b_{kj} \right|$$

$$\leq \sum_{j<k} \left| \frac{\lambda_j + \lambda_k}{2} - m_f(\lambda_j, \lambda_k) \right| |a_{jk}||b_{kj}| + \sum_{j<k} \left| \frac{\lambda_k + \lambda_j}{2} - m_f(\lambda_k, \lambda_j) \right| |a_{kj}||b_{kj}|$$

$$= 2 \sum_{j<k} \left| \frac{\lambda_j + \lambda_k}{2} - m_f(\lambda_j, \lambda_k) \right| |a_{jk}||b_{kj}|$$

$$\leq \sum_{j<k} |\lambda_j - \lambda_k||a_{jk}||b_{kj}|.$$

Then we have

$$f(0)|Corr_\rho^f(A, B)|^2 \leq \left( \sum_{j<k} f(0)^{1/2} |\lambda_j - \lambda_k||a_{jk}||b_{kj}| \right)^2.$$
\[
\begin{align*}
&\leq \left( \sum_{j<k} \left\{ \left( \frac{\lambda_j + \lambda_k}{2} \right)^2 - m_f(\lambda_j, \lambda_k)^2 \right\}\left| a_{jk} \right| \left| b_{kj} \right| \right) \frac{1}{2} \\
&\leq \left( \sum_{j<k} \left\{ \lambda_j + \lambda_k - m_f(\lambda_j, \lambda_k) \right\} \left| a_{jk} \right|^2 \right) \\
&\times \left( \sum_{j<k} \left\{ \lambda_j + \lambda_k + m_f(\lambda_j, \lambda_k) \right\} \left| b_{kj} \right|^2 \right) \\
&\leq \frac{1}{4} I^f_\rho(A) J^f_\rho(B).
\end{align*}
\]

By the similar way, we also have
\[
I^f_\rho(B) J^f_\rho(A) \geq 4f(0) \left| \text{Corr}^f_\rho(A, B) \right|^2.
\]

Hence we have the desired inequality (18).

**Remark 4.10** Under the same assumptions with Theorem 4.7, we have the following Heisenberg-type uncertainty relation [35]:
\[
U^f_\rho(A) U^f_\rho(B) \geq f(0) \left| \text{Tr} \left[ \rho [A, B] \right] \right|^2
\]
by the similar way to the proof of Theorem 4.7, since we have
\[
\left| \text{Tr} \left[ \rho [A, B] \right] \right| \leq 2 \sum_{j<k} \left| \lambda_j - \lambda_k \right| \left| a_{jk} \right| \left| b_{kj} \right|.
\]

As stated in Remark 2.6, there is no ordering between the right hand side of the inequality (18) and that of the inequality (20), in general.

If we use the function
\[
f_{\text{WYD}}(x) = \alpha (1 - \alpha) \frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1),
\]
then we obtain the following uncertainty relation.

**Corollary 4.11** For \( A, B \in M_{n,sa}(\mathbb{C}) \) and \( \rho \in M_{n,++}(\mathbb{C}) \), we have
\[
U^f_\rho^{\text{WYD}}(A) U^f_\rho^{\text{WYD}}(B) \geq 4\alpha (1 - \alpha) \left| \text{Corr}^f_\rho^{\text{WYD}}(A, B) \right|^2.
\]

**Proof:** From the definition
\[
f_{\text{WYD}}(x) = \alpha (1 - \alpha) \frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},
\]
it is clear that
\[
\tilde{f}_{\text{WYD}}(x) = \frac{1}{2} \{ x + 1 - (x^\alpha - 1)(x^{1-\alpha} - 1) \}.
\]
By Lemma 2.3, we have for \( 0 \leq \alpha \leq 1 \) and \( x > 0 \),
\[
(1 - 2\alpha)^2(x - 1)^2 - (x^\alpha - x^{1-\alpha})^2 \geq 0.
\]
This inequality can be rewritten by

\[(x^{2\alpha} - 1)(x^{2(1-\alpha)} - 1) \geq 4\alpha(1 - \alpha)(x - 1)^2.\]

Thus we have

\[
\frac{x + 1}{2} + f_{WYD}(x) = x + 1 - \frac{1}{2}(x^\alpha - 1)(x^{1-\alpha} - 1) \\
= \frac{1}{2}(x^\alpha + 1)(x^{1-\alpha} + 1) \\
\geq 2\alpha(1 - \alpha)\frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)} \\
= 2f_{WYD}(x).
\]

Thus we obtain the aimed result from Theorem 4.7.

Note that Corollary 3.3 coincides with Corollary 4.11, since we have \(U_{\rho,\alpha}(A) = U_{\rho}^{f_{WYD}}(A)\) which is obtained by the fact the function \(f_{WYD}(x)\) corresponds to the Wigner-Yanase-Dyson skew information. We also note that we have \(Corr_{\rho}^{f_{WYD}}(A, B) = Corr_{\rho,\alpha,\frac{1}{2}}^{(sym)}(A, B)\) and \(Corr_{\rho}^{f_{WYD}}(A, B) \neq Corr_{\rho,\alpha,\frac{1}{2}}(A, B)\) in general.

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